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Global Stability for a Asymptotically Periodic Cooperative Lotka-Volterra System with Time Delays

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Abstract

In this paper a class of cooperative Lotka-Volterra population system with time delay is considered. Some sufficient conditions on the existence and globally asymptotically stability for the asymptotically periodic solution of the system are established by using the Lyapunov function method and the method given in Fengying Wei and Wang Ke (Applied Mathematics and Computation 182 (2006) 161-165).

Keywords

Lotka-Volterra Cooperative System, Asymptotically Periodic Function, Global Asymptotic Stability, Time Delay

1. Introduction

Since Lotka-Volterra system has been established and was accepted by many scientists, it becomes the most important means to explain the ecological phenomenon now. For many years, a lot of extensive research results were made in mathematical biology and mathematical ecology [1]-[8], during this time Lotka-Volterra system has played an important role in theses research field of mathematical biology and mathematical ecology. Still now many research work mostly discussed periodic Lotka-Volterra systems [2] [3] [4] [5] [6] and the references cited therein. In fact asymptotically periodic systems [3] [4] describe our world more realistic and more accurate than periodic ones.

As is well known, Lotka-Volterra Cooperative system is one of the most important classe of interaction model which is discussed widely in mathematical biology and mathematical ecology.

In this paper we consider the following Lotka-Volterra cooperative system with time delay:

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) + a_{12}^1(t)x_2(t-\tau)\right] \\ \dot{x}_2(t) = x_2(t)\left[r_2(t) + a_{21}^0(t)x_1(t) + a_{21}^1(t)x_1(t-\tau) - a_{22}^0(t)x_2(t) - a_{22}^1(t)x_2(t-\tau)\right] \end{cases} \quad (1)$$

where $x_1(t)$, $x_2(t)$ are the density of two cooperative species at time t respectively, $r_i(t)(i=1,2)$ are intrinsic growth rate of two cooperative species at time t respectively, $a_{11}^l(t)(l=1,2)$, $a_{22}^l(t)(l=0,1)$ are the intra patch restriction density of species x_1, x_2, at time t respectively, and $a_{21}^l(t)(l=0,1)$, $a_{12}^1(t)$ are the are cooperative coefficients between two species at time t respectively. In this paper we assume that system (1) satisfies the following assumption

(**H1**) τ is a positive constant and $r_i(t)(i=1,2)$, $a_{11}^l(t)(l=1,2)$, $a_{22}^l(t)(l=0,1)$, $a_{21}^l(t)(l=0,1)$ and $a_{21}^1(t)$ are continuous, asymptotically periodic, bounded and strictly positive functions on $[-\tau,+\infty)$.

From the viewpoint of mathematical biology, in this paper, for system (1) we consider the solution with the following initial condition

$$x_1(t) = \phi_1(t) \ge 0, \quad \text{for } t \in [-2\tau, 0] \text{ and } \phi_1(0) > 0, \quad (2)$$

$$x_2(t) = \phi_2(t) \ge 0 \quad \text{for } t \in [-\tau, 0] \text{ and } \phi_2(0) > 0 \quad (3)$$

then for any (t_0,ϕ), $\phi \in C = C\left((-2\tau,0], R_+^n\right)$, System (1) with initial conditions has a unique solution denoted by $X(t,t_0,\phi)$.

For a continuous and bounded function $f(t)$, we define

$$f^L = \inf_{t\in[0,+\infty]}\{f(t)\} \quad \text{and} \quad f^M = \sup_{t\in[0,+\infty]}\{f(t)\}$$

Y. Nakata and Y. Muroya have proved in [1] that the system (1) is permanent under the following conditions

$$b_1^{lL} > 0(l=1,2), b_2^{lL} > 0 \quad \text{and} \quad b_{22}^{lL} > 0$$

where

$$b_1^l(t) = a_{11}^l - a_{21}^{l-1}(t-\tau) \quad \text{for} \quad l=1,2 \quad \text{and} \quad b_2^1(t) = a_{22}^0(t-\tau) - a_{21}^1(t)$$

which means that the system (1) had a bounded region that is

$$\Phi = \left\{\left(x_1(t), x_2(t)\right) : 0 < m_i \le x_i(t) \le M_i(t) \le +\infty (i=1,2)\right\} \quad (4)$$

In particularly,

$$M_1 = -\frac{a_{12}^{1M}P}{r_1^M} + \left\{\frac{a_{12}^{1M}P}{r_1^M} + x^*\right\}\exp\left(r_1^M 2\tau\right) \quad (5)$$

$$M_2 = \frac{r_2^M + \left(a_{21}^{0M} + a_{21}^{1M}\right)M_1}{a_{22}^{0L} + a_{22}^{1L}}\exp\left\{\left(r_2^M + \left(a_{21}^{0M} + a_{21}^{1M}\right)M_1\right)\tau\right\} \quad (6)$$

$$m_1 = \frac{r_1^L}{a_{11}^{1M} + a_{11}^{2M}}\exp\left\{\left(r_1^L - \left(a_{11}^{1M} + a_{11}^{2M}\right)M_1\right)2\tau\right\} \quad (7)$$

$$m_2 = \frac{r_2^L}{a_{22}^{0M} + a_{22}^{1M}}\exp\left\{\left(r_2^L - \left(a_{22}^{0M} + a_{22}^{1M}\right)M_2\right)\tau\right\} \quad (8)$$

where $x = x_1^*$ is the unique positive solution of $x\left(r_1^M - \left(a_{11}^{1L} + a_{11}^{2L}\right)x\right) + a_{12}^{1M}P = 0$, and p is a positive constant such that,

$$\lim_{t\to+\infty}\sup x_1(t)x_2(t-\tau)\le P=\frac{\left(r_1^M+r_2^M\right)^2}{b_1^{1L}a_{22}^{1L}}\exp\left\{\left(r_1^M+r_2^M\right)\tau\right\}<+\infty$$

Let the set

$$\Gamma=\left\{\left(x_1(t),x_2(t)\right)\in R_+^2:0<m_i\le M_i(t)<+\infty(i=1,2),R_+=[0,+\infty)\right\}$$

where $m_i,M_i(i=1,2)$ are given above, then set Γ is the ultimately bounded set of system (1)

Following is the adjoin system (2) of system (1)

$$\begin{cases}\dot{x}_1(t)=x_1(t)\left[r_1(t)-a_{11}^1(t)x_1(t-\tau)-a_{11}^2(t)x_1(t-2\tau)+a_{12}^1(t)x_2(t-\tau)\right]\\ \dot{x}_2(t)=x_2(t)\left[r_2(t)+a_{21}^0(t)x_1(t)+a_{21}^1(t)x_1(t-\tau)-a_{22}^0(t)x_2(t)-a_{22}^1(t)x_2(t-\tau)\right]\\ \dot{y}_1(t)=y_1(t)\left[r_1(t)-a_{11}^1(t)y_1(t-\tau)-a_{11}^2(t)y_1(t-2\tau)+a_{12}^1(t)y_2(t-\tau)\right]\\ \dot{y}_2(t)=y_2(t)\left[r_2(t)+a_{21}^0(t)y_1(t)+a_{21}^1(t)y_1(t-\tau)-a_{22}^0(t)y_2(t)-a_{22}^1(t)y_2(t-\tau)\right]\end{cases}\tag{9}$$

Now, we present a useful definition

Definition 1.1 (see [[3] Definition 1.1]) $f(t)$ is called asymptotically periodic function, if $f(t)$ is a continuous function mapping from R^+ to R, and satisfies

$$f(t)=\overline{f}(t)+a(t),\tag{10}$$

where $\overline{f}(t)$ is a continuous periodic function and $\lim_{t\to\infty}a(t)=0$.

Now, we present some useful lemmas.

Lemma 2.1 The set $R_+^2=\left\{(x_1,x_2)\mid x_i>0,i=1,2\right\}$ is the positively invariant set of system (1)

Proof. We can obtain for $x_i(0)=\phi_i(0)>0,i=1,2$

$$x_1(t)=x_1(0)\exp\left\{\int_0^t\left[r_1(s)-a_{11}^1(s)x_1(s-\tau)-a_{11}^2(s)x_1(s-2\tau)+a_{12}^1(s)x_2(s-\tau)\right]\mathrm{d}s\right\}$$

$$x_2(t)=x_2(0)\exp\left\{\int_0^t\left[r_2(s)+a_{21}^0(s)x_1(s)+a_{21}^1(s)x_1(s-\tau)-a_{22}^0(s)x_2(s)-a_{22}^1(s)x_2(s-\tau)\right]\mathrm{d}s\right\}$$

our results will be discussed in the positively invariant set R_+^2.

Let the set

$$\Gamma=\left\{\left(x_1(t),x_2(t)\right)\in R_+^2:0<m_i\le x_i(t)\le M_i(t)<+\infty(i=1,2),R_+=[0,+\infty)\right\}$$

where $m_i,M_i(i=1,2)$ are given above (in Introduction).

Lemma 2.2 Assume that $b_1^{lL}>0(l=1,2),b_2^{lL}>0,b_{22}^{lL}>0,$ then system (1) is permanent, where $b_1^l=a_{11}^l(t)-a_{21}^{l-1}(t-\tau),l=1,2$ and $b_2^1=a_{22}^0(t-\tau)-a_{12}^1(t-\tau)$.

Lemma 2.3 ([4]) Let $V\in C\left(R_+\times S_H\times S_H,R_+\right)$ satisfy

1) $a(|x-y|)\le V(t,x,y)\le b(|x-y|)$, where $a(r)$ are $b(r)$ are continuously positively increasing functions;

2) $\left|V(t,x_1,y_1)-V(t,x_2,y_2)\right|\le l\left(|x_1-x_2|+|y_1-y_2|\right)$, l is a constant and satisfies $l>0$;

3) There exists continuous function $p(s)$, such that for $s>0$, $p(s)>s$. And as $P\left(V(t,\phi(0))\right)>V\left(t+\theta,\phi(\theta),\phi(\theta)\right)$, $\theta\in[-\tau,0]$, it follows that

$V'_{(2.8)}(t,\phi(0),\phi(0)) \le -\delta V(t,\phi(0),\phi(0))$, where δ is a constant and satisfies $\delta > 0$.

Furthermore, system (2.7) has a solution $\xi(t)$ for $t \ge t_0$ and satisfies $\|\xi_t\| \le H$. Then system (2.7) has a unique asymptotically periodic solution, which is uniformly asymptotically stable.

Our main purpose is to establish some sufficient conditions on the existence and globally asymptotically stability for the asymptotically periodic solution of the system (1). The method used in this paper is motivated by the work done by Fengying Wei and Wang Ke in [4] and the Lyapunov function method.

2. Main Results

Theorem 2.1 Assume that the condition of lemma 2.2 is hold and $a_{11}^{1L} > a_{21}^{1M}, a_{22}^{1L} > a_{12}^{1M}, W > 0$, then there exists a unique asymptotically periodic solution of system (1), which is uniformly asymptotically stable. (W defined in the proof)

Proof. From Lemma 2.2, we know that the solution of system (1) is ultimately bounded. Γ is the region of ultimately bounded. We consider the adjoint system (2) of system (1)

For $X(t) = (x_1(t), x_2(t))$ and $Y(t) = (y_1(t), y_2(t))$ are the solution of system (2) in $\Gamma \times \Gamma$. Let $x_i^*(t) = \ln x_i(t), y_i^*(t) = \ln y_i(t), (i = 1,2)$. Next we construct a Lyapunov functional as follows:

$$V(t) = \sum_{i=1}^{2} \left| x_i^*(t) - y_i^*(t) \right| \tag{11}$$

Take $a(r) = b(r) = \sum_{i=1}^{2} \left| x_i^*(t) - y_i^*(t) \right|$ and by using the inequality $\left\| a \right| - \left| b \right\| \le \left| a - b \right|$, we can easily prove 1) and 2). To check the condition 3) of Lemma 2.3, we need to calculate upper-right derivative of system (2):

$$\begin{aligned}
D^+V(t) &= D^+\left(\left| x_1^*(t) - y_1^*(t) \right| + \left| x_2^*(t) - y_2^*(t) \right| \right) \\
&= \left(\frac{\dot{x}_1}{x_1} - \frac{\dot{y}_1}{y_1} \right) sign\left(x_1(t) - y_1(t)\right) + \left(\frac{\dot{x}_2}{x_2} - \frac{\dot{y}_2}{y_2} \right) sign\left(x_2(t) - y_2(t)\right) \\
&\le a_{21}^{0M} \left| x_1(t) - y_1(t) \right| + \left(-a_{11}^{1L} + a_{21}^{1M} \right) \left| x_1(t-\tau) - y_1(t-\tau) \right| \\
&\quad - a_{21}^{1L} \left| x_1(t-2\tau) - y_1(t-2\tau) \right| - a_{22}^{0L} \left| x_2(t) - y_2(t) \right| \\
&\quad + \left(-a_{22}^{1L} + a_{12}^{1M} \right) \left| x_2(t-2\tau) - y_2(t-2\tau) \right| \\
&\le a_{21}^{0M} \left| x_1(t) - y_1(t) \right| - A_1 \left| x_1(t-\tau) - y_1(t-\tau) \right| \\
&\quad + a_{22}^{0M} \left| x_2(t) - y_2(t) \right| - A_2 \left| x_2(t-\tau) - y_2(t-\tau) \right|
\end{aligned}$$

where $A_1 = a_{11}^{1L} - a_{21}^{1M}, A_2 = a_{22}^{1L} - a_{12}^{1M}$ and we take

$$\lambda_1 = \max\left\{ a_{21}^{0M}, a_{22}^{0M} \right\}, A = \min\left\{ A_1, A_2 \right\}$$

Then we have

$$\begin{aligned}
D^+V(t) \le \lambda &\left(\left| x_1(t) - y_1(t) \right| + \left| x_2(t) - y_2(t) \right| \right) \\
&- A\left(\left| x_1(t-\tau) - y_1(t-\tau) \right| + \left| x_2(t-\tau) - y_2(t-\tau) \right| \right)
\end{aligned}$$

By the following formula:

$$\left|x_i(t)-y_i(t)\right|=\left|e^{x_i^*}-e^{y_i^*}\right|=\psi_i(t)\geq m_i\left|x_i^*-y_i^*\right|,(i=1,2) \tag{12}$$

$$\left|x_i(t)-y_i(t)\right|=\left|e^{x_i^*}-e^{y_i^*}\right|=\psi_i(t)\leq M_i\left|x_i^*-y_i^*\right|,(i=1,2) \tag{13}$$

where $\psi_i(t)(i=1,2)$ lie in between $x_i(t)$ and $y_i(t)$ respectively, then $\psi_i(t)\in\Gamma$. let $M=\max\{M_1,M_2\}$, $m=\min\{m_1,m_2\}$ and if $\lambda_2 V(t)\leq V(t-\tau)$, where $\lambda_2>0$ is a constant ,then we have

$$\begin{aligned} D^+V(t) &\leq \lambda_1 M\left(\left|x_1^*-y_1^*\right|+\left|x_2^*-y_2^*\right|\right) \\ &\quad -Am\left(\left|x_1^*(t-\tau)-y_1^*(t-\tau)\right|+\left|x_2^*(t-\tau)-y_2^*(t-\tau)\right|\right) \\ &= \lambda MV(t)-AmV(t-\tau)\leq -V(t)\left(Am\lambda_2-\lambda_1 M\right):=-WV(t) \end{aligned}$$

where $W=\left(Am\lambda_2-\lambda_1 M\right)$.

From the known condition of Theorem 2.1, we obtain that $W>0$, $D^+V(t)\leq WV(t)$. Then 3) of Lemma 2.3 is satisfied. has system (1) has a unique positive asymptotically periodic solution in domain Γ, which is uniformly asymptotically stable. The proof is complete.

3. Conclusions

In [1] the author's discussed system (1) and derived some sufficient conditions on the permanence of system (1). However, in this paper, based on the permanence of the system (1), we further study system (1) in a asymptotically periodic environment and established conditions on the existence and globally asymptotically stability for the asymptotically periodic solution of the system (1) by using the Lyapunov function method and the method given in Fengying Wei and Wang Ke (Applied Mathematics and Computation 182 (2006) 161 - 165).

We have more interesting topics deserve further investigation, such as the dynamical behaviors of n-species Lotka-Volterra cooperative systems with discrete time delays.

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References

[1] Nakata, Y. and Muroya, Y. (2010) Permanence for Nonautonomous Lotka-Volterra Cooperative Systems with Delays. *Nonlinear Analysis*: *Real World Applications*, **11**, 528-534.

[2] Lu, S. (2008) On the Existence of Positive Periodic Solutions to a Lotka Volterra Cooperative Population Model with Multiple Delays. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **68**, 1746-1753.

[3] Wei, F. and Wang, K. (2006) Asymptotically Periodic Solution of N-Species Cooperation System with Time Delay. *Nonlinear Analysis*: *Real World Applications*, **7**, 591-596.

[4] Wei, F. and Wang, K. (2006) Global Stability and Asymptotically Periodic Solution

for Nonautonomous Cooperative Lotka-Volterra Diffusion System. *Applied Mathematics and Computation*, **182**, 161-165.

[5] Wei, F. and Wang, K. (2002) Almost Periodic Solution and Stability for Nonautonmous Cooperative Lotka-Volterra Diffusion System. *Songliao Journal* (*Natural Science Edition*), **3**.

[6] Liu, C. and Chen, L. (1997) Periodic Solution and Global Stability for Nonautonomous Cooperative Lotka-Volterra Diffusion System. *Journal of Lanzhou University* (*Natural Science*), **33**, 33-37.

[7] Zhang, J. and Chen, L. (1996) Permanence and Global Stability for Two-Species Co-Operative System with Delays in Two-Patch Environment. *Mathematical and Computer Modelling*, **23**, 17-27.

[8] Chen, F. (2003) Persistence and Global Stability for Nonautonomous Co-Operative System with Diffusion and Time Delay. *Acta Scientiarum Naturalium Universitatis Pekinensis*, **39**, 22-28.