

Mixed-Type Reverse Order Laws for Generalized Inverses over Hilbert Space

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Abstract

In this paper, by using a block-operator matrix technique, we study mixed-type reverse order laws for {1,3}-, {1,2,3}- and {1,3,4}-generalized inverses over Hilbert spaces. It is shown that $\{(AB)^{(1,3)}\} = \{B^{(1,3)}(ABB^{(1,3)})^{(1,3)}\}$ and $\{(AB)^{(1,2,i)}\} = \{B^{(1,2,i)}(ABB^{(1,2,i)})^{(1,2,i)}\}, (i = 3, 4)$ when the ranges of A, B, AB are closed. Moreover, a new equivalent condition of $\{(AB)^{(1,3,4)}\} = \{B^{(1,3,4)}(ABB^{(1,3,4)})^{(1,3,4)}\}$ is given.

Keywords

{1,2,3}-Reverse, {1,3,4}-Reverse, Reverse Order Law, Block-Operator Matrix

1. Introduction

The reverse order law of generalized inverses plays an important role in theoretic research and numerical computations in many areas, including the singular matrix problem and optimization problem. They have attracted considerable attention since the middle 1960s, and many interesting results have been studied, see [1]-[10].

For convenience, we firstly introduce some notations. Let H and K be infinite dimensional Hilbert spaces and B(H, K) be the set of all bounded linear operators from H to K and abbreviate B(K, H) to B(H) if K = H. For an operator $A \in B(H, K)$, N(A) and R(A) are the null space and the range of A, respectively. Denote by A^* the adjoint of A. Recall that $A \in B(H, K)$ has a Moore-Penrose inverse if there exists an operator $G \in B(K, H)$ satisfies the following four equations, which is said to be the Moore-Penrose conditions:

(1)
$$AGA = A$$
; (2) $GAG = G$; (3) $(AG)^* = AG$; (4) $(GA)^* = GA$.

If one exists, the Moore-Penrose inverse of A is unique and it is denoted by A^+ . And let $A\{i, j, \dots, k\}$ denote the set of all operator $G \in B(K, H)$ which satisfy equations $(i), (j), \dots, (k)$ from among the above Moore-Penrose equations. Such G will be called a $\{i, j, k\}$ -inverse of A and will be denoted by $A^{(i, j, k)}$. evidently, $A\{1, 2, 3, 4\} = A^+$ when A has closed range.

For the Moore-Penrose inverse, Greville [2] gave the necessary and sufficient conditions for $(AB)^+ = B^+A^+$ on matrix algebra, this result was extended to bounded operators on Hilbert space by Izumino [4]. Subsequently, some researcher discussed the reverse order laws of other type generalized inverses, such as $(AB)\theta = B\theta A\theta, \theta \subset \{1, 2, 3, 4\}$ [5] [6] [8] [10]. The mixed-type reverse-order laws for AB like $(AB)^+ = B^+(A^+AB)^+$ and $(AB)^+ = (A^+AB)^+A^+$ were considered in [3] [4] when A and B are matrices. Motivated by this, Wang *et al.* [7] studied the mixed-type reverse-order laws for $AB^{(1,2,i)}$ = $\{B^{(1,2,i)}(ABB^{(1,2,i)})^{(1,2,i)}\}$, (i = 3, 4), by using

the extremal ranks of generalized Schur complements, when A and B are matrices. The mixed-type reverse order laws of $(AB)^{(1,3,4)}$ were discussed on operator space over Hilbert space [5].

In this article, we study the mixed-type reverse order laws of $(AB)^{(1,2,i)}$, $(AB)^{(1,3)}$ and $(AB)^{(1,3,4)}$ over infinite Hilbert space by using a block-operator matrix technique. For given A, B, it is shown that

$$\left\{ \left(AB\right)^{(1,3)} \right\} = \left\{ B^{(1,3)} \left(ABB^{(1,3)}\right)^{(1,3)} \right\}$$

and

$$\left\{ \left(AB\right)^{(1,2,i)} \right\} = \left\{ B^{(1,2,i)} \left(ABB^{(1,2,i)}\right)^{(1,2,i)} \right\}, (i = 3, 4)$$

when the ranges of *A*, *B*, *AB* are closed. We generalized the results from [7] and [9] to the case of bounded linear operators on Hilbert spaces. Moreover, a new equivalent condition of $\{(AB)^{(1,3,4)}\} = \{B^{(1,3,4)}(ABB^{(1,3,4)})^{(1,3,4)}\}$ is given.

2. Main Results

To obtain our main results, we begin with some notations and lemmas. Let $A \in B(H, K)$ with closed range. It is well known that A has the following matrix decomposition

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix} \rightarrow \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix},$$
(2.1)

where A_1 is invertible. Also, A^+ has the form

$$A^{+} = \begin{bmatrix} A_{1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^{*}) \end{bmatrix} \rightarrow \begin{bmatrix} R(A^{*}) \\ N(A) \end{bmatrix}.$$
 (2.2)



The {1,2,3}- and {1,3,4}-inverse has also similarly matrix form. Lemma 1 ([5]). Let $A \in B(H, K)$ have closed range. Then

$$A\{1,3\} = \left\{ \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix} \right\}$$
$$A\{1,2,3\} = \left\{ \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} R(A^*) \\ N(A) \end{bmatrix} \right\}$$

and

$$A\{1,3,4\} = \left\{ \begin{bmatrix} A_1^{-1} & 0\\ 0 & X_4 \end{bmatrix} : \begin{bmatrix} R(A)\\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} R(A^*)\\ N(A) \end{bmatrix} \right\}$$

Let $A \in B(H, K)$, $B \in B(L, H)$ and $AB \in B(L, K)$ with closed ranges. For convenience, denote by

$$\begin{cases} H_1 = R(B) \cap N(A), \\ H_2 = R(B)\Theta^{\perp}H_1, \\ H_3 = N(B^*) \cap N(A), \\ H_4 = N(B^*)\Theta^{\perp}H_3, \end{cases} \begin{cases} K_1 = R(AB) \\ K_2 = R(A)\Theta^{\perp}R(AB)^* \end{cases}$$

and

$$L_{1} = B^{+}(H_{1}), L_{2} = R(B^{*})\Theta^{\perp}L_{1}.$$

then

$$H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$$
, $K = K_1 \oplus K_2 \oplus N(A^*)$

and

$$L = L_1 \oplus L_2 \oplus N(B).$$

Under these space decomposition, we get two useful representations of operators $A \in B(H, K)$ and $B \in B(L, K)$.

Lemma 2 ([9] [10]). Let $A \in B(H, K)$, $B \in B(L, K)$ such that R(A), R(B) and R(AB) are closed.

If $AB \neq 0$, the following statements hold,

(1) When $H_1 \neq \{0\}$, A and B have the matrix form as follows, respectively,

$$A = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix} \rightarrow \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix},$$
(2.3)

$$B = \begin{bmatrix} B_{11} & B_{12} & 0\\ 0 & B_{22} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} L_1\\ L_2\\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} H_1\\ H_2\\ H_3\\ H_4 \end{bmatrix},$$
(2.4)

where A_{11} , B_{11} , B_{22} are invertible and A_{22} is surjective. (2) When $H_1 = \{0\}$,

$$A = \begin{bmatrix} A_{12} & 0 & A_{14} \\ 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} H_2 \\ H_3 \\ H_4 \end{bmatrix} \rightarrow \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix},$$
(2.5)

$$B = \begin{bmatrix} B_{22} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} L_2 \\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} H_2 \\ H_3 \\ H_4 \end{bmatrix}, \qquad (2.6)$$

where A_{12} , B_{22} are invertible and A_{24} is surjective.

Theorem 3. Let $A \in B(H, K)$, $B \in B(L, H)$ such that R(A), R(B) and R(AB) are closed. Then

$$\left\{ \left(AB\right)^{(1,2,3)} \right\} = \left\{ B^{(1,2,3)} \left(ABB^{(1,2,3)}\right)^{(1,2,3)} \right\}.$$

Proof If AB = 0, then $(AB)\{1, 2, 3\} = \{0\}$, the result holds. So assume that $AB \neq 0$. Next, we divide the proof into two cases.

Case 1. $H_1 \neq \{0\}$.

In this case, A, B have matrix forms (2.3) and (2.4), respectively. This implies that

$$AB = \begin{bmatrix} 0 & A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} J_1 \\ J_2 \\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix}.$$
 (2.7)

Using Lemma 1, we get

$$B^{(123)} = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0\\ 0 & B_{22}^{-1} & 0 & 0\\ F_{31} & F_{32} & 0 & 0 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix} \rightarrow \begin{bmatrix} L_1 \\ L_2 \\ N(B) \end{bmatrix},$$
(2.8)

and

$$(AB)^{(123)} = \begin{bmatrix} M_{11} & 0 & 0 \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ M_{31} & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} L_1 \\ L_2 \\ N(B) \end{bmatrix}.$$
 (2.9)

where F_{31} , F_{32} , M_{11} , M_{31} are arbitrary. Combining formulae (2.7) with (2.8), it is easy to get

Using Lemma 1 again, we have

$$(ABB^{(123)})^{(123)} = \begin{bmatrix} G_{11} & 0 & 0 \\ A_{12}^{-1} & 0 & 0 \\ G_{31} & 0 & 0 \\ G_{41} & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix},$$
(2.10)



where G_{i1} , i = 1, 3, 4 are arbitrary. By direct computation, it is clearly from (2.8) and (2.10) that

$$B^{(123)} \left(ABB^{(123)}\right)^{(123)} = \begin{bmatrix} P_{11} & 0 & 0 \\ B_{22}^{-1} & 0 & 0 \\ P_{31} & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} L_1 \\ L_2 \\ N(B) \end{bmatrix}, \quad (2.11)$$

where $P_{11} = B_{11}^{-1}G_{11} - B_{11}^{-1}B_{12}B_{22}^{-1}A_{12}^{-1}$, $P_{31} = F_{31}G_{11} + F_{32}A_{12}^{-1}$. Thus, by the arbitrariness of G_{11}, F_{31}, F_{32} , it follows from fromulae (2.9) and (2.11) that

$$\left\{ \left(AB\right)^{(1,2,3)} \right\} = \left\{ B^{(1,2,3)} \left(ABB^{(1,2,3)}\right)^{(1,2,3)} \right\}$$

holds.

Case 2 $H_1 = \{0\}$. Obviously, $L_1 = \{0\}$. Consequently, $H = H_2 \oplus H_3 \oplus H_4$ and $L = L_2 \oplus N(B)$. By Lemma 2, A, B have matrix forms (2.5) and (2.6), respectively. This follows that

$$AB = \begin{bmatrix} A_{12}B_{22} & 0\\ 0 & 0\\ 0 & 0 \end{bmatrix} : \begin{bmatrix} L_2\\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} K_1\\ K_2\\ N(A^*) \end{bmatrix}.$$
 (2.12)

By Lemma 1, we get

$$B^{(123)} = \begin{bmatrix} B_{22}^{-1} & 0 & 0 \\ F_{21} & 0 & 0 \end{bmatrix} : \begin{bmatrix} H_2 \\ H_3 \\ H_4 \end{bmatrix} \rightarrow \begin{bmatrix} L_2 \\ N(B) \end{bmatrix},$$
 (2.13)

and

$$(AB)^{(123)} = \begin{bmatrix} B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ M_{21} & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} L_2 \\ N(B) \end{bmatrix}.$$
 (2.14)

where F_{21}, M_{21} are arbitrary. Combining formulae (2.12) with (2.13),

$$ABB^{(123)} = \begin{bmatrix} A_{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} H_2 \\ H_3 \\ H_4 \end{bmatrix} \to \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix}.$$

Again from Lemma 1,

$$(ABB^{(123)})^{(123)} = \begin{bmatrix} A_{12}^{-1} & 0 & 0 \\ Q_{21} & 0 & 0 \\ Q_{31} & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} H_2 \\ H_3 \\ H_4 \end{bmatrix},$$

where Q_{21}, Q_{31} are arbitrary. By direct computation, it is clearly that

$$B^{(123)} \left(ABB^{(123)}\right)^{(123)} = \begin{bmatrix} B_{22}^{-1}A_{12}^{-1} & 0 & 0\\ F_{21}A_{12}^{-1} & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_1\\ K_2\\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} L_2\\ N(B) \end{bmatrix}.$$
(2.15)

Thus, from fromulae (2.14) and (2.15), it is clear that

$$\left\{ \left(AB\right)^{(1,2,3)} \right\} = \left\{ B^{(1,2,3)} \left(ABB^{(1,2,3)}\right)^{(1,2,3)} \right\}$$

also holds in this case. The proof is completed.

From the relationship of $\{1,2,3\}$ -inverse and $\{1,2,4\}$ -inverse, we can obtain the following result without proof.

Corollary 4. Let $A \in B(H, K)$, $B \in B(L, H)$ such that R(A), R(B) and R(AB) are closed. Then

$$\left\{ \left(AB\right)^{(1,2,4)} \right\} = \left\{ B^{(1,2,4)} \left(ABB^{(1,2,4)}\right)^{(1,2,4)} \right\}.$$

Similar to the proof of Theorem 3, we also can get the following result.

Theorem 5. Let $A \in B(H, K)$, $B \in B(L, H)$ such that R(A), R(B) and R(AB) are closed. Then

$$\left\{ \left(AB\right)^{(1,3)} \right\} = \left\{ B^{(1,3)} \left(ABB^{(1,3)}\right)^{(1,3)} \right\}.$$

In [5], the author gave a necessary and sufficient condition of

 $\left\{ \left(AB\right)^{(1,3,4)} \right\} = \left\{ B^{(1,3,4)} \left(ABB^{(1,3,4)}\right)^{(1,3,4)} \right\}.$ Next, we give a new equivalent condition of the mixed-type reverse order law for $(AB)^{(134)}.$

Theorem 6. Let $A \in B(H, K)$, $B \in B(L, H)$ such that R(A), R(B) and R(AB) are closed. If $AB \neq 0$, the following statements are equivalent,

- (1) $\left\{ \left(AB \right)^{(1,3,4)} \right\} = \left\{ B^{(1,3,4)} \left(ABB^{(1,3,4)} \right)^{(1,3,4)} \right\};$ (2) $R \left(BB^*A^* \right) \subset R \left(A^* \right).$
- **Proof** We divide the proof into two cases.

Case 1. $H_1 \neq \{0\}$. Using Lemma 2, A, B have matrix forms (2.3) and (2.4),

respectively. The operator AB has the matrix decomposition (2.7). Then

$$A^{*} = \begin{bmatrix} 0 & 0 & 0 \\ A_{12}^{*} & 0 & 0 \\ 0 & 0 & 0 \\ A_{14}^{*} & A_{24}^{*} & 0 \end{bmatrix} : \begin{bmatrix} K_{1} \\ K_{2} \\ N(A^{*}) \end{bmatrix} \rightarrow \begin{bmatrix} H_{1} \\ H_{2} \\ H_{3} \\ H_{4} \end{bmatrix}$$
(2.16)

and

$$BB^{*}A^{*} = \begin{bmatrix} B_{12}B_{12}^{*}A_{12}^{*} & 0 & 0\\ B_{22}B_{12}^{*}A_{12}^{*} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} K_{1} \\ K_{2} \\ N(A^{*}) \end{bmatrix} \rightarrow \begin{bmatrix} H_{1} \\ H_{2} \\ H_{3} \\ H_{4} \end{bmatrix}$$
(2.17)

by direct computation from (2.3) and (2.4). Therefore, comparing (2.16) with (2.17), it is natural that $R(BB^*A^*) \subset R(A^*)$ if and only if $B_{12}B_{12}^*A_{12}^* = 0$. So $R(BB^*A^*) \subset R(A^*)$ if and only if $B_{12} = 0$ since A_{12} is invertible.

On the other hand, it follows from Lemma 1 that



$$B^{(134)} = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0\\ 0 & B_{22}^{-1} & 0 & 0\\ 0 & 0 & F_{33} & F_{34} \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix} \rightarrow \begin{bmatrix} L_1 \\ L_2 \\ N(B) \end{bmatrix},$$
(2.18)

and

$$(AB)^{(134)} = \begin{bmatrix} 0 & M_{12} & M_{13} \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ 0 & M_{32} & M_{33} \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} L_1 \\ L_2 \\ N(B) \end{bmatrix}.$$
 (2.19)

where F_{33} , F_{34} , M_{ij} , (i = 1, 3, j = 2, 3) are arbitrary.

Combining formulae (2.7) with (2.18), it is easy to get

Using Lemma 1 again, we have

$$\left(ABB^{(134)}\right)^{(134)} = \begin{bmatrix} 0 & G_{12} & G_{13} \\ A_{12}^{-1} & 0 & 0 \\ 0 & G_{32} & G_{33} \\ 0 & G_{42} & G_{43} \end{bmatrix} : \begin{bmatrix} K_1 \\ K_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix}.$$
 (2.20)

where G_{ij} (i = 1, 3, 4, j = 2, 3) are arbitrary. By direct computation, it is clearly from (2.18) and (2.20) that

$$B^{(134)} \left(ABB^{(134)}\right)^{(134)} = \begin{bmatrix} -B_{11}^{-1}B_{12}B_{22}^{-1}A_{12}^{-1} & B_{11}^{-1}G_{12} & B_{11}^{-1}G_{13} \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ 0 & F_{33}G_{32} + F_{34}G_{42} & F_{33}G_{33} + F_{34}G_{43} \end{bmatrix}.$$
 (2.21)

Comparing (2.21) with (2.19), we have

$$\left\{ \left(AB\right)^{(1,3,4)} \right\} = \left\{ B^{(1,3,4)} \left(ABB^{(1,3,4)}\right)^{(1,3,4)} \right\}$$

if and only if $-B_{11}^{-1}B_{12}B_{22}^{-1}A_{12}^{-1} = 0$, that is, $B_{12} = 0$. Therefore, $\{(AB)^{(1,3,4)}\} = \{B^{(1,3,4)}(ABB^{(1,3,4)})^{(1,3,4)}\}$ if and only if $R(BB^*A^*) \subset R(A^*)$.

Case 2 $H_1 = \{0\}$. Then *A*, *B* have matrix forms (2.5) and (2.6), respectively. By similarly discussing to case 1 and case 2 in the proof of Theorem 3.2, it is easy to get that $\{(AB)^{(1,3,4)}\} = \{B^{(1,3,4)}(ABB^{(1,3,4)})^{(1,3,4)}\}$ and $R(BB^*A^*) \subset R(A^*)$ always hold in this case.

So the proof is completed.

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