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Alternative Infinitesimal Generator of Invertible Evolution Families

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Abstract

A logarithm representation of evolution operators is defined. Generators of invertible evolution families are characterized by the logarithm representation. In this article, using the logarithm representation, a concept of evolution operators without satisfying the semigroup property is introduced. In conclusion the existence of alternative infinitesimal generator is clarified.

Keywords

Invertible Evolution Family, Abstract Cauchy Problem

1. Introduction

For a given evolution operator let its logarithm function be well-defined. A simple question arises here; "is there any difference between the logarithm of evolution operator and the infinitesimal generator?" This question is associated with the unboundedness of infinitesimal generator. However a role of the unboundedness of the infinitesimal generator has not been understood well so far. Indeed, in the standard theory of linear evolution equation (for example, see [1]), the evolution operator is treated as a bounded operator in a given functional space X regardless of whether the infinitesimal generator is bounded or unbounded in X.

This question is considered in a concrete framework of abstract Cauchy problem. Partial differential equations are regarded as ordinary differential equations in functional spaces. The initial value problems of autonomous evolution equations are written by

$$\begin{cases} \mathrm{d}u(t)/\mathrm{d}t = A(t)u(t), \\ u(0) = u_0 \end{cases}$$

in X, where an initial value u_0 is given in X, and $A(t)$ is generally unbounded in X. If there is a solution for this initial value problem, its solution is formally represented by

$$u(t) = e^{\int A(t)dt} u_0,$$

under the well-definedness of the indefinite integral $\int A(t) dt$ and its exponential function. The evolution operators appearing in the following discussion correspond to the above exponential function. Note that, since $A(t)$ is generally given as an unbounded operator in X, the exponential of $A(t)$ is not necessarily well-defined even if $A(t)$ is independent of t (cf. Hille-Yosida theorem).

There is a long history of studying logarithm of operators [2]-[8]. The logarithm of $e^{\int A(t)dt}$ is defined under a certain setting and such a logarithm is clarified to play a role of extracting an essential and bounded part of infinitesimal generator [9]. In this article the logarithm representation of evolution operator is shown to lead to the concept of evolution operator without satisfying the semigroup property.

2. Evolution Operator and Its Infinitesimal Generator

2.1. Invertible Evolution Operator

An evolution operator is assumed to be defined on a Banach space X. Although evolution operator is not necessarily invertible, here we confine ourselves to invertible evolution operators. The reason for this restriction can be found in Sec. 3.2.

For $T > 0$, let a certain time interval $[-T, T]$ satisfy $-T \le t, r, s \le T$. Let a family of two-parameter evolution operator on X be $\{U(t,s)\}_{-T \le t,s \le T}$ satisfying

$$\begin{aligned} &U(t,r)U(r,s) = U(t,s), \\ &U(s,s) = I. \end{aligned} \tag{1}$$

The property (1) is called the semigroup property. In addition the inverse operator is assumed to exist:

$$U(t,s)U(s,t) = U(t,t) = I.$$

If $U(t,s)$ is generated by the operator independent of t (and s), one-parameter group $U(\cdot)$ can be defined by

$$U(t-s) := U(t,s).$$

utilizing this two-parameter evolution operator. Since $t-s$ is a real number only satisfying $-T \le t, s \le T$, it can be negative. If a solution at certain times t and s are represented by $u(t) = U(t,0)u_0$ and $u(s) = U(s,0)u_0$ respectively, it is trivially seen from the definition that $U(t-s)$ is a mapping from $u(s)$ to $u(t)$. $U(t)$ is nothing but a C_0-group generated by t-independent infinitesimal generator, and the following properties are satisfied:

$$\begin{aligned} &U(t)U(r) = U(r)U(t) = U(t+r) \\ &U(0) = I \end{aligned}$$

Indeed the C_0-group property can be confirmed by

$$U(r)U(t) = U(r+t-t)U(t) = U(r+t,t)U(t,0)$$
$$= U(r+t,0) = U(t+r).$$

Furthermore, for $r \neq s$, application of $U(t-s)$ to $u(r) = U(r,0)u_0$ leads to

$$\begin{aligned} U(t-s)u(r) &= U(t-s+r-r)u(r) = U(t-s+r,r)U(r,0)u_0 \\ &= U(t-s+r,0)u_0 = u(t-s+r). \end{aligned} \tag{2}$$

Consequently the C_0-group (C_0-semigroup) property of $U(t)$ is derived from the definition of $U(t,s)$.

According to the standard theory of linear evolution equations [1], the following boundedness is assumed; there exists real numbers M and β such that

$$\|U(t)\| \le Me^{\beta t}$$

for $t \in [-T,T]$, where $\|\cdot\|$ denotes an operator norm. This assumption restricts the time evolution to be linearly bounded. Note that, using the equality $U(t) = U(t,0)$, the assumption can be replaced with

$$\|U(t,s)\| \le Me^{\beta t}$$

without the essential difference.

2.2. Pre-Infinitesimal Generator

The infinitesimal generator is defined using the evolution operator. Let the dense subspace Y of X be non-empty space admitting the definition of the following weak limit:

$$w\lim_{h\to 0} h^{-1}\left(U(t+h,s) - U(t,s)\right)u$$

for $u \in Y \subset X$. Since there is an arbitrariness of choosing the dense subspace of *X*, *Y* can be different depending on the detail of $U(t,s)$. Under the existence of the above weak limit, the infinitesimal generator is defined by

$$A(t)u := w\lim_{h\to 0} h^{-1}\left(U(t+h,t) - I\right)u$$

Since $A(t)$ is defined under a weaker assumption compared to the standard theory of evolution equations, we call this operator the pre-infinitesimal generator. The definition of weak *t*-differential, which is denoted as ∂_t, follows as

$$\partial_t U(t,s)u = A(t)U(t,s)u,$$

where the relation $U(t+h,s) - U(t,s) = \left(U(t+h,t) - I\right)U(t,s)$ is used. In this article we consider the subspace Y as a natural choice of domain space $D(A(t))$. Generally speaking, if $A(t)$ is dependent on *t*, $D(A(t))$ does depend on *t*. Here, by considering sufficiently small interval $t,s \in [-T,T]$, $D(A(t))$ is assumed to be independent of *t* and *s*. Furthermore, by taking $u = u(s)$,

$$\partial_t u(t) = A(t)u(t) \tag{3}$$

follows. This is a linear evolution equation of autonomous type. In this manner the pre-infinitesimal generator of $U(t,s)$ is obtained as the operator $A(t)$ satisfying Equation (3).

3. Logarithmic Representation of Infinitesimal Generator

3.1. Function of Operator

It is sufficient to consider the function of bounded operators, since $U(t,s)$ is bounded on X. As a framework of defining functions of bounded operator, the Dunford-Riesz integral [10]

$$f\left(U(t,s)\right) = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\left(\lambda - U(t,s)\right)\mathrm{d}\lambda \tag{4}$$

is utilized. Note that functions of bounded operator on X are not necessarily bounded operators on X. For drawing an integral path on the complex plain,

- the integral path Γ consists of Jordan curves including all the spectral sets of $U(t,s)$,
- the integral path Γ must not include singular points of $f(\lambda)$.

That is, for the definition of logarithm of operators, it is necessary to take an integral path not to include the origin, since the origin is the singular point of logarithm function.

3.2. Logarithmic Function of Operator

The logarithm of $U(t,s)$ is defined using the Dunford-Riesz integral. Let the principal branch of logarithm be denoted by Log. For a certain complex number $\kappa \neq 0$, the logarithm of $U(t,s)$ is defined by

$$\mathrm{Log}\left(U(t,s)+\kappa I\right) = \frac{1}{2\pi i}\int_{\Gamma}\mathrm{Log}(\lambda)\left(\lambda - U(t,s) - \kappa I\right)\mathrm{d}\lambda,$$

where κI plays a role of moving the spectral set of $U(t,s)+\kappa I$ not to include the origin. In addition, according to the preceding discussion, it is necessary for an integral path Γ to include the spectral set of $U(t,s)+\kappa I$. Since the boundedness of $U(t,s)$ is assumed, it is necessarily possible to take an appropriate integral path Γ by adjusting the amplitude of κ. If $\mathrm{Log}\left(U(t,s)+\kappa\right)$ is well-defined, then the definition of $-\mathrm{Log}\left(U(t,s)+\kappa\right)$ trivially follows. That is, for the present definition manner, the sign of the logarithm of operator can not be a matter. This fact is essentially arises from limiting the time interval $[-T,T]$ as finite. This provides the reason why we assume $U(t,s)$ as invertible.

The relation

$$A(t) = \left(I + \kappa U(s,t)\right)\partial_t \mathrm{Log}\left(U(t,s)+\kappa I\right) \tag{5}$$

between the infinitesimal generator and the logarithm of operator has been proved in Ref. [9]. Let us introduce a notation:

$$a(t,s) := \mathrm{Log}\left(U(t,s)+\kappa I\right). \tag{6}$$

Since $a(t,s)$ is bounded on *X*, it is possible to define the exponential function of $a(t,s)$ by a convergent power series. Meanwhile $\partial_t a(t,s)$ is obtained by applying the resolvent operator of $U(t,s)$ to the pre-infinitesimal generator $A(t)$.

3.3. Evolution Operator without Satisfying the Semigroup Property

If $a(t,s)$ with different *t* and *s* are further assumed to commute, the exponential function $\mathrm{e}^{a(t,s)}$ satisfies

$$\partial_t \mathrm{e}^{a(t,s)} u_s = \partial_t \left(a(t,s)\right) \mathrm{e}^{a(t,s)} u_s, \tag{7}$$

where $\partial_t \left(a(t,s)\right)$ is generally an unbounded operator in *X*, and it is well-defined by considering the previously-defined subspace *Y*. Although $u_s \in X$ stands for $u(s)$, it can be arbitrarily taken from *X*. Therefore, if we take $v(t) = \mathrm{e}^{a(t,s)} u_s$,

$$\partial_t v(t) = \partial_t \left(a(t,s)\right) v(t), \tag{8}$$

is obtained, where note that *s* in u_s must be common to *s* in $a(t,s)$. On the other hand, the exponential function of $a(t,s)$ leads to

$$\begin{aligned} U(t,s) &= \mathrm{e}^{a(t,s)} - \kappa I \\ &= \sum_{n=0}^{\infty} \frac{a(t,s)^n}{n!} - \kappa I. \end{aligned} \tag{9}$$

This means that a group $U(t,s)$ is generated by possibly unbounded operators being represented by the convergent power series. Here it is clear that the logarithmic representation is not simply a paraphrase of Hille-Yosida theorem. Equation (8) is an alternative equation of Equation (3), where the described evolutions are not exactly the same but connected by Equation (9).

It is notable here that exponential function of $a(t,s)$ with a certain complex number κ:

$$\mathrm{e}^{a(t,s)} = U(t,s) + \kappa I$$

does not satisfy the semigroup property:

$$\begin{aligned} &\mathrm{e}^{a(t,r)} \mathrm{e}^{a(r,s)} = \mathrm{e}^{a(t,s)} \\ &\mathrm{e}^{a(s,s)} = I. \end{aligned}$$

First of all $\mathrm{e}^{a(s,s)} \neq I$ is seen by

$$\begin{aligned} \mathrm{e}^{a(s,s)} &= \mathrm{e}^{\mathrm{Log}\left(U(s,s)+\kappa I\right)} \\ &= \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{\mathrm{Log}\lambda} \left(\lambda - (1+\kappa) I\right) \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma \lambda \left(\lambda - (1+\kappa) I\right) \mathrm{d}\lambda \\ &= (1+\kappa) I \end{aligned} \tag{10}$$

Next $\mathrm{e}^{a(t,r)} \mathrm{e}^{a(r,s)} \neq \mathrm{e}^{a(t,s)}$ is seen by

$$
\begin{aligned}
\mathrm{e}^{a(t,r)}\mathrm{e}^{a(r,s)} &= \mathrm{e}^{\mathrm{Log}(U(t,r)+\kappa I)}\ \mathrm{e}^{\mathrm{Log}(U(r,s)+\kappa I)} \\
&= \left(U(t,r)+\kappa I\right)\left(U(r,s)+\kappa I\right) \\
&= U(t,r)U(r,s)+\kappa\left(U(t,r)+U(r,s)\right)+\kappa^2 I \\
&= U(t,s)+\kappa\left(U(t,r)\right)+U(r,s))+\kappa^2 I \\
&= \left(U(t,s)+\kappa I\right)+\kappa\left(U(t,r)+\kappa I\right)+\kappa\left(U(r,s)+\kappa I\right)-\left(\kappa^2 I+\kappa I\right) \\
&= \mathrm{e}^{a(t,s)}+\kappa\left(\mathrm{e}^{a(t,r)}\mathrm{e}^{a(r,s)}\right)-\kappa(\kappa+1)I
\end{aligned}
\tag{11}
$$

where $\mathrm{e}^{a(t,s)}$ and $\mathrm{e}^{a(r,s)}$ satisfy

$$\partial_t \mathrm{e}^{a(t,s)}u_s = \partial_t\left(a(t,s)\right)\mathrm{e}^{a(t,s)}u_s,$$

while $\mathrm{e}^{a(t,r)}$ satisfies

$$\partial_t \mathrm{e}^{a(t,r)}u_r = \partial_t\left(a(t,r)\right)\mathrm{e}^{a(t,r)}u_r.$$

That is, the master equations of $\mathrm{e}^{a(t,s)}$ and $\mathrm{e}^{a(t,r)}$ are different, and this fact is associated with the insufficiency of semigroup property. Consequently the evolution operator without satisfying the semigroup property is clarified to be generated by $\left\{\partial_t a(t,s)\right\}_{t,s\in[-T.T]}$.

4. Main Result

Introduction of $\kappa \neq 0$ is the key to obtain the logarithmic representation, as well as to find the operator $\mathrm{e}^{a(t,s)}$. Indeed it is always possible to define $\mathrm{e}^{a(t,s)}$ for a certain $\kappa \in \mathbb{C}$. As seen in the preceding discussion, the singularity treatment depends on the boundedness property, which results from the finiteness of the interval $[-T,T]$ in this article, and assumed in the standard theory of evolution equations. Here, under the existence of $\left\{\mathrm{e}^{a(t,s)}\right\}_{t,s\in[-T.T]}$, we study algebraic properties of $\mathrm{e}^{a(t,s)}$ with focusing on the replacement of the original semigroup property.

Theorem 1. *For the operator* $\mathrm{e}^{a(t,s)}$ *on X, the semigroup property is replaced with*

$$
\begin{aligned}
&\mathrm{e}^{a(t,s)}-\mathrm{e}^{a(t,r)}\mathrm{e}^{a(r,s)} = \kappa(\kappa+1)I-\kappa\left(\mathrm{e}^{a(t,r)}+\mathrm{e}^{a(r,s)}\right), \\
&\mathrm{e}^{a(s,s)}-I = \kappa I.
\end{aligned}
\tag{12}
$$

The inverse relation is replaced with

$$\mathrm{e}^{a(s,t)}\mathrm{e}^{a(t,s)}-\mathrm{e}^{a(s,s)} = \kappa\left(\mathrm{e}^{a(t,s)}+\mathrm{e}^{a(s,t)}\right)-\kappa(\kappa+1)I. \tag{13}$$

In particular the commutation

$$\mathrm{e}^{a(s,t)}\mathrm{e}^{a(t,s)}-\mathrm{e}^{a(t,s)}\mathrm{e}^{a(s,t)} = 0 \tag{14}$$

i*s necessarily valid.*

Proof. Substitution of $U(t,s)=\mathrm{e}^{a(t,s)}-\kappa I$ to $U(t,r)U(r,s)=U(t,s)$ leads to the following relation (see Equation (11)):

$$U(t,r)U(r,s)=\mathrm{e}^{a(t,s)}-\kappa I,$$

and

$$\left(\mathrm{e}^{a(t,r)} - \kappa I\right)\left(\mathrm{e}^{a(r,s)} - \kappa I\right) = \mathrm{e}^{a(t,s)} - \kappa I.$$

where, by taking κ with a large $|\kappa|$, κ is possible to be taken as common to $U(t,s)$ with different *t* and *s*. Meanwhile the replacement of $U(t,s) = \mathrm{e}^{a(t,s)} - \kappa I$ with $U(s,s) = I$ leads to the following relation (see Equation (10)):

$$\mathrm{e}^{a(s,s)} = (\kappa+1) I$$

That is, for $\kappa \neq 1$, $(\kappa+1)^{-1}\mathrm{e}^{a(s,s)}$ behaves as the unit operator. Modified version of semigroup property (*i.e.*, (12)) has been proved. The inverse relation (13) follows readily from Equation (12). According to Equation (12),

$$\mathrm{e}^{a(t,t)} - \mathrm{e}^{a(t,s)}\mathrm{e}^{a(s,t)} = \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,s)} + \mathrm{e}^{a(s,t)}\right),$$

is valid. Combination with another relation

$$\mathrm{e}^{a(s,s)} - \mathrm{e}^{a(s,t)}\mathrm{e}^{a(t,s)} = \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(s,t)} + \mathrm{e}^{a(t,s)}\right),$$

leads to the commutation:

$$\mathrm{e}^{a(t,s)}\mathrm{e}^{a(s,t)} - \mathrm{e}^{a(s,t)}\mathrm{e}^{a(t,s)} = 0$$

where $\mathrm{e}^{a(t,t)} = \mathrm{e}^{a(s,s)} = (1+\kappa)I$ is utilized.

Equations (12) and (13) show the commutativity and violation of semigroup property by $\mathrm{e}^{a(t,s)}$. The right hand sides of Equations (12) and (13) are equal to zero for $\kappa = 0$. These situations correspond to the cases when the semigroup property is satisfied by $\mathrm{e}^{a(t,s)}$, and we see that the insufficiency of semigroup property is ultimately reduced to the introduction of nonzero κ.

The decomposition is obtained by the following constitution theorem for the evolution operator. Note that the decomposition of $\mathrm{e}^{a(t,s)}$ also provides a certain relation between the time-discretization and the violation of semigroup property.

Theorem 2. *For a given decomposition* $s < r_1, r_2, \cdots, r_n < t$ *of the interval* $[s,t]$ *with* $n \geq 2$*, the operator* $\mathrm{e}^{a(t,s)}$ *on X is represented by*

$$\begin{aligned}\mathrm{e}^{a(t,s)} &= \mathrm{e}^{a(t,r_n)}\mathrm{e}^{a(r_n,r_{n-1})}\cdots\mathrm{e}^{a(r_2,s_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)}\right)\\ &\quad + \sum_{k=2}^{n}\left[\left\{\kappa(\kappa+1) - \kappa\left(\mathrm{e}^{a(t,r_k)} + \mathrm{e}^{a(r_k,r_{k-1})}\right)\right\}\mathrm{e}^{a(r_{k-1},r_{k-2})}\cdots\mathrm{e}^{a(r_2,s_1)}\mathrm{e}^{a(r_1,s)}\right]\end{aligned} \tag{15}$$

where r_0 *and* r_{n+1} *in the sum are denoted as* $s = r_0$ *and* $t = r_{n+1}$ *respectively. Note that* $\mathrm{e}^{a(r_n,r_{n-1})}$ *and* $\mathrm{e}^{a(r_{n-1},r_{n-2})}$ *are the solutions of Equation* (7) *with different coefficients.*

Proof. According to Equation (12), a decomposition

$$\mathrm{e}^{a(t,s)} = \mathrm{e}^{a(t,r_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_1)}\mathrm{e}^{a(r_1,s)}\right)$$

is true. Another decomposition

$$\mathrm{e}^{a(t,r_1)} = \mathrm{e}^{a(t,r_2)}\mathrm{e}^{a(r_2,s_1)} + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,r_1)}\right),$$

is also true, and then

$$\begin{aligned}
\mathrm{e}^{a(t,s)} &= \left\{\mathrm{e}^{a(t,r_2)}\mathrm{e}^{a(r_2,r_1)} + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,s_1)}\right)\right\}\mathrm{e}^{a(r_1,s)} \\
&\quad + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)}\right) \\
&= \mathrm{e}^{a(t,r_2)}\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)\left(I + \mathrm{e}^{a(r_1,s)}\right) \\
&\quad - \kappa\left\{\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)} + \left(\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,s_1)}\right)\mathrm{e}^{a(r_1,s)}\right\}
\end{aligned}$$

follows by sorting based on κ and $\kappa(\kappa+1)$ dependence. Further decomposition shows

$$\mathrm{e}^{a(t,r_2)} = \mathrm{e}^{a(t,r_3)}\mathrm{e}^{a(r_3,r_2)} + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_3)} + \mathrm{e}^{a(r_3,r_2)}\right),$$

and then

$$\begin{aligned}
\mathrm{e}^{a(t,s)} &= \mathrm{e}^{a(t,r_2)}\mathrm{e}^{a(r_2,s_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)\left(I + \mathrm{e}^{a(r_1,s)}\right) \\
&\quad - \kappa\left\{\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)} + \left(\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,r_1)}\right)\mathrm{e}^{a(r_1,s)}\right\} \\
&= \left\{\mathrm{e}^{a(t,r_3)}\mathrm{e}^{a(r_3,r_2)} + \kappa(\kappa+1)I - \kappa\left(\mathrm{e}^{a(t,r_3)} + \mathrm{e}^{a(r_3,r_2)}\right)\right\}\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} \\
&\quad + \kappa(\kappa+1)\left(I + \mathrm{e}^{a(r_1,s)}\right) - \kappa\left\{\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)} + (\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,r_1)})\mathrm{e}^{a(r_1,s)}\right\} \\
&= \mathrm{e}^{a(t,r_3)}\mathrm{e}^{a(r_3,r_2)}\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} \\
&\quad - \kappa\left(\mathrm{e}^{a(t,r_3)} + \mathrm{e}^{a(r_3,r_2)}\right)\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)\left(I + \mathrm{e}^{a(r_1,s)}\right) \\
&\quad - \kappa\left\{\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)} + \left(\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,r_1)}\right)\mathrm{e}^{a(r_1,s)}\right\} \\
&= \mathrm{e}^{a(t,r_3)}\mathrm{e}^{a(r_3,r_2)}\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} + \kappa(\kappa+1)\left\{I + \left(I + \mathrm{e}^{a(r_2,r_1)}\right)\mathrm{e}^{a(r_1,s)}\right\} \\
&\quad - \kappa\left\{\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)} + \left(\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,r_1)}\right)\mathrm{e}^{a(r_1,s)} + \left(\mathrm{e}^{a(t,r_3)} + \mathrm{e}^{a(r_3,r_2)}\right)\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)}\right\}
\end{aligned}$$

follows. For a certain $n \ge 2$, a constitutional representation is suggested by the deduction:

$$\begin{aligned}
\mathrm{e}^{a(t,s)} &= \mathrm{e}^{a(t,r_n)}\mathrm{e}^{a(r_n,r_{n-1})}\cdots\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)} \\
&\quad + \kappa(\kappa+1)\left[I + \mathrm{e}^{a(r_1,s)} + \left(\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)}\right) + \cdots + \left(\mathrm{e}^{a(r_n,r_{n-1})}\cdots\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)}\right)\right] \\
&\quad - \kappa\Big[\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)} + (\mathrm{e}^{a(t,r_2)} + \mathrm{e}^{a(r_2,r_1)})\mathrm{e}^{a(r_1,s)} \\
&\qquad + \left(\mathrm{e}^{a(t,r_3)} + \mathrm{e}^{a(r_3,r_2)}\right)\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)}\cdots \\
&\qquad + \left(\mathrm{e}^{a(t,r_n)} + \mathrm{e}^{a(r_n,r_{n-1})}\right)\mathrm{e}^{a(r_{n-1},r_{n-2})}\cdots\mathrm{e}^{a(r_2,r_1)}\mathrm{e}^{a(r_1,s)}\Big]
\end{aligned}$$

Consequently

$$\begin{aligned}
\mathrm{e}^{a(t,s)} &= \mathrm{e}^{a(t,r_n)}\mathrm{e}^{a(r_n,r_{n-1})}\cdots\mathrm{e}^{a(r_2,s_1)}\mathrm{e}^{a(r_1,s)} \\
&\quad + \kappa(\kappa+1)\left(I + \sum_{k=2}^{n}\left[\mathrm{e}^{a(r_{k-1},r_{k-2})}\cdots\mathrm{e}^{a(r_2,s_1)}\mathrm{e}^{a(r_1,s)}\right]\right) \\
&\quad - \kappa\left(\left(\mathrm{e}^{a(t,r_1)} + \mathrm{e}^{a(r_1,s)}\right) + \sum_{k=2}^{n}\left[\left(\mathrm{e}^{a(t,r_k)} + \mathrm{e}^{a(r_k,r_{k-1})}\right)\mathrm{e}^{a(r_{k-1},r_{k-2})}\cdots\mathrm{e}^{a(r_2,s_1)}\mathrm{e}^{a(r_1,s)}\right]\right)
\end{aligned}$$

is obtained. The statement is proved by sorting terms.

5. Summary

Logarithm of invertible evolution families is defined by introducing nonzero κ. By comparing the logarithm of evolution operator to the infinitesimal generator, the difference has been found in the generated evolution operators (cf. Equation (9)). In conclusion, using the logarithmic representation, a concept of the evolution operator without satisfying the semigroup property is introduced. The violation of semigroup property has been quantitatively shown. Such an evolution operator is the alternative of original evolution operator without any loss of information.

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