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An Iterative Algorithm for Generalized Mixed Equilibrium Problems and Fixed Points of Nonexpansive Semigroups

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Abstract

In this works, by using the modified viscosity approximation method associated with Meir-Keeler contractions, we proved the convergence theorem for solving the fixed point problem of a nonexpansive semigroup and generalized mixed equilibrium problems in Hilbert spaces.

Keywords

Meir-Keeler Contraction Mappings, Left Regular, Generalized Mixed Equilibrium Problems, Variational Inequalities, α-Inverse Strongly Monotone Mappings, Nonexpansive Semigroups

1. Introduction

As you know, there are many problems that are reduced to find solutions of equilibrium problems which cover variational inequalities, fixed point problems, saddle point problems, complementarity problems as special cases. Equilibrium problem which was first introduced by Blum and Oettli [1] has been extensively studied as effective and powerful tools for a wide class of real world problems, which arises in economics, finance, image reconstruction, ecology, transportation network and related optimization problems.

From now on, we assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and K is a nonempty closed convex subset of H. $\mathbb{R}$ is denoted by the set of real numbers. Let $G: K\times K\to\mathbb{R}$ be a bifunction. Blum and Oettli [1] consider the equilibrium problem of finding $x\in K$ such that

$$G(x,y)\geq 0, \forall y\in K. \tag{1.1}$$

The solution set of problem (1.1) is denoted by $EP(G)$, *i.e.,*

$$EP(G) = \{x \in K : G(x, y) \geq 0, \forall y \in K\}$$

Recently the so-called generalized mixed equilibrium problem has been investigated by many authors [2] [3]. The generalized mixed equilibrium problem is to find $x \in K.$ such that

$$G(x, y) + \varphi(y) + \langle Ax, y - x\rangle \geq \varphi(x), \forall y \in K, \tag{1.2}$$

where $A : K \to H$ is a mapping and $\varphi : K \to \mathbb{R} \cup \{+\infty\}$ is a real valued function. We use $GMEP(G, A, \varphi)$ to denote the solution set of generalized mixed equilibrium problem *i.e.*,

$$GMEP(G, A, \varphi) = \{x \in K : G(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in K\}.$$

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequality problem, minimax problems, the Nash equilibrium problems in noncooperative games and others (see [4] [5] [6] [7] [8] [9] [10] [11] [12]).

Special Cases: The following problems are the special cases of problem (1.2).

1) If $A = 0$ then (1.2) is equivalent to finding $x \in K$ such that

$$G(x, y) + \varphi(y) - \varphi(x) \geq 0, \forall y \in K, \tag{1.3}$$

is called mixed equilibrium problems.

2) If $G = 0$ then (1.2) is equivalent to finding $x \in K$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in K, \tag{1.4}$$

is called mixed variational inequality of Browder type [13].

3) If $\varphi = 0$ then (1.2) is equivalent to find $x \in K$ such that

$$G(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in K, \tag{1.5}$$

is called generalized equilibrium problems (shortly, (GEP)). We denote GEP(G,A) the solution set of problem (GEP).

4) If $A = 0$ and $\varphi = 0$ then (1.2) is equivalent to (1.1).

5) Let $G(x, y) = \langle Ax, y - x\rangle,$ for all $x, y \in K$. Then we see that (1.1) is reduces to the following classical variational inequalities for finding $x \in K$ such that

$$\langle Ax, y - x\rangle \geq 0, \forall y \in K. \tag{1.6}$$

It is known that $x \in K$ is a solution to (1.6) if and only if x is a fixed point of the mapping $P_K(I - \rho A)$, where $\rho > 0$ is a constant and I is an identity mapping.

Let $T : K \to K$ be a mapping from K into itself. Let denote $F(T)$ the set of fixed points of the mapping T. A mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K.$$

A mapping T is said to be contractive if there exists a constant $\alpha \in [0,1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in K.$$

A mapping T is called α -inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|Tx - Ty\|^2, \forall x, y \in K.$$

Remark 1.1 Every α-inverse strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

In 1967, Halpern [14] introduced the following iterative method for a nonexpansive mapping $T : K \to K$ in a real Hilbert space, for finding $x_1 \in K$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Tx_n, n \geq 1 \tag{1.7}$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$ is fixed.

Moudafi [15] introduced the viscosity approximation method for a nonexpansive mapping T as follows: For finding $x_1 \in K$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Tx_n, n \geq 1 \tag{1.8}$$

where $\{\alpha_n\} \subset (0,1)$ and f is a contraction mapping.

A viscosity approximation method with Meir-Keeler contraction was first studied by Suzuki [16]. Very recently Petrusel and Yao [17] studied the following viscosity approximation method with a generalized contraction: for finding $x_0 \in K$ and

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1}) T_{n+1} x_n, n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{T_n\}_{n=1}^{\infty}$ is a family of nonexpansive mappings on K.

Takahashi and Takahashi [18] introduced the following iterative scheme for solving a generalized equilibrium problems and a fixed point problems of a nonexpansive mapping T in a Hilbert spaces H: Finding $x_1, u \in K$ and

$$\begin{cases} u_n \in K \text{ such that} \\ G(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, y \in K, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T\left[\alpha_n u + (1 - \alpha_n) u_n\right], \ n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1), \{r_n\} \subset (0, \infty)$ and A is an α-inverse strongly monotone mapping. They proved that the sequence $\{x_n\}$ generated by (1.9) strongly converges to an element in $F(T) \cap GEP(G, A)$ under suitable conditions.

In this paper, from the recent works [19] [20] [21] [22] [23] [24] [25] [26], we introduced an iterative scheme by the modified viscosity approximation method associated with Meir-Keeler contraction (see [27]) for solving the generalized mixed equilibrium problems and fixed point problem of a nonexpansive semigroup in Hilbert spaces, and also we discussed a convergence theorem. Finally we apply our main results for commutative nonexpansive mappings and semigroup of strongly continuous mappings.

2. Preliminaries

Let S be a semigroup and $\ell^{\infty}(S)$ be the Banach space of all bounded real valued functionals on S with superimum norm. For each $s \in S$, we define the left and right translation operators l_s and r_s on $\ell^{\infty}(S)$ by $(l_s f)(t) = f(st)$

and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in \ell^\infty(S)$, respectively. Let X be a subspace of $\ell^\infty(s)$ containing 1. An element μ in the dual space X^* of X is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We denote the value of μ at the function f by $\mu(f)$. According to the time and circumstances, we write the value $\mu(f)$ by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$. It is well known that μ is a mean of X if and only if for each $f \in X$,

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s).$$

Let X be a translation invariant subspace of $\ell^\infty(S)$ (*i.e.*, $l_s X \subset X$ and $r_s X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be invariant if μ is both left and right invariant [28] [29]. S is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. S is amenable if S is left and right amenable [30]. In this case $\ell^\infty(S)$ also has an invariant mean. It is known that $\ell^\infty(S)$ is amenable when S is commutative semigroup or solvable group. However the free group or semigroup of two generators is not left or right amenable (see [31]). A net $\{\mu_\alpha\}$ of mean on X is said to be left regular if

$$\lim_\alpha \left\| l_s^* \mu_\alpha - \mu_\alpha \right\| = 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let K be a nonempty closed convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a nonexpansive semigroup on S if for each $s \in S$, the mapping $T(s): K \to K$ is nonexpansive and $T(st) = T(s)T(t)$ for each $s,t \in S$ (see [30] [30]). We denote by $F(\mathcal{S})$ the set of common fixed point of S, *i.e.*,

$$F(\mathcal{S}) = \bigcap_{s\in S} F\left(T(s)\right) = \bigcap_{s\in S} \left\{x \in K : T(s)x = x\right\}.$$

Assume that B_r is a open ball of radius r centered at 0 and $\overline{co}A$ is a closed convex hull of $A \subset H$. For $\epsilon > 0$ and a mapping $T: D \to H$, the set of ϵ-approximate fixed points of T will be denoted by $F_\epsilon(T, D)$, *i.e.*,

$$F_\epsilon(T, D) = \left\{x \in D : \|x - Tx\| \le \epsilon\right\}.$$

Lemma 2.1 [32] *Let* f *be a function of a semigroup* S *into a Banach space* E *such that the weak closure of* $\{f(t) : t \in S\}$ *is weakly compact and* X *a subspace of* $\ell^\infty(S)$ *containing all the function* $t \to \langle f(t), x^*\rangle$ *with* $x^* \in E^*$. *Then for any* $\mu \in X^*$ *there exists a unique element* f_μ *in* E *such that for all* $x^* \in E^*$,

$$\left\langle f_\mu, x^*\right\rangle = \mu_t \left\langle f(t), x^*\right\rangle.$$

Moreover if μ *is a mean on* X *then*

$$\int f(t)d\mu(t) \in \overline{co}\left\{f(t) : t \in S\right\}.$$

We can write f_μ *by* $\int f(t)d\mu(t)$.

Lemma 2.2 [32] *Let* K *be a closed convex subset of a Hilbert space H.Let*

$\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup from K into itself such that $F(\mathcal{S}) \neq \emptyset$, X be a subspace of $\ell^\infty(S)$ containing 1, the mapping $t \to \langle T(t)x, y\rangle$ be an element of X for each $x \in K$ and $y \in H$ and μ be a mean on X. If we write $T_\mu x$ instead of $\epsilon T(t)x d\mu(t)$, then the following statements hold:

1) *T_μ is a nonexpansive mapping from K into K,*
2) *$T_\mu x = x$ for each $x \in F(\mathcal{S})$,*
3) *$T_\mu x \in \overline{co}\{T(t)x : t \in S\}$, for each $x \in K$;*
4) *if μ is left invariant then T_μ is a nonexpansive retraction from K into $F(\mathcal{S})$.*

Let K be a nonempty closed convex subset of a real Hilbert space H. Then for any $x \in H$ there exists a unique nearest point in K, denoted by $P_K(x)$ such that for all $y \in K$,

$$\|x - P_K(x)\| \le \|x - y\|,$$

where P_K is the metric projection of H onto K. We also know that for $x \in H$ and $z \in K, z = P_K x$ if and only if for all $y \in K$,

$$\langle x - z, y - z\rangle \le 0.$$

A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an L-function if $\psi(0) = 0, \psi(t) > 0$ for each $t > 0$ and for every $s > 0$ there exists $u > s$ such that $\psi(t) \le s$ for all $t \in [s, u]$. As a consequence, every L-function ψ satisfies $\psi(t) < t$ for each $t > 0$.

Definition 2.3 Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a

1) (ψ, L)-contraction if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an L-function and

$$d(f(x), f(y)) < \psi(d(x, y))$$

for all $x, y \in X$ with $x \neq y$;

2) Meir-Keeler type mapping if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in X$ with $d(x, y) < \epsilon + \delta$ we have $d\left(f(x), f(y)\right) < \epsilon$ (see [33] [34]).

Theorem 2.4 [34] *Let (X, d) be a complete metric space and $f : X \to X$ is a Meir-Keeler type mapping. Then f has a unique fixed point.*

Theorem 2.5 [35] *Let (X, d) be a complete metric space and $f : X \to X$ is a mapping. Then the following statements are equivalent.*

1) *f is a Meir-Keeler type mapping;*
2) *there exists an L-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that f is a (ψ, L)-contraction.*

Theorem 2.6 [16] *Let K be a convex subset of a Banach space E and let $f : K \to K$ be a Meir-Keeler type mapping. Then for each $\epsilon > 0$ there exists $r \in (0,1)$ such that for each $x, y \in K$ with $\|x - y\| \ge \epsilon$ we have*

$$\|f(x) - f(y)\| \le r\|x - y\|.$$

Proposition 2.7 [31] *Let K be a convex subset of a Banach space E, T be a nonexpansive mapping on K and $f : K \to K$ be a Meir-Keeler type* *mapping. Then the following statements hold*:

1) $T \circ f$ *is a Meir-Keeler type mapping on* K.

2) *For each* $\alpha \in (0,1)$, *the mapping* $x \to \alpha f(x) + (1-\alpha)T(x)$ *is a Meir-Keeler type mapping on* K.

Lemma 2.8 [36] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real number such that*

$$a_{n+1} \le (1-\rho_n)a_n + \rho_n \delta_n, n \ge 1,$$

where $\{\rho_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *satisfying*

1) $\sum_{n=1}^{\infty} \rho_n = \infty$;

2) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\rho_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 [37] *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* E *such that*

$$x_{n+1} = (1-\beta_n)z_n + \beta_n x_n, \forall n \ge 1,$$

where $\{\beta_n\}$ *is a real sequence in* $(0,1)$ *with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

If

$$\limsup_{n\to\infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \le 0,$$

then

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Lemma 2.10 [38] *Let* $0 < p \le t_n \le q < 1$ *for all* $n \ge 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* H *such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$$

for some $r \ge 0$. *Then we have*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.11 [39] *Let* K *be a nonempty closed convex subset of a real Hilbert space* H *and* $T: K \to K$ *be a nonexpansive mapping with* $F(T) \ne \emptyset$. *Then* $I - T$ *is demiclosed at zero, that is, for all sequence* $\{x_n\} \subset K$ *with* $x_n \rightharpoonup y$ *and* $\|x_n - Tx_n\| \to 0$ *it follows that* $y = Ty$.

For solving the equilibrium problem we assume that bifunction G satisfies the following conditions:

(A1) $G(x,x) = 0, \forall x \in K$;

(A2) G is monotone, *i.e.*, $G(x,y) + G(y,x) \le 0, \forall x, y \in K$;

(A3) for each $x, y, z \in K$, $\lim_{t\to 0} G(tz + (1-t)x, y) \le G(x,y)$;

(A4) for each $x \in K$, $y \to G(x,y)$ is convex and lower semicontinuous.

Lemma 2.12 [1] *Let* K *be a nonempty closed convex subset of a real Hilbert* *space* H *and* G *be a bifunction from* $K \times K$ *to* $\mathbb{R}$ *satisfying* $(A1)$-$(A4)$. *Then for any* $r > 0$ *and* $x \in H$, *there exists* $z \in K$ *such that*

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K.$$

Further, if

$$T_r x = \left\{ z \in K : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K \right\},$$

then we have the followings:

1) T_r *is single-valued*;
2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$
$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$
3) $F(T_r) = EP(G)$;
4) $EP(G)$ *is closed and convex.*

Lemma 2.13 [18] *Let* H, K, G *and* $T_r x$ *be as in Lemma 2.12. Then we have*

$$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle,$$

for all $s, t > 0$ *and* $x \in H$.

3. Main Results

Theorem 3.1 *Let K be a nonempty closed convex subset of a Hilbert space* H. *Let* S *be a semigroup,* $\mathcal{S} = \{T(t) : t \in S\}$ *be a nonexpansive semigroup on* S, $G : K \times K \to \mathbb{R}$ *be a bifunction satisfying* $(A1)$-$(A4)$ *and* $A : K \to H$ *be an* α-*inverse strongly monotone mapping with*

$$\mathcal{F} := F(\mathcal{S}) \cap GMEP(G, A, \varphi) \neq \emptyset.$$

Let $\varphi : K \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function, X be a left invariant subspace of* $\ell^\infty(S)$ *such that* $1 \in X$ *and the function* $t \to \langle T(t)x, y\rangle$ *be an element of X for each* $x, y \in K$. *Let* $\{\mu_n\}$ *be a left regular sequence of means on X such that* $\|\mu_{n+1} - \mu_n\| \to 0$ *as* $n \to \infty$ *and* $f : K \to K$ *be a Meir-Keeler contraction. Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in K$ *and*

$$\begin{cases} u_n \in K \text{ such that} \\ G(u_n, y) + \langle Ax_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \varphi(y) - \varphi(u_n) \geq 0, \forall y \in K, \\ x_{n+1} = \beta_n x_n + \beta_n' T_{\mu_n}\left[\alpha_n f(x_n) + (1-\alpha_n)u_n\right] + \beta_n'' e_n, n \geq 1, \end{cases}$$

where $\{e_n\}$ *is bounded sequence in* K, $\{\alpha_n\}, \{\beta_n\}, \{\beta_n'\}$ *and* $\{\beta_n''\}$ *are real number sequences in* $(0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying the conditions*:

(C1) $\beta_n + \beta_n' + \beta_n'' = 1,\ 0 < a \leq \beta_n \leq b < 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(C4) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha, \sum_{n=1}^{\infty} |\beta_n''| < \infty, \lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ *strongly converges to* $p \in \mathcal{F}$ *which is also solves the following variational inequality problem*:

$$\langle f(p)-p, q-p\rangle \le 0, \forall q \in \mathcal{F}. \tag{3.1}$$

Proof. We give the several steps for the proof.

Step 1: First we show that $\{x_n\}$ is bounded. Put $u_n = T_{r_n}(x_n - r_n A x_n)$ and $y_n = \alpha_n f(x_n) + (1-\alpha_n)u_n$ for all $n \ge 1$. Then for $w \in \mathcal{F}$, we have

$$\begin{aligned}
\|u_n - w\|^2 &= \left\|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(w - r_n A w)\right\|^2 \\
&\le \|(x_n - r_n A x_n) - (w - r_n A w)\|^2 \\
&\le \|(x_n - w) - r_n(Ax_n - Aw)\|^2 \\
&\le \|x_n - w\|^2 - 2r_n\langle x_n - w, Ax_n - Aw\rangle + r_n^2\|Ax_n - Aw\|^2 \\
&\le \|x_n - w\|^2 - 2r_n\alpha\|Ax_n - Aw\|^2 + r_n^2\|Ax_n - Aw\|^2 \\
&\le \|x_n - w\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2 \\
&\le \|x_n - w\|^2.
\end{aligned} \tag{3.2}$$

Set $T_n = \alpha_n I + (1-\alpha_n)T$, then T_n is nonexpansive and $F(T_n) = F(T)$. Hence we have

$$\begin{aligned}
&\|x_{n+1} - w\| \\
&\le \beta_n\|x_n - w\| + \beta_n'\left\|T_{\mu_n} y_n - w\right\| + \beta_n''\|e_n - w\| \\
&\le \beta_n\|x_n - w\| + \beta_n'\|y_n - w\| + \beta_n''\|e_n - w\| \\
&\le \beta_n\|x_n - w\| + \beta_n'\left(\alpha_n\|f(x_n) - w\| + (1-\alpha_n)\|u_n - w\|\right) + \beta_n''\|e_n - w\| \\
&\le \beta_n\|x_n - w\| + \beta'\left(\alpha_n\|f(x_n) - f(w)\| + \alpha_n\|f(w) - w\| + (1-\alpha_n)\|x_n - w\|\right) \\
&\quad + \beta_n''\|e_n - w\| \\
&\le \beta_n\|x_n - w\| + \beta'\left(\alpha_n\psi\left(\|x_n - w\|\right) + \alpha_n\|f(w) - w\| + (1-\alpha_n)\|x_n - w\|\right) \\
&\quad + \beta_n''\|e_n - w\| \\
&\le \|x_n - w\| - \alpha_n(\beta'(\eta(\|x_n - w\|)) + \alpha_n\beta'\eta(\eta^{-1}(\|f(w) - w\|)) + \beta_n''\|e_n - w\| \\
&\le \max\left\{\|x_n - w\|, \eta^{-1}\left(\|f(w) - w\|\right), \|e_n - w\|\right\}.
\end{aligned}$$

By induction, we can prove that

$$\|x_n - w\| \le \max\left\{\|x_1 - w\|, \eta^{-1}\left(\|f(w) - w\|\right), \|e_n - w\|\right\}, \forall n \ge 1.$$

Hence the sequence $\{x_n\}$ is bounded. So $\{f(x_n)\}, \{u_n\}, \{y_n\}$ and $\{T_{\mu_n} y_n\}$ are all bounded.

Step 2: We next show that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Observe that

$$\lim_{n\to\infty}\left\|T_{\mu_{n+1}} y_n - T_{\mu_n} y_n\right\| = 0. \tag{3.3}$$

Indeed

$$\begin{aligned}
&\left\|T_{\mu_{n+1}} y_n - T_{\mu_n} y_n\right\| \sup_{\|z\|=1}\left|\langle T_{\mu_{n+1}} y_n - T_{\mu_n} y_n, z\rangle\right| \\
&= \sup_{\|z\|=1}\left|(\mu_{n+1})_s\langle T(s) y_n, z\rangle - (\mu_n)_s\langle T(s) y_n, z\rangle\right| \\
&\le \|\mu_{n+1} - \mu_n\| \sup_{s\in S}\|T(s) y_n\|.
\end{aligned}$$

Since $\{y_n\}$ is bounded and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\|=0$, (3.3) holds. Since $u_n=T_{r_n}(x_n-r_nAx_n)$ and $u_{n+1}=T_{r_{n+1}}(x_{n+1}-r_{n+1}Ax_{n+1})$, we have

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|T_{r_{n+1}}(x_{n+1}-r_{n+1}Ax_{n+1})-T_{r_n}(x_n-r_nAx_n)\|\\ &\le \|T_{r_{n+1}}(x_{n+1}-r_{n+1}Ax_{n+1})-T_{r_{n+1}}(x_n-r_nAx_n)\|\\ &\quad+\|T_{r_{n+1}}(x_n-r_nAx_n)-T_{r_n}(x_n-r_nAx_n)\|\\ &\le \|(x_{n+1}-r_{n+1}Ax_{n+1})-(x_n-r_nAx_n)\|+\|T_{r_{n+1}}(x_n-r_nAx_n)-T_{r_n}(x_n-r_nAx_n)\|\\ &\le \|(x_{n+1}-r_{n+1}Ax_{n+1})-(x_n-r_{n+1}Ax_n)\|+\|(x_n-r_{n+1}Ax_n)-(x_n-r_nAx_n)\|\\ &\quad+\|T_{r_{n+1}}(x_n-r_nAx_n)-T_{r_n}(x_n-r_nAx_n)\|\\ &\le \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|Ax_n\|+\|T_{r_{n+1}}(x_n-r_nAx_n)-T_{r_n}(x_n-r_nAx_n)\|.\end{aligned}\tag{3.4}$$

From $y_n=\alpha_nf(x_n)+(1-\alpha_n)u_n$ and $y_{n+1}=\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})u_{n+1}$, we have

$$\begin{aligned}y_{n+1}-y_n &= \alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})u_{n+1}-(\alpha_nf(x_n)+(1-\alpha_n)u_n)\\ &=\alpha_{n+1}(f(x_{n+1})-u_{n+1})+\alpha_n(u_n-f(x_n))+(u_{n+1}-u_n),\end{aligned}$$

it follows that

$$\|y_{n+1}-y_n\|\le\alpha_{n+1}(\|f(x_{n+1})\|+\|u_{n+1}\|)+\alpha_n(\|u_n\|+\|f(x_n)\|)+\|u_{n+1}-u_n\|.\tag{3.5}$$

We see that

$$\begin{aligned}\|T_{\mu_{n+1}}y_{n+1}-T_{\mu_n}y_n\| &\le \|T_{\mu_{n+1}}y_{n+1}-T_{\mu_{n+1}}y_n\|+\|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\|\\ &\le \|y_{n+1}-y_n\|+\|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\|.\end{aligned}\tag{3.6}$$

Combining (3.4) and (3.5) with (3.6), we obtain

$$\begin{aligned}\|T_{\mu_{n+1}}y_{n+1}-T_{\mu_n}y_n\| &\le \alpha_{n+1}(\|f(x_{n+1})\|+\|u_{n+1}\|)+\alpha_n(\|u_n\|+\|f(x_n)\|)\\ &\quad+\|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|Ax_n\|\\ &\quad+\|T_{r_{n+1}}(x_n-r_nAx_n)-T_{r_n}(x_n-r_nAx_n)\|+\|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\|.\end{aligned}$$

Using Lemma 2.13, (3.3),($C1$) and ($C4$), then we have

$$\limsup_{n\to\infty}\left(\|T_{\mu_{n+1}}y_{n+1}-T_{\mu_n}y_n\|-\|x_{n+1}-x_n\|\right)\le 0.$$

From this inequality and ($C3$), it follows from Lemma 2.9 that

$$\lim_{n\to\infty}\|T_{\mu_n}y_n-x_n\|=0.\tag{3.7}$$

It implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.8}$$

Step 3: Next we prove that for all $t\in S$,

$$\lim_{n\to\infty}\|x_{n+1}-T(t)x_n\|=0.$$

Put

$$M=\max\left\{\|x-w\|,\eta^{-1}(\|f(w)-w\|),\|e_n-w\|\right\}.$$

Set $D=\{y\in K:\|y-w\|\le M\}$. It is easily seen that D is a nonempty bounded closed convex subset of K. Further $\{x_n\},\{y_n\}$ and $\{u_n\}$ are in D. To complete our proof, we follows that proof line as in [30]. From [40], for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in S$,

$$\overline{co}F_{\delta}\left(T(t);D\right)+B_{\delta}\subseteq F_{\epsilon}\left(T(t);D\right). \tag{3.9}$$

From Corollary 1.1 in [40], there exists a natural number N such that for all $t,s\in S, y\in D$,

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N}T(t^i s)y-T(t)\left(\frac{1}{N+1}\sum_{i=0}^{N}T(t^i s)y\right)\right\|\le\delta. \tag{3.10}$$

Since $\{\mu_n\}$ is left regular, for $t\in S$ there exists $n_0\in N$ such that

$$\left\|\mu_n-\ell_{t^i}^{*}\mu_n\right\|\le\frac{\delta}{2\left(M+\|w\|\right)},$$

for all $n\ge n_0$ and $i=1,2,\cdots,N$. Therefore, we have for all $n\ge n_0$,

$$\begin{aligned}
&\sup_{y\in D}\left\|T_{\mu_n}y-\int\frac{1}{N+1}\sum_{i=0}^{N}T(t^i s)y\,d\mu_n(s)\right\|\\
&=\sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\left\langle T(s)y,z\right\rangle-(\mu_n)_s\left\langle\frac{1}{N+1}\sum_{i=0}^{N}T(t^i s)y,z\right\rangle\right|\\
&\le\frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\left\langle T(s)y,z\right\rangle-\left(\ell_{t^i}^{*}\mu_n\right)_s\left\langle T(s)y,z\right\rangle\right|\\
&\le\max_{i=1,2,\cdots,N}\left\|\mu_n-\ell_{t^i}^{*}\mu_n\right\|\left(M+\|w\|\right)\le\frac{\delta}{3}.
\end{aligned} \tag{3.11}$$

We observe from Lemma 2.2 (iii) that

$$\int\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^i s\right)y\,d\mu_n(s)\in\overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^i\right)T(s)y:s\in S\right\}. \tag{3.12}$$

Combining (3.10), (3.12) and (3.12), we have for all $y\in D, n\ge n_0$,

$$\begin{aligned}
T_{\mu_n}y&=\int\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^i s\right)y\,d\mu_n(s)+\left(T_{\mu_n}y-\int\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^i s\right)y\,d\mu_n(s)\right)\\
&\in\overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N}T\left(t^i\right)T(s)y:s\in S\right\}+B_{\frac{\delta}{3}}\\
&\subseteq\overline{co}F_{\delta}\left(T(t);D\right)+B_{\frac{\delta}{3}}.
\end{aligned} \tag{3.13}$$

Let $t\in S$ and $\epsilon>0$. Then there exists $\delta>0$ which satisfies (3.9). From ($C3$) there exist $a,b\in(0,1)$ such that $0<a\le\beta_n\le b<1$. From (3.7) there exists $k_0\in N$ such that $\left\|x_n-T_{\mu_n}y_n\right\|<\frac{\delta}{3b}$ and $\left\|e_n-T_{\mu_n}y_n\right\|<\frac{\delta}{3b}$, for all $n>k_0$. So from (3.9) and (3.13), we have

$$\begin{aligned}
x_{n+1}&=\beta_n x_n+\beta_n' T_{\mu_n}y_n+\beta_n'' e_n\\
&=\beta_n x_n+\left(1-\beta_n\right)T_{\mu_n}y_n+\beta_n''\left(e_n-T_{\mu_n}y_n\right)\\
&=T_{\mu_n}y_n+\beta_n\left(x_n-T_{\mu_n}y_n\right)+\beta_n''\left(e_n-T_{\mu_n}y_n\right)\\
&\in\overline{co}F_{\delta}\left(T(t);D\right)+B_{\frac{\delta}{3}}+B_{\frac{\delta}{3}}+B_{\frac{\delta}{3}}\\
&\subseteq\overline{co}F_{\delta}\left(T(t);D\right)+B_{\delta}\subseteq F_{\varepsilon}(T(t);D).
\end{aligned}$$

Hence $\limsup_{n\to\infty}\|x_n - T(t)x_n\| \le \epsilon$. Since $\epsilon > 0$ is arbitrary,

$$\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0.$$

Step 4: We next show that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.14}$$

Using inequality (3.2), we obtain

$$\begin{aligned}
\|x_{n+1} - w\|^2 &\le \beta_n\|x_n - w\|^2 + \beta_n'\|T_{\mu_n}y_n - w\|^2 + \beta_n''\|e_n - w\|^2 \\
&\le \beta_n\|x_n - w\|^2 + \beta_n'\|y_n - w\|^2 + \beta_n''\|e_n - w\|^2 \\
&\le \beta_n\|x_n - w\|^2 + \beta_n'\left(\alpha_n\|f(x_n) - w\|^2 + (1-\alpha_n)\|u_n - w\|^2\right) + \beta_n''\|e_n - w\|^2 \\
&\le \beta_n\|x_n - w\|^2 + \beta_n'\Big(\alpha_n\|f(x_n) - w\|^2 \\
&\quad + (1-\alpha_n)\left(\|x_n - w\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2\right)\Big) + \beta_n''\|e_n - w\|^2 \\
&\le \|x_n - w\|^2 + \beta_n'\alpha_n\|f(x_n) - w\|^2 \\
&\quad + \beta_n'(1-\alpha_n)r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2 + \beta_n''\|e_n - w\|^2,
\end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned}
&\beta_n'(1-\alpha_n)r_n(2\alpha - r_n)\|Ax_n - Aw\|^2 \\
&\le \|x_n - w\|^2 - \|x_{n+1} - w\|^2 + \beta_n'\alpha_n\|f(x_n) - w\|^2 + \beta_n''\|e_n - w\|^2.
\end{aligned}$$

From (C1)-(C4) and (3.8), we obtain

$$\lim_{N\to\infty}\|Ax_n - Aw\| = 0. \tag{3.16}$$

Since T_{r_n} is firmly nonexpansive,

$$\begin{aligned}
\|u_n - w\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(w - r_nAw)\|^2 \\
&\le \langle x_n - r_nAx_n - (w - r_nAw), u_n - w\rangle \\
&= \frac{1}{2}\Big(\|(x_n - r_nAx_n) - (w - r_nAw)\|^2 + \|u_n - w\|^2 \\
&\quad - \|(x_n - r_nAx_n) - (w - r_nAw) - (u_n - w)\|^2\Big) \\
&\le \frac{1}{2}\left(\|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2\right) \\
&\le \frac{1}{2}\Big(\|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u_n - w\|^2 &\le \|x_n - w\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|x_{n+1}-w\|^2 &\le \beta_n\|x_n-w\|^2+\beta_n'\left\|T_{\mu_n}y_n-w\right\|^2+\beta_n''\|e_n-w\|^2\\
&\le \beta_n\|x_n-w\|^2+\beta_n'\|y_n-w\|^2+\beta_n''\|e_n-w\|^2\\
&\le \beta_n\|x_n-w\|^2+\beta_n'\left(\alpha_n\|f(x_n)-w\|^2+(1-\alpha_n)\|u_n-w\|^2\right)+\beta_n''\|e_n-w\|^2\\
&\le \beta_n\|x_n-w\|^2+\beta_n'\Big(\alpha_n\|f(x_n)-w\|^2\\
&\quad+(1-\alpha_n)\left(\|x_n-w\|^2-\|x_n-u_n\|^2+2r_n\langle x_n-u_n,Ax_n-Aw\rangle-r_n^2\|Ax_n-Aw\|^2\right)\Big)\\
&\quad+\beta_n''\|e_n-w\|^2\\
&\le \|x_n-w\|^2+\alpha_n\|f(x_n)-w\|^2-\beta_n'\|x_n-u_n\|^2\\
&\quad+2(1-\alpha_n)r_n\|x_n-u_n\|\|Ax_n-Aw\|+\beta_n''\|e_n-w\|^2,
\end{aligned}$$

which yields

$$\begin{aligned}
\beta_n'\|x_n-u_n\|^2 &\le \|x_n-w\|^2-\|x_{n+1}-w\|^2+\alpha_n\|f(x_n)-w\|^2\\
&\quad+2\beta_n'r_n\|x_n-u_n\|\|Ax_n-Aw\|+\beta_n''\|e_n-w\|^2.
\end{aligned}$$

Hence, from $(C2)$, $(C3)$ and (3.16) we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.17}$$

Since $y_n=\alpha_n f(x_n)+(1-\alpha_n)u_n$, we have $y_n-u_n=\alpha_n(f(x_n)-u_n)$ and hence

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.18}$$

On the other hand, by Proposition 2.7 (i), we know that $P_{\mathcal{F}}f$ is a Meir-Keeler contraction. From Theorem 2.4, there exists a unique element p such that $P_{\mathcal{F}}f(p)=p$ which is equivalent to

$$\langle f(p)-p,q-p\rangle\le 0,\forall q\in\mathcal{F}.$$

Step 5: We next show that

$$\limsup_{n\to\infty}\langle f(p)-p,y_n-p\rangle\le 0.$$

To see this, we chose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p)-p,y_n-p\rangle=\lim_{k\to\infty}\langle f(p)-p,y_{n_k}-p\rangle.$$

Since $\{x_n\}$ is a bounded, K is closed and H is reflexive, there exists a point $z\in K$ such that $x_{n_k}\rightharpoonup z\in K$. From (3.17) and (3.18) there exists a corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ (resp. $\{y_{n_k}\}$ of $\{y_n\}$) such that $u_{n_k}\rightharpoonup z\in K$ (resp. $y_{n_k}\rightharpoonup z\in K$). We next show that $z\in GMEP(G,A,\varphi)$. Since $u_n=T_{r_n}(x_n-r_nAx_n)$. We can write

$$G(u_n,y)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle+\varphi(y)-\varphi(u_n)\ge 0,\forall y\in K.$$

From $(A2)$, we have

$$\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge G(y,u_n)-\varphi(y)+\varphi(u_n)\ge 0,\forall y\in K.$$

Then

$$\left\langle Ax_{n_k}, y-u_{n_k}\right\rangle+\frac{1}{r_{n_k}}\left\langle y-u_{n_k}, u_{n_k}-x_{n_k}\right\rangle \\ \geq G\left(y, u_{n_k}\right)-\varphi(y)+\varphi\left(u_{n_k}\right)\geq 0, \forall y\in K. \tag{3.19}$$

Put $y_t = ty+(1-t)z$, for $t\in(0,1]$ and $y\in K$. Since $y\in K$ and $z\in K$, $y_t\in K$. So from (3.19) we have

$$\begin{aligned}\left\langle y_t-u_{n_k}, Ay_t\right\rangle &\geq \left\langle y_t-u_{n_k}, Ay_t\right\rangle-\left\langle y_t-u_{n_k}, Ax_{n_k}\right\rangle-\left\langle y_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \\ &\quad+G\left(y_t, u_{n_k}\right)+\varphi(y_t)-\varphi\left(u_{n_k}\right) \\ &=\left\langle y_t-u_{n_k}, Ay_t-Au_{n_k}\right\rangle+\left\langle y_t-u_{n_k}, Au_{n_k}-Ax_{n_k}\right\rangle \\ &\quad-\left\langle y_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle+G\left(y_t, u_{n_k}\right)+\varphi(y_t)-\varphi\left(u_{n_k}\right) \\ &\geq \left\langle y_t-u_{n_k}, Au_{n_k}-Ax_{n_k}\right\rangle-\left\langle y_t-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \\ &\quad+G\left(y_t, u_{n_k}\right)+\varphi(y_t)-\varphi\left(u_{n_k}\right)\end{aligned}$$

From (A4), we have

$$\langle y_t - z, Ay_t\rangle \geq G(y_t, z)+\varphi(y_t)-\varphi(z). \tag{3.20}$$

From (A1)-(A4) and (3.20), we have

$$\begin{aligned}0 &= G(y_t, y_t) \\ &\leq tG(y_t, y)+(1-t)G(y_t, z)\square \\ &\leq tG(y_t, y)+(1-t)\langle y_t - z, Ay_t\rangle \\ &\leq tG(y_t, y)+(1-t)\langle y - z, Ay_t\rangle.\end{aligned}$$

It follows that

$$0\leq tG(y_t, y)+(1-t)\langle y-z, Ay_t\rangle,$$

letting $t\to 0$ by $(A3)$, we have

$$0\leq G(z, y)+\langle y-z, Az\rangle, \forall y\in K.$$

Hence $z\in GMEP(G, A, \varphi)$. It is easily seen that $z\in F(\mathcal{S})$. Indeed, since $x_{n_k}\rightharpoonup z$ and $\|x_n - T(t)x_n\|\to 0$, for all $t\in S$, we conclude from Lemma 2.1 that $z\in F(\mathcal{S})$. Consequently, we have $z\in\mathcal{F}=F(\mathcal{S})\cap GMEP(G, A, \varphi)$ and hence

$$\begin{aligned}\limsup_{n\to\infty}\left\langle f(p)-p, y_n-p\right\rangle &= \lim_{k\to\infty}\left\langle f(p)-p, y_{n_k}-p\right\rangle \\ &=\left\langle f(p)-p, z-p\right\rangle \\ &\leq 0.\end{aligned} \tag{3.21}$$

Step 6: Now we are in a position to show that x is a fixed point of T.

Let $\lim_{n\to\infty}\|x_n - w\| = d > 0$. Then we have

$$\begin{aligned}&\lim_{n\to\infty}\|x_{n+1}-w\| \\ &=\lim_{n\to\infty}\left\|\beta_n\left(x_n - w+\beta_n''\left(e_n - T_{\mu_n}u_n\right)\right)+\left(1-\beta_n\right)\left(T_{\mu_n}u_n - w+\beta_n''\left(e_n - T_{\mu_n}u_n\right)\right)\right\| \\ &= d.\end{aligned}$$

We note that

$$\begin{aligned}&\limsup_{n\to\infty}\left\|x_n - w + \beta_n''\left(e_n - T_{\mu_n}u_n\right)\right\| \\ &\le \limsup_{n\to\infty}\left\|x_n - w\right\| + \limsup_{n\to\infty}\beta''\left\|e_n - T_{\mu_n}u_n\right\| \\ &\le d\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\left\|T_{\mu_n}u_n - w + \beta_n''\left(e_n - T_{\mu_n}u_n\right)\right\| \\ &\le \limsup_{n\to\infty}\left\|T_{\mu_n}u_n - w\right\| + \limsup_{n\to\infty}\beta_n''\left\|e_n - T_{\mu_n}u_n\right\| \\ &\le \limsup_{n\to\infty}\left\|u_n - w\right\| + \limsup_{n\to\infty}\beta_n''\left\|e_n - u_n\right\| \le d.\end{aligned}$$

It follows from Lemma 2.10 that

$$\lim_{n\to\infty}\left\|x_n - T_{\mu_n}u_n\right\| = 0. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}\left\|T_{\mu_n}x_n - x_n\right\| &\le \left\|T_{\mu_n}x_n - T_{\mu_n}u_n\right\| + \left\|T_{\mu_n}u_n - x_n\right\| \\ &\le \left\|x_n - u_n\right\| + \left\|T_{\mu_n}u_n - x_n\right\|.\end{aligned}$$

It follows from (3.17) and (3.22) that

$$\lim_{n\to\infty}\left\|x_n - T_{\mu_n}x_n\right\| = 0. \tag{3.23}$$

Therefore $x \in F(T)$. Let $\{x_{n_j}\}$ be an another subsequence of $\{x_n\}$ converging to x_0 with $x_0 \ne x$. Similarly, we can find $x_0 \in F(T)$. Hence we have

$$\begin{aligned}d &= \liminf_{i\to\infty}\left\|x_{n_i} - x\right\| < \liminf_{i\to\infty}\left\|x_{n_i} - x_0\right\| = \liminf_{j\to\infty}\left\|x_j - x_0\right\| \\ &< \liminf_{j\to\infty}\left\|x_j - x\right\| = d.\end{aligned}$$

This is a contradiction. Hence we have $x = x_0$.

Step 7: We finally show that $x_n \to p$ as $n \to \infty$.

Suppose that $\{x_n\}$ does not strongly converge to $p \in \mathcal{F}$. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\left\|x_{n_j} - p\right\| > \epsilon$ for all $j \in \{0,-1,\cdots\}$. By Proposition 2.7, for this ϵ there exists $r \in (0,1)$ such that

$$\left\|f\left(x_{n_j}\right) - f(p)\right\| \le r\left\|x_{n_j} - p\right\|.$$

So we have

$$\begin{aligned}&\left\|y_{n_j} - p\right\|^2 = \left\|\alpha_{n_j}\left(f\left(x_{n_j}\right) - p\right) + \left(1 - \alpha_{n_j}\right)\left(u_{n_j} - p\right)\right\|^2 \\ &\le \left(1 - \alpha_{n_j}\right)^2\left\|u_{n_j} - p\right\|^2 + 2\alpha_{n_j}\left\langle f\left(x_{n_j}\right) - p, y_{n_j} - p\right\rangle \\ &\le \left(1 - \alpha_{n_j}\right)^2\left\|x_{n_j} - p\right\|^2 + 2\alpha_{n_j}\left\langle f\left(x_{n_j}\right) - f(p), y_{n_j} - p\right\rangle + 2\alpha_{n_j}\left\langle f(p) - p, y_{n_j} - p\right\rangle \\ &\le \left(1 - \alpha_{n_j}\right)^2\left\|x_{n_j} - p\right\|^2 + 2\alpha_{n_j}r\left\|x_{n_j} - p\right\|\left\|y_{n_j} - p\right\| + 2\alpha_{n_j}\left\langle f(p) - p, y_{n_j} - p\right\rangle \\ &\le \left(1 - \alpha_{n_j}\right)^2\left\|x_{n_j} - p\right\|^2 + 2\alpha_{n_j}r\left(\left\|x_{n_j} - p\right\|^2 + \left\|y_{n_j} - p\right\|^2\right) + 2\alpha_{n_j}\left\langle f(p) - p, y_{n_j} - p\right\rangle.\end{aligned}$$

This implies that

$$\left\|y_{n_j}-p\right\|^2 \le \frac{\left(1-\alpha_{n_j}\right)^2+\alpha_{n_j} r}{1-\alpha_{n_j} r}\left\|x_{n_j}-p\right\|^2+\frac{2\alpha_{n_j}}{1-\alpha_{n_j} r}\left\langle f(p)-p, y_{n_j}-p\right\rangle.$$

Hence

$$\begin{aligned}
&\left\|x_{n_j+1}-p\right\|^2 \le \beta_{n_j}\left\|x_{n_j}-p\right\|^2+\beta'_{n_j}\left\|T_{\mu_{n_j}} y_{n_j}-p\right\|^2+\beta''_{n_j}\left\|e_{n_j}-p\right\|^2 \\
&\le \beta_{n_j}\left\|x_{n_j}-p\right\|^2+\beta'_{n_j}\left\|y_{n_j}-p\right\|^2+\beta''_{n_j}\left\|e_{n_j}-p\right\|^2 \\
&\le \beta_{n_j}\left\|x_{n_j}-p\right\|^2+\beta'_{n_j}\left\{\frac{\left(1-\alpha_{n_j}\right)^2+\alpha_{n_j} r}{1-\alpha_{n_j} r}\left\|x_{n_j}-p\right\|^2+\frac{2\alpha_{n_j}}{1-\alpha_{n_j} r}\left\langle f(p)-p, y_{n_j}-p\right\rangle\right\}+\beta''_{n_j}\left\|e_{n_j}-p\right\|^2 \\
&\le \beta_{n_j}\left\|x_{n_j}-p\right\|^2+\beta'_{n_j}\left\{\frac{1-\alpha_{n_j} r-2(1-r)\alpha_{n_j}+\alpha_{n_j}^2}{1-\alpha_{n_j} r}\right\}\left\|x_{n_j}-p\right\|^2+\frac{2\alpha_{n_j}\beta'_{n_j}}{1-\alpha_{n_j} r}\left\langle f(p)-p, y_{n_j}-p\right\rangle+\beta''_{n_j}\left\|e_{n_j}-p\right\|^2 \\
&= \beta_{n_j}\left\|x_{n_j}-p\right\|^2+\beta'_{n_j}\left\{\left(1-\frac{2(1-r)\alpha_{n_j}}{1-\alpha_{n_j} r}\right)+\frac{\alpha_{n_j}^2}{1-\alpha_{n_j} r}\right\}\left\|x_{n_j}-p\right\|^2+\frac{2\alpha_{n_j}\beta'_{n_j}}{1-\alpha_{n_j} r}\left\langle f(p)-p, y_{n_j}-p\right\rangle+\beta''_{n_j}\left\|e_{n_j}-p\right\|^2 \\
&\le \left(1-\frac{2(1-r)\alpha_{n_j}\beta'_{n_j}}{1-\alpha_{n_j r}}\right)\left\|x_{n_j}-p\right\|^2+\frac{2(1-r)\alpha_{n_j}\beta'_{n_j}}{1-\alpha_{n_j} r}\left\{\frac{\alpha_{n_j}}{2(1-r)}\left\|x_{n_j}-p\right\|^2+\frac{1}{1-r}\left\langle f(p)-p, y_{n_j}-p\right\rangle\right\} \\
&\quad +\beta''_{n_j}\left(\left\|e_{n_j}-p\right\|^2-\left\|x_{n_j}-p\right\|^2\right) \\
&= \left(1-\frac{2(1-r)\alpha_{n_j}\beta'_{n_j}}{1-\alpha_{n_j} r}\right)\left\|x_{n_j}-p\right\|^2+\frac{2(1-r)\alpha_{n_j}\beta'_{n_j}}{1-\alpha_{n_j} r}\left\{\left(\frac{\alpha_{n_j}}{2(1-r)}+\frac{\beta''_{n_j}\left(1-\alpha_{n_j} r\right)}{2(1-r)\alpha_{n_j}\beta'_{n_j}}\right)\left\|x_{n_j}-p\right\|^2\right. \\
&\quad \left.+\frac{1}{1-r}\left\langle f(p)-p, y_{n_j}-p\right\rangle-\frac{\left(1-\alpha_{n_j} r\right)\beta''_{n_j}}{2(1-r)\alpha_{n_j}\beta'_{n_j}}\left\|e_{n_j}-p\right\|^2\right\}.
\end{aligned}$$

Using (3.21) and (C2), we can conclude by Lemma 2.8 that $x_{n_j} \to p$ as $j \to \infty$. This is a contradiction and hence the sequence $\{x_n\}$ converges to $p \in \mathcal{F}$. Thus we completes the proof. ■

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