Retraction Notice

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History

Expression of Concern:

☐ yes, date: yyyy-mm-dd

X no

Correction:

☐ yes, date: yyyy-mm-dd

☐ no

Comment:

Free style text with summary of information from above and more details that can not be expressed by ticking boxes.

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Eigenvalues of the p-Laplacian and Evolution under the Ricci-Harmonic Map Flowc

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Abstract

Properties of eigenvalues of the *p*-Laplacian operator on a finite dimensional compact Riemannian manifold are studied for the case in which the metric of the manifold evolves under the Ricci-harmonic map flow. It will be shown that the first nonzero eigenvalue is monotonically nondecreasing along the flow and differentiable almost everywhere.

Keywords

Manifold, Operators, *p*-Laplacian, Eigenvalues

1. Introduction

Let (M,g) and (N,h) be two compact Riemannian manifolds without boun- dary with dimensions m and n, respectively. Let $\varphi: M \to N$ be a smooth map that is a critical point of the Dirichlet energy integral

$$E(\varphi) = \int_M |\nabla \varphi|^2 \, \mathrm{d}\mu_g,$$

where $\mathrm{d}\mu_g$ is the integration measure on the manifold. Nash's embedding theorem implies N is isometrically embedded in R^d for $d \geq n$. The configuration $(g(x,t), \varphi(x,t))$ for $t \in [0,T]$ of a one-parameter family of Riemannian metrics $g(x,t)$ and a family of smooth maps $\varphi(x,t)$ is defined to be a Ricci-harmonic map flow if it satisfies the coupled system of nonlinear parabolic equations

$$\begin{aligned} &\frac{\partial}{\partial t} g(x,t) = -2\mathrm{Ric}(x,t) + 2\alpha \nabla \varphi(x,t) \otimes \nabla \varphi(x,t), \\ &\frac{\partial}{\partial t} \varphi(x,t) = \tau_g \varphi(x,t), \end{aligned} \tag{1.1}$$

where $(x,t) \in M \times [0,T)$, $\otimes$ denotes tensor product, Ric is the Ricci curvature tensor corresponding to g and $\alpha(t) > 0$ is a parameter-dependent coupling constant such that $\tau_g \varphi$ is the intrinsic Laplacian of φ [1] [2] [3].

The problem to be investigated here is the p-eigenvalue problem where $p \in [2, \infty)$ defined by the following nonlinear equation which is constructed from the p-Laplacian

$$\Delta_{p,g} w(x) = -\lambda_p |w(x)|^{p-2} w(x), \tag{1.2}$$

with $w(x) \neq 0$ for $x \in M$ and such that $w(x) = 0$ on ∂M. In local coordinates, the p-Laplacian is given by [1] [2]

$$\Delta_{p,g} w(x) = \frac{1}{\sqrt{|g(x)|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{|g(x)|} g^{ij}(x) |\nabla w(x)|^{p-2} \frac{\partial w}{\partial x^j} \right). \tag{1.3}$$

where $|g| = \det(g_{ij})$ and inverse metric g^{ij}. When $p = 2$, the operator $\Delta_{p,g}$ reduces to the usual Laplace-Beltrami operator $\Delta_{2,g} w = \operatorname{div} \operatorname{grad} w$ [4] [5]. It can be verified that the principal symbol of (1.2) is nonnegative everywhere and strictly positive on the neighborhood of a point at which $\nabla w \neq 0$. It is also known that (1.2) has weak solutions with only partial regularity: in general, they are of class $C^{1,\alpha}$ $(0 < \alpha < 1)$. Notice that the least eigenvalue of a compact manifold without boundary or with Dirichlet boundary condition is zero with corresponding eigenfunction a constant. It is known that the first eigenvalue of $\Delta_{p,g}$ is obtained by means of the formula [2]

$$\lambda_{p,1}(M) = \inf_{0 \neq w \in W_0^{1,p}(M)} \left\{ \frac{\int_M |\nabla w|_g^p \, \mathrm{d}\mu_g}{\int_M |w|_g^p \, \mathrm{d}\mu_g} \,|\, w \neq 0, w \in W_0^{1,p}(M) \right\}, \tag{1.4}$$

while satisfying the constraint $\int_M |w|_g^{p-2} w \mathrm{d}\mu_g = 0$. The infimum does not c h a n g e when $W_0^{1,p}(M)$ is replaced by $C_0^\infty(M)$. The corresponding eigenfunction w_1 is the energy minimizer of the p-Rayleigh quotient (1.4) such that the infimum runs over all $w \in W_0^{1,p}(M)$.

The objective is to present a new concise proof of the general evolution of the first eigenvalue as a function of t under the Ricci flow (1.1). The proof is based on the work of Cao [6] [7] and Abolarinwa [8]. A monotonicity formula without differentiability assumption on the eigenfunction can also be obtained. The differentiability of a p-eigenvalue is a consequence of the monotonicity for- mula.

For the most part, a local coordinate system $\{x^i\}$ on M is adopted. The Riemannian metric $g(x)$ at any point $x \in M$ is a bilinear symmetric positive definite matrix $g_{ij}(x)$ with inverse written $g^{ij}(x)$. This induces a norm, the metric norm

$$|\nabla w|_g^2 = g^{ij} \nabla_i w \nabla_j w = \nabla^i w \nabla_i w.$$

The Riemannian structure on the manifold M allows a Riemannian volume measure $\mathrm{d}\mu_g(t)$ to be defined on M by the expression

$$\mathrm{d}\mu_g(t) = \sqrt{|g|} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{1.5}$$

The fact that the Riemannian metric is parallel, $\nabla g = 0$, will be used frequently without further mention as well as integration by parts, which for example takes the form,

$$\int_M \langle -\operatorname{div} X, w \rangle_g \,\mathrm{d}\mu_g = \int_M \langle X, \nabla w \rangle_g \,\mathrm{d}\mu_g$$
$$= -\int_M \frac{1}{\sqrt{\det g}} w \partial_i \left(X^i \sqrt{|g|} \right) \sqrt{|g|} \,\mathrm{d}x^i,$$

and for functions $u, w \in C^2(M)$,

$$\int_M u \Delta w \mathrm{d}\mu_g = -\int_M \langle \nabla u, \nabla w \rangle_g \mathrm{d}\mu_g = \int_M \Delta_g u w \mathrm{d}\mu_g.$$

Also the following notations for the Ricci-harmonic map flow [1] will be used in the following form,

$$S = \mathrm{Ric}_g - \alpha \nabla \varphi \otimes \nabla \varphi, \; S_{ij} = R_{ij} - \alpha \nabla_i \varphi \nabla_j \varphi, \; S_g = g^{ij} S_{ij}. \tag{1.6}$$

2. The Ricci Flow

All the geometric quantities associated with the manifold M evolve as the Riemannian metric on M evolves along the Ricci-harmonic map flow.

Lemma 1. Let a one-parameter family of smooth metrics $g(t)$ solve the Ricci-harmonic map flow (1.1). Then the following evolutions hold:

$$\frac{\partial}{\partial t} g^{ij} = 2S^{ij}, \tag{2.1}$$

$$\frac{\partial}{\partial t} |\nabla w|^2 = 2S^{ij} \nabla_i w \nabla_j w + 2g^{ij} \nabla_i w \nabla_j w_t, \tag{2.2}$$

$$\frac{\partial}{\partial t} \mathrm{d}\mu_g = -S_g \mathrm{d}\mu_g. \tag{2.3}$$

Here w is a smooth function defined on M and S_g the metric trace of the symmetric 2-tensor S_{ij} as in (1.6).

Proof: To prove equation (2.1), recall the metric satisfies $g^{ij} g_{jl} = \delta^i_l$. Differentiating both sides of this with respect to t and using (1.1), we have

$$\left(\frac{\partial}{\partial t} g^{ij} \right)_{jl} = -g^{ij} \left(-2R_{jl} + 2\alpha \nabla_j \varphi \nabla_l \varphi \right).$$

To obtain the second result (2.2), differentiate $|\nabla w|^2$ with respect to t and substitute the first result,

$$\frac{\partial}{\partial t} |\nabla w|^2 = \frac{\partial}{\partial t} \left(g^{ij} \partial_i w \partial_j w \right)$$
$$= \left(\frac{\partial}{\partial t} \right) \partial_i w \partial_j w + 2g^{ij} \partial_i w \partial_j w_t$$
$$= 2S^{ij} \partial_i w \partial_j w + 2g^{ij} \partial_i w \partial_j w_t.$$

To obtain (2.3), differentiate both sides of the volume form on M with respect to t to obtain,

$$\frac{\partial}{\partial t} \mathrm{d}\mu_g(t) = \frac{\partial}{\partial t} \left(\sqrt{g} \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \right).$$

By the chain rule, we get,

$$\frac{\partial}{\partial t}\sqrt{g} = \frac{1}{2\sqrt{g}}\frac{\partial|g|}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial t} = \frac{1}{2\sqrt{g}}\left(-2S_{ij}\right)\frac{\partial|g|}{\partial g_{ij}} = -\sqrt{g}\,g^{ij}S_{ij} = -\sqrt{g}\,S_g.$$

Therefore, it follows that

$$\frac{\partial}{\partial t}\mathrm{d}\mu_g(t) = -S_g\,\mathrm{d}\mu_g(t).$$

□

To obtain the results for the p-Laplacian, the following Lemma will be very important.

Lemma 2. Suppose a one-parameter family of smooth metrics $g(t)$ solves Ricci-harmonic map flow (1.1). Then there are the following evolutions

(a) $$\frac{\partial}{\partial t}|\nabla w|^p = p|\nabla w|^{p-2}\left[S^{ij}\nabla_i w\nabla_j w + g^{ij}\nabla_i w\nabla_j w_t\right], \tag{2.4}$$

(b) $$\frac{\partial}{\partial t}|\nabla w|^{p-2} = (p-2)|\nabla w|^{p-4}\left[S^{ij}\nabla_i w\nabla_j w + g^{ij}\nabla_i w\nabla_j w_t\right], \tag{2.5}$$

(c) $$\begin{aligned}\frac{\partial}{\partial t}\left(\Delta_{p,g}w\right) &= 2S^{ij}\nabla_i\left(\eta\nabla_j w\right) + g^{ij}\nabla_i\left(\eta_t\nabla_j w\right) + g^{ij}\nabla_i\left(\eta\nabla_j w_t\right)\\ &\quad +2\eta\left[g^{kl}g^{ij}\nabla_i S_{jl} - \frac{1}{2}gkl\nabla_l S_g\right]\nabla_k w,\end{aligned} \tag{2.6}$$

where $\eta = |\nabla w|^{p-2}$ and w is a smooth function on M.

Proof: (a) Using (2.1) from Lemma 1,

$$\begin{aligned}\frac{\partial}{\partial t}\left(|\nabla w|^p\right) &= \frac{\partial}{\partial t}\left(|\nabla w|^2\right)^{p/2} = \frac{p}{2}\left(|\nabla w|^2\right)^{p/2-1}\frac{\partial}{\partial t}|\nabla w|^2\\ &= \frac{p}{2}|\nabla w|^{p-2}\left[2S^{ij}\nabla_i w\nabla_j w + 2g^{ij}\nabla_i w\nabla_j w_t\right].\end{aligned}$$

Replace p by $p-2$ in (a) and the result in (b) follows immediately. (c)

$$\begin{aligned}\frac{\partial}{\partial t}\Delta_{p,g}w &= \frac{\partial}{\partial t}\left(g^{ij}\nabla_i\left(\eta\nabla_j w\right)\right) = \frac{\partial}{\partial t}\left(g^{ij}\nabla_i\eta\nabla_j w + \eta g^{ij}\nabla_i\nabla_j w\right)\\ &= \left(\frac{\partial g^{ij}}{\partial t}\right)\nabla_i\eta\nabla_j w + g^{ij}\nabla_i\eta_t\nabla_j w + g^{ij}\nabla_i\eta\nabla_j w_t + \eta_t\Delta w + \eta\left(\Delta w\right)_t\\ &= 2S^{ij}\nabla_i\eta\nabla_j w + g^{ij}\nabla_i\eta_t\nabla_j w + g^{ij}\nabla_i\eta\nabla_j w_t + \eta_t\Delta w\\ &\quad +\eta\left[\Delta w_t + 2S^{ij}\nabla_i\nabla_j w + 2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k w - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k w\right]\\ &= 2S^{ij}\nabla_i\eta\nabla_j w + 2S^{ij}\eta\nabla_i\nabla_j w + g^{ij}\nabla_l w_t\nabla_j w + w_t\Delta w_t + g^{ij}\nabla_i\eta\nabla_j w_t\\ &\quad +\eta\Delta w_t + w\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k w - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k w\right]\\ &= 2S^{ij}\nabla_i\left(\eta\nabla_j w\right) + g^{ij}\nabla_i\left(\eta_t\nabla_j w\right) + g^{ij}\left(\nabla_i\left(\eta\nabla_j w_t\right)\right)\\ &\quad +2\eta\left[g^{kl}g^{ij}\nabla_i S_{jl} - \frac{1}{2}g^{kl}\nabla_l S_g\right]\nabla_k w.\end{aligned}$$

3. Study of the Eigenvalue Problem

A nonlinear eigenvalue problem is introduced which involves the p-Laplacian (1.3) and is defined as

$$\Delta_{p,g} u = -\lambda_p \, | u |^{p-2} \, u, \tag{3.1}$$

with $u \neq 0$ and subject to the normalization condition

$$\int_M |u|^p \, \mathrm{d}\mu_g = 1. \tag{3.2}$$

One of the main objectives is to derive a general evolution equation for the p- eigenvalues of the p-Laplacian. Out of this, it can be shown that $\lambda_{p,1}$ is monotone on those metrics which evolve under the Ricci-harmonic map flow. The continuity and differentiability of $\lambda_{p,1}$ can be derived from its evolution by using Cao's approach. To study this, begin by multiplying (3.1) by the function u on both sides and then integrating over M using (3.2) to obtain

$$\lambda_p(t) = -\int_M u(x,t) \Delta_p u(x,t) \mathrm{d}\mu_g. \tag{3.3}$$

Integrating this by parts once, it follows that

$$\lambda_p(t) = \int_M |\nabla u|^2 \, \mathrm{d}\mu_g. \tag{3.4}$$

Equation (3.4) implies that the eigenvalues from (3.1) are all positive. Suppose now that $u(x,t)$ is the eigenfunction that corresponds to the first p- eigenvalue $\lambda_{p,1}(t)$ from (3.1). An equation which specifies the evolution of $\lambda_{p,1}(t)$ can be obtained by differentiating (3.3),

$$\frac{\partial \lambda_{p,1}}{\partial t} = -\frac{\partial}{\partial t} \int_M u(x,t) \Delta_{p,g} u(x,t) \mathrm{d}\mu_g. \tag{3.5}$$

The function u will satisfy the following integrability condition

$$\frac{\partial}{\partial t} \int_M |u|^p \, \mathrm{d}\mu_g = 0.$$

This can be developed by direct computation,

$$\frac{\partial}{\partial t}\left(\int_M |u|^{p-2} u^2 \, \mathrm{d}\mu_g \right) = (p-1) \int_M |u|^{p-2} \frac{\partial u}{\partial t} u^2 \, \mathrm{d}\mu_g + \int_M |u|^{p-1} \frac{\partial}{\partial t}\left(u \, \mathrm{d}\mu_g \right)$$

so

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |u|^p \, \mathrm{d}\mu_g &= \int_M p |u|^{p-1} \frac{\partial u}{\partial t} \mathrm{d}\mu_g + \int_M |u|^{p-1} u \frac{\partial}{\partial t}\left(\mathrm{d}\mu_g \right) \\ &= \int_M |u|^{p-1} \left(p \frac{\partial u}{\partial t} \mathrm{d}\mu_g + \mu \frac{\partial}{\partial t}\left(\mathrm{d}\mu_g \right) \right) \\ &= 0. \end{aligned}$$

This implies the following constraint holds for $u \neq 0$,

$$p \frac{\partial u}{\partial t} \mathrm{d}\mu_g + u \frac{\partial}{\partial t}\left(\mathrm{d}\mu_g \right) = 0. \tag{3.6}$$

At this point, it is possible to prove a theorem with regard to the evolution, monotonicity and differentiability of the first eigenvalue of the p-Laplacian under the Ricci-harmonic map flow.

Theorem 1. Let (M,g) be an m-dimensional, closed Riemannian manifold evolving by the Ricci-harmonic map flow. Let $\lambda_{p,1}(t)$ be the first eigenvalue of the p-Laplacian on M corresponding to the eigenfunction u at time $t \in [0,T]$. Then the evolution of $\lambda_{p,1}(t)$ is governed by the expression

$$\frac{\partial}{\partial t}\lambda_{p,1}(t) = \lambda_{p,1}(t)\int_M S_g |u|^p \mathrm{d}\mu_g - \int_M S_g |\nabla u|^p \mathrm{d}\mu_g + p\int_M |\nabla u|^{p-2} S^{ij}\nabla_i u \nabla_j u \mathrm{d}\mu_g. \quad (3.7)$$

Moreover, if it is the case that

$$S_{ij} - \beta S_g g_{ij} \geq 0, \quad \beta \in \left[\frac{1}{p}, \frac{1}{m}\right),$$

then $\lambda_{p,1}(t)$ is monotonically nondecreasing along the flow it is differentiable almost everywhere and

$$\frac{\partial}{\partial t}\lambda_{p,1}(t) \geq \lambda_{p,1}(t)\int_M S_g |u|^p \mathrm{d}\mu_g + (\beta p - 1)\int_M S_g |\nabla u|^p \mathrm{d}\mu_g \geq 0, \quad (3.8)$$

provided that S_g is nonnegative, that is, when $R_g \geq \alpha |\nabla\varphi|^2$.

Proof: Working in local coordinates and denoting $\eta = |\nabla u|^{p-2}$, it is the case that

$$\begin{aligned}\frac{\partial}{\partial t}\int_M u\Delta_p u \mathrm{d}\mu_g &= \frac{\partial}{\partial t}\int_M g^{ij}\nabla_i\left[\eta\nabla_j u\right] u \mathrm{d}\mu_g \\ &= \frac{\partial}{\partial t}\int_M \left(g^{ij}\nabla_i\eta\nabla_j u + \eta\Delta_p u\right) u \mathrm{d}\mu_g \\ &= \int_M \frac{\partial}{\partial t}\left(g^{ij}\nabla_i\eta\nabla_j u + \eta\Delta_p u\right) u \mathrm{d}\mu_g + \int_M \Delta_p u \frac{\partial}{\partial t}\left(u \mathrm{d}\mu_g\right).\end{aligned} \quad (3.9)$$

By the third part of Lemma 2, by the evolution of $\Delta_{p,g}$, the first part of this takes the form,

$$\begin{aligned}I = \int_M \Big\{&2S^{ij}\nabla_i\left(\eta\nabla_j u\right) + g^{ij}\nabla_i\left(\eta_t\nabla_j u\right) + g^{ij}\nabla_i\left(\eta\nabla_j u_t\right) \\ &+\eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right]\Big\} u \mathrm{d}\mu_g.\end{aligned}$$

Integrating the second and third terms in I by parts gives

$$\begin{aligned}I = \int_M \Big\{&2S^{ij}\nabla_i\left(\eta\nabla_j u\right) u - g^{ij}\left(\eta_t\nabla_j u\right)\nabla_i u - g^{ij}\left(\eta\nabla_j u_t\right)\nabla_i u \\ &+\eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right] u\Big\} \mathrm{d}\mu_g.\end{aligned}$$

Now, recall the fact that

$$\frac{\partial}{\partial t}\eta = (p-2)|\nabla u|^{p-4}\left\{S^{ij}\nabla_i u\nabla_j u + g^{ij}\nabla_i u\nabla_j u\right\},$$

so the integral I takes the form,

$$\begin{aligned}I &= \int_M \Big\{2S^{ij}\nabla_i\left(\eta\nabla_j u\right)u - (p-2)\eta S^{ij}\nabla_i u\nabla_j u - (p-2)\eta g^{ij}\nabla_i u\nabla_j u_t - \eta g^{ij}\nabla_i u\nabla_j u_t \\ &\quad + \eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right]u\Big\}\mathrm{d}\mu_g \\ &= \int_M \Big\{2S^{ij}\nabla_i\left(\eta\nabla_j u\right)u - (p-2)\eta S^{ij}\nabla_i u\nabla_j u - (p-1)\eta g^{ij}\nabla_i u\nabla_j u_t \\ &\quad + \eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right]u\Big\}\mathrm{d}\mu_g.\end{aligned}$$

Computing the first term in I, we get

$$\int_M 2S^{ij}\nabla_i\left(\eta\nabla_j u\right)u\,\mathrm{d}\mu_g = -2\int_M \nabla_i\left(S^{ij}u\right)\eta\nabla_j u\,\mathrm{d}\mu_g$$
$$= -2\int_M \eta\nabla_i S^{ij}\nabla_j u u d\mu_g - 2\int_M \eta S^{ij}\nabla_i u\nabla_j u\,\mathrm{d}\mu_g$$
$$= -2\int_M \eta\langle \mathrm{div}S,\nabla u\rangle u d\mu_g - 2\int_M \eta S^{ij}\nabla_i u\nabla_j u\,\mathrm{d}\mu_g.$$

Computing the third term in I,

$$-\left(p-1\right)\int_M g^{ij}\left|\nabla u\right|^{p-1}\nabla_i u\nabla_j u_t\,\mathrm{d}\mu_g = \left(p-1\right)\int_M g^{ij}\nabla_j\left(\eta\nabla_i u\right)u_t\,\mathrm{d}\mu_g$$
$$= \left(p-1\right)\int_M \Delta_p u\cdot u_t\,\mathrm{d}\mu_g.$$

Therefore, putting all of these into (3.9) for the time derivative, it has been found that

$$\frac{\partial}{\partial t}\int_M u\Delta_p u\,\mathrm{d}\mu_g = -2\int_M \eta\nabla_i S^{ij}\nabla_j u\cdot u\,\mathrm{d}\mu_g - 2\int_M \eta S^{ij}\nabla_i u\nabla_j u\,\mathrm{d}\mu_g$$
$$- p\int_M \eta S^{ij}\nabla_i u\nabla_j u d\mu_g + 2\int_M \eta S^{ij}\nabla_i u\nabla_j u d\mu_g + \left(p-1\right)\int_M \Delta_p u u_t\,\mathrm{d}\mu_g$$
$$+\int_M \eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right]u\,\mathrm{d}\mu_g + \int_M \Delta_p u\frac{\partial}{\partial t}\left(u\,\mathrm{d}\mu_g\right).$$

Using integrability condition (3.5),

$$\left(p-1\right)\int_M \Delta_p u\cdot u_t\,\mathrm{d}\mu_g + \int_M \Delta_p u\cdot u_t\,\mathrm{d}\mu_g + \int_M \Delta_p u\cdot u\frac{\partial}{\partial t}\mathrm{d}\mu_g$$
$$= \int_M\left\{p\Delta_p u u_t\,\mathrm{d}\mu_g + \Delta_p u u\frac{\partial}{\partial t}\mathrm{d}\mu_g\right\}$$
$$= \int_M \Delta_p u\left\{p\frac{\partial u}{\partial t}\mathrm{d}\mu_g + u\frac{\partial}{\partial t}\mathrm{d}\mu_g\right\}$$
$$= 0.$$

Therefore, the result simplifies considerably to the form,

$$\frac{\partial}{\partial t}\int_M u\Delta_p u\,\mathrm{d}\mu_g = -p\int_M \eta S^{ij}\nabla_i u\nabla_j u\,\mathrm{d}\mu_g - 2\int_M \eta\nabla_i S^{ij}\nabla_j u\cdot u\,\mathrm{d}\mu_g$$
$$+\int_M \eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right]u\,\mathrm{d}\mu_g.$$

The last pair of integrals can be simplified in the following way,

$$-2\int_M \eta\nabla_i S^{ij}\nabla_j u u\,\mathrm{d}\mu_g + \int_M \eta\left[2g^{kl}g^{ij}\nabla_i S_{jl}\nabla_k u - g^{kl}g^{ij}\nabla_l S_{ij}\nabla_k u\right]u\,\mathrm{d}\mu_g$$
$$= -2\int_M \nabla_i S^{ij}\nabla_j u\cdot u\,\mathrm{d}\mu_g + 2\int_M \eta\nabla_i S^{ij}\nabla_j u\cdot u\,\mathrm{d}\mu_g - \int_M g^{kl}\nabla_l S_g\eta\nabla_k u\cdot u\,\mathrm{d}\mu_g$$

The first two terms cancel out and so integrating the last term by parts using the definition $\Delta_p u = g^{kl}\nabla_l\left[\eta\nabla_k u\right]$,

$$\int_M g^{kl}S_g\nabla_l\left[\eta\nabla_k u\cdot u\right]\mathrm{d}\mu_g = \int_M g^{kl}S_g\nabla_l\left[\eta\nabla_k u\right]u\,\mathrm{d}\mu_g + \int_M g^{kl}S_g\eta\nabla_k u\nabla_l u\,\mathrm{d}\mu_g$$
$$= \int_M S_g\Delta_p u\cdot u\,\mathrm{d}\mu_g + \int_M S_g\left|\nabla u\right|^{p-2}g^{kl}\nabla_k u\nabla_l u\,\mathrm{d}\mu_g.$$

Making use of eigenvalue Equation (3.1), this integral simplifies to the form,

$$-\lambda_p\int_M S_g\left|u\right|^p d\mu_g + \int_M S_g\left|\nabla u\right|^p\mathrm{d}\mu_g.$$

Substituting this result into Equation (3.9) for the derivative, the final result becomes

$$\frac{\partial}{\partial t}\int_M u\Delta_p u \mathrm{d}\mu_g = -p\int_M |\nabla u|^{p-2} S^{ij}\nabla_i u\nabla_j u \mathrm{d}\mu_g - \lambda_p \int_M S_g |u|^p \mathrm{d}\mu_g + \int_M S_g |\nabla u|^p \mathrm{d}\mu_g .$$

However, the eigenvalue Equation (3.1) using (3.2) implies that

$$\frac{\partial}{\partial t}\lambda_p(t) = p\int_M |\nabla u|^{p-2} S^{ij}\nabla_i u\nabla_j u \mathrm{d}\mu_g + \lambda_p(t)\int_M S_g |u|^p \mathrm{d}\mu_g - \int_M S_g |\nabla u|^p \mathrm{d}\mu_g ,$$

for all $t \in [0,T]$.

Suppose the constraint with regard to S_{ij} is satisfied, then from the first and third terms,

$$\begin{aligned}
&\int_M p|\nabla u|^{p-2} S^{ij}\nabla_i u\nabla_j u \mathrm{d}\mu_g - \int_M S_g |\nabla u|^{p-2} g^{ij}\nabla_i u\nabla_j u \mathrm{d}\mu_g \\
&= \int_M |\nabla u|^{p-2} \nabla_i u\nabla_j u \left(pS^{ij} - S_g g^{ij}\right)\mathrm{d}\mu_g \\
&\geq \int_M |\nabla u|^{p-2} \nabla_i u\nabla_j u \left(pS_g g^{ij} - S_g g^{ij}\right)\mathrm{d}\mu_g \\
&= (p\beta - 1)\int_M |\nabla u|^p S_g \mathrm{d}\mu_g .
\end{aligned}$$

Hence, $\lambda_{p,1}(t)$ is monotonically nondecreasing along the flow and differentiable almost everywhere, thus

$$\frac{\partial}{\partial t}\lambda_{p,1}(t) \geq \lambda_{p,1}(t)\int_M S_g |u|^p \mathrm{d}\mu_g + (p\beta - 1)\int_M S_g |\nabla u|^p \mathrm{d}\mu_g \geq 0,$$

provided that S_g is nonnegative, or when $R_g \geq \alpha|\nabla\varphi|^2$. □

In the case where $p = 2$, $\Delta_{2,g} \equiv \Delta$, which is the usual Laplace-Beltrami operator. Thus this theorem implies that the first eigenvalue of Δ and the corresponding eigenfunction are smoothly differentiable for this operator as well.

4. Evolution of the First Eigenvalue

There are some important consequences of Theorem 1 with regard to the evolution of the first eigenvalue that will be discussed now.

Corollary 1. Under the conditions of Theorem 1, it is the case that

$$\lambda_{p,1}(t_2) \geq \lambda_{p,1}(t_1) + \int_{t_1}^{t_2} \Psi\left(g(t), u(x,t)\right)\mathrm{d}t, \tag{4.1}$$

where Ψ is defined to as

$$\Psi\left(g(t), u(x,t)\right) = \lambda_{p,1}(t)\int_M S_g |u|^p \mathrm{d}\mu_g + (p\beta - 1)\int_M S_g |\nabla u|^p \mathrm{d}\mu_g . \tag{4.2}$$

Furthermore, if $S_g \geq S_{\min} > 0$ and satisfies the governing inequality

$$S_{g(t)} \geq \theta(t) = \frac{S_{\min}(0)}{1 - \frac{2}{m}S_{\min}(0)t}, \tag{4.3}$$

then it holds for all $t_1 < t_2$ that

$$\lambda_{p,1}(t_2) \geq \lambda_{p,1}(t_1)\exp\left(p\beta\int_{t_1}^{t_2}\theta(s)\mathrm{d}s\right). \tag{4.4}$$

Proof: Integrating both sides of the inequality

$$\frac{\partial}{\partial t}\lambda_{p,1}(t) \geq \Psi\left(g(t), u(x,t)\right)$$

from t_1 to t_2 on a sufficiently small time interval, it follows that

$$\lambda_{p,1}(t_2)-\lambda_{p,1}(t_1)\geq\int_{t_1}^{t_2}\Psi(g(t),u)\mathrm{d}t,$$

as required. Now suppose that $S_{g(t)}\geq\theta(t)$ where $\theta(t)$ is independent of the manifold coordinates

$$\begin{aligned}\frac{\partial}{\partial t}\lambda_{p,1}(t)&\geq\lambda_{p,1}(t)\theta(t)\int_M|u|^p\,\mathrm{d}\mu_g+(p\beta-1)\theta(t)\int_M|\nabla u|^p\,\mathrm{d}\mu_g\\&=\lambda_{p,1}(t)\theta(t)\int_M|u|^p\,d\mu_g+(p\beta-1)\theta(t)\int_M|\nabla u|^{p-2}g^{ij}\nabla_i u\nabla_j u\mathrm{d}\mu_g\\&=\lambda_{p,1}(t)\theta(t)\int_M|u|^p\,\mathrm{d}\mu_g-(p\beta-1)\theta(t)\int_M\Delta_p u\cdot u\mathrm{d}\mu_g\end{aligned}$$

Therefore, it follows that

$$\frac{\partial}{\partial t}\lambda_{p,1}(t)\geq\beta p\theta(t)\lambda_{p,1}(t)$$

which is equivalent to

$$\int_{t_1}^{t_2}\mathrm{d}\log(\lambda_{p,1}(t))\geq\beta p\int_{t_1}^{t_2}\theta(s)\mathrm{d}s$$

Completing the integral on the left, this immediately gives (4.3). □

Note that both $\lambda_{p,1}(t)$ and $\theta(t)$ depend only on the parameter t, therefore, denoting

$$\theta(0)=S_{\min}(0)=\theta_0,$$

the following integral can be evaluated

$$\int_{t_1}^{t_2}\theta(t)\mathrm{d}t=\int_{t_1}^{t_2}\left(\frac{\theta_0}{1-\frac{2}{m}\theta_0 t}\right)\mathrm{d}t=\int_{t_1}^{t_2}\left(\frac{\mathrm{d}t}{\theta_0^{-1}-\frac{2}{m}t}\right)=\log\left(\frac{\theta_0^{-1}-\frac{2}{m}t_1}{\theta_0^{-1}-\frac{2}{m}t_2}\right).$$

Substituting this result into (4.3), it is found that

$$\log\left(\frac{\lambda_{p,1}(t_2)}{\lambda_{p,1}(t_1)}\right)=\log\left(\frac{\theta_0^{-1}-\frac{2}{m}t_1}{\theta_0^{-1}-\frac{2}{m}t_2}\right)^{\frac{m}{2}p\beta},$$

for $t_1<t_2$ and t_1 sufficiently close to t_2. However, this implies that

$$\lambda_{p,1}(t_2)\left(\theta_0^{-1}-\frac{2}{m}t_2\right)^{\frac{m}{2}p\beta}=\lambda_{p,1}(t_1)\left(\theta_0^{-1}-\frac{2}{m}t_1\right)^{\frac{m}{2}p\beta}.$$

This has the implication that the function $\lambda_{p,1}(t)\left(\theta_0^{-1}-\frac{2}{m}t\right)^{\frac{m}{2}p\beta}$ is non-decreasing along the Ricci-harmonic map flow, and this is important enough to be summarized in the form of Theorem 2.

Theorem 2. Under the assumptions of Theorem 1, the function

$$\lambda_{p,1}(t)\cdot\left(\theta_0^{-1}-\frac{2}{m}t\right)^{\frac{m}{2}\beta p}\tag{4.5}$$

is nondecreasing and $\lambda_{p,1}(t)$ is differentiable almost everywhere along the flow. □

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