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The Solution of Yang-Mills Equations on the Surface

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Abstract

We show that Yang-Mills equation in 3 dimensions is local well-posedness in H^s if the norm is sufficiently. Here, we construct a solution on the quadric that is independent of the time. And we also construct a solution of the polynomial form. In the process of solving, the polynomial is used to solve the problem before solving.

Keywords

H^s-Space, Well-Posedness, Polynomial, Quadric

1. Introduction and Preliminaries

This paper is concerned with the solution of the Yang-Mills equation.

We shall denote g-valued tensors define on Minkowski space-time $A_\alpha : R^{3+1} \to g$ by bold character A_α, where α ranges over 0, 1, 2, 3. We use the usual summation conventions on α, and raise and lower indices with respect to the Minkowski metric $\eta^{\alpha\beta} := \text{diag}(-1,1,1,1)$; for more details, see [1] [2] [3]. Given an arbitrary g-valued tensor $F_{\alpha\beta} : R^{3+1} \to g$.

The curvature of a connection $F_{\alpha\beta}$ by

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha + \left[A_\alpha, A_\beta\right]$$

Here [,] denotes the Lie bracket of g. It appears in calculations whenever we commute covariant derivatives [4] [5], or more precisely that

$$\partial_\alpha F^{\alpha\beta} + \left[A_\alpha, F^{\alpha\beta}\right] = 0$$

We can expand this as

$$\square A^\beta - \partial^\beta\left(\partial_\alpha A^\beta\right) + \left[A_\alpha, \partial^\alpha A^\beta\right] - \left[A_\alpha, \partial^\beta A^\alpha\right] + \left[A_\alpha, \left[A^\alpha, A^\beta\right]\right] = 0$$

where $\square := -\partial_t^2 + \Delta, \alpha, \beta = 0,1,2,3.$

The Cauchy problem for Yang-mills equation is not well-posed because of gauge invariance (see [6] [7]). However, if one fixes the connection to lie in the temporal gauge $A_0 = 0$, the Yang-Mills equations become essentially hyperbolic [8] [9], and simplify to

$$\partial_t(\mathrm{div}A) + [A_i, \partial_t A_i] = 0 \tag{1}$$

and

$$\Box A_j - \partial_j(\mathrm{div}A) + +[A_i, \partial_i A_j] - [A_i, \partial_j A_i] + [A_i, [A_i, A_j]] = 0 \tag{2}$$

where $i, j = 1,2,3$.

The local well-posedness of the Equations (1) and (2) have already proved in [10]. Here in not described in detail. This paper will show that the solution of operator and polynomial type.

2. Exact Solution of Equation

Below we will construct the exact solution of the equation on the general quadric that denotes by

$$A_i = \partial_{x_i} + a_i \qquad i = 1,2,3. \tag{3}$$

where $a_i = a_i(x_1, x_2, x_3)$.

We bring (3) to Equation (2), because the equation is used in the two general surfaces, we define the general quadric by

$$f = \sum_{\alpha_1+\alpha_2+\alpha_3\le 2} c_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

$\alpha_1, \alpha_2, \alpha_3 = 0,1,2. c_{\alpha_1\alpha_2\alpha_3}$ as coefficient and $c_{\alpha_1\alpha_2\alpha_3} \in R$. So we calculate the equation. The first calculation can be

$$(\Box A_j) f = (\Delta - \partial_t^2)(\partial_{x_j} + a_j) f = \left[\Delta\partial_{x_j} + (\Delta a_j) + a_j\Delta - \partial_t^2\partial_{x_j} - \frac{\partial^2 a_j}{\partial t^2} - a_j\partial_t^2\right] f$$
$$= [(\Delta a_j) + a_j\Delta] f = (\Delta a_j) f + 2a_j(c_{200} + c_{020} + c_{002})$$

Divergence terms can be

$$[\partial_j(\mathrm{div}A)] f = \left[\partial_{x_j}\left(\partial_{x_1}\partial_{x_1} + a_1\partial_{x_1} + \frac{\partial a_1}{\partial_{x_1}} + \partial_{x_2}\partial_{x_2} + a_2\partial_{x_2} + \frac{\partial a_2}{\partial_{x_2}} + \partial_{x_3}\partial_{x_3} + a_3\partial_{x_3} + \frac{\partial a_3}{\partial_{x_3}}\right)\right] f$$
$$= \left[a_1\partial_{x_j}\partial_{x_1} + \frac{\partial a_1}{\partial_{x_j}}\partial_{x_1} + \frac{\partial^2 a_1}{\partial_{x_j}\partial_{x_1}} + \frac{\partial a_1}{\partial_{x_1}}\partial_{x_j} + \frac{\partial a_2}{\partial_{x_j}}\partial_{x_2} + a_2\partial_{x_j}\partial_{x_2} + \frac{\partial^2 a_2}{\partial_{x_j}\partial_{x_2}} + \frac{\partial a_2}{\partial_{x_2}}\partial_{x_j}\right.$$
$$\left. + \frac{\partial a_3}{\partial_{x_j}}\partial_{x_3} + a_3\partial_{x_j}\partial_{x_3} + \frac{\partial^2 a_3}{\partial_{x_j}\partial_{x_3}} + \frac{\partial a_3}{\partial_{x_3}}\partial_{x_j}\right] f$$
$$= \left(\frac{\partial a_1}{\partial_{x_j}} + a_1\partial_{x_j}\right)(c_{100} + c_{110}x_2 + c_{101}x_3 + 2c_{200}x_1) + \left(\frac{\partial a_2}{\partial_{x_j}} + a_2\partial_{x_j}\right)\cdot(c_{010} + c_{110}x_1$$
$$+ c_{011}x_3 + 2c_{020}x_2) + \left(\frac{\partial a_3}{\partial_{x_j}} + a_3\partial_{x_j}\right)(c_{001} + c_{101}x_1 + c_{011}x_2 + 2c_{002}x_3) + \left(\frac{\partial^2 a_1}{\partial_{x_j}\partial_{x_1}}\right.$$
$$\left. + \frac{\partial a_1}{\partial_{x_1}}\partial_{x_j} + \frac{\partial^2 a_2}{\partial_{x_j}\partial_{x_2}} + \frac{\partial a_2}{\partial_{x_2}}\partial_{x_j} + \frac{\partial^2 a_3}{\partial_{x_j}\partial_{x_3}} + \frac{\partial a_3}{\partial_{x_3}}\partial_{x_j}\right) f$$

Finally, the sections of Lie bracket can be

$$\begin{aligned}
\left[A_i,\partial_i A_j\right] f &= \left[\left(A_i\cdot\partial_i A_j-\partial_i A_j\cdot A_i\right)\right] f \\
&= \left[\left(\partial_{x_i}+a_i\right)\left(\partial_{x_j}\partial_{x_i}+\frac{\partial a_j}{\partial_{x_i}}+a_j\partial_{x_i}\right)-\left(\partial_{x_j}\partial_{x_i}+\frac{\partial a_j}{\partial_{x_i}}+a_j\partial_{x_i}\right)\left(\partial_{x_i}+a_i\right)\right] f \\
&= \left(\frac{\partial^2 a_j}{\partial_{x_i}\partial_{x_i}}+\frac{\partial a_j}{\partial_{x_i}}\partial_{x_i}-\frac{\partial^2 a_i}{\partial_{x_j}\partial_{x_i}}-\frac{\partial a_i}{\partial_{x_i}}\partial_{x_j}-\frac{\partial a_i}{\partial_{x_j}}\partial_{x_i}-a_j\frac{\partial a_i}{\partial_{x_i}}\right) f \\
&= \left(\frac{\partial^2 a_j}{\partial_{x_i}\partial_{x_i}}-a_j\frac{\partial a_i}{\partial_{x_i}}-\frac{\partial a_i}{\partial_{x_i}}\partial_{x_j}\right) f+\left(\frac{\partial a_j}{\partial_{x_1}}+\frac{\partial a_1}{\partial_{x_1}}\right)\left(c_{100}+c_{110}x_2+c_{101}x_3+2c_{200}x_1\right) \\
&= \left(\frac{\partial a_j}{\partial_{x_2}}+\frac{\partial a_2}{\partial_{x_2}}\right)\left(c_{010}+c_{110}x_1+c_{011}x_3+2c_{020}x_2\right)+\left(\frac{\partial a_j}{\partial_{x_3}}+\frac{\partial a_3}{\partial_{x_3}}\right)\left(c_{001}+c_{101}x_1\right. \\
&\quad \left.+c_{011}x_2+2c_{002}x_3\right)
\end{aligned}$$

$$\begin{aligned}
\left[A_i,\partial_j A_i\right] f &= \left[\left(A_i\cdot\partial_j A_i-\partial_j A_i\cdot A_i\right)\right] f \\
&= \left[\left(\partial_{x_i}+a_i\right)\left(\partial_{x_j}\partial_{x_i}+\frac{\partial a_i}{\partial_{x_j}}+a_i\partial_{x_j}\right)-\left(\partial_{x_j}\partial_{x_i}+\frac{\partial a_i}{\partial_{x_j}}+a_i\partial_{x_j}\right)\left(\partial_{x_i}+a_i\right)\right] f \\
&= -\left(\frac{\partial a_i}{\partial_{x_j}}\partial_{x_i}+a_i\frac{\partial a_i}{\partial_{x_j}}\right) f \\
&= -\left[\frac{\partial a_1}{\partial_{x_j}}\left(c_{100}+c_{110}x_2+c_{101}x_3+2c_{200}x_1\right)+\frac{\partial a_2}{\partial_{x_j}}\left(c_{010}+c_{110}x_1+c_{011}x_3+2c_{020}x_2\right)\right. \\
&\quad \left.+\frac{\partial a_3}{\partial_{x_j}}\left(c_{001}+c_{101}x_1+c_{011}x_2+2c_{002}x_3\right)+\left(a_1\frac{\partial a_1}{\partial_{x_j}}+a_2\frac{\partial a_2}{\partial_{x_j}}+a_3\frac{\partial a_3}{\partial_{x_j}}\right) f\right]
\end{aligned}$$

$$\begin{aligned}
\left[A_i,\left[A_i,A_j\right]\right] f &= \left[A_i,A_iA_j-A_jA_i\right] f \\
&= \left[A_i,\left(\partial_{x_i}+a_i\right)\left(\partial_{x_j}+a_j\right)-\left(\partial_{x_j}+a_j\right)\left(\partial_{x_i}+a_i\right)\right] f \\
&= \left[A_i,\frac{\partial a_j}{\partial_{x_i}}-\frac{\partial a_i}{\partial_{x_j}}\right] f \\
&= \left[\left(\partial_{x_i}+a_i\right)\left(\frac{\partial a_j}{\partial_{x_i}}-\frac{\partial a_i}{\partial_{x_j}}\right)-\left(\frac{\partial a_j}{\partial_{x_i}}-\frac{\partial a_i}{\partial_{x_j}}\right)\left(\partial_{x_i}+a_i\right)\right] f \\
&= \left(\frac{\partial^2 a_j}{\partial_{x_i}\partial_{x_i}}-\frac{\partial^2 a_i}{\partial_{x_j}\partial_{x_i}}\right) f
\end{aligned}$$

Combining the above calculations we have

$$\begin{aligned}
&2a_j\left(c_{200}+c_{020}+c_{002}\right)+\left[\Delta a_j-\frac{\partial^2 a_j}{\partial t^2}+2\frac{\partial^2 a_j}{\partial x_i^2}-2\left(\frac{\partial^2 a_1}{\partial x_j\partial x_1}+\frac{\partial^2 a_2}{\partial x_j\partial x_2}+\frac{\partial^2 a_3}{\partial x_j\partial x_3}\right)\right. \\
&\left.+a_j\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}\right)+a_1\frac{\partial a_1}{\partial x_j}+a_2\frac{\partial a_2}{\partial x_j}+a_3\frac{\partial a_3}{\partial x_j}\right] f+\frac{\partial a_j}{\partial x_1}\left(c_{100}+c_{110}x_2+c_{101}x_3\right. \\
&\left.+2c_{200}x_1\right)+\frac{\partial a_j}{\partial x_2}\left(c_{010}+c_{110}x_1+c_{011}x_3+2c_{020}x_2\right)+\frac{\partial a_j}{\partial x_3}\left(c_{001}+c_{101}x_1+c_{011}x_2\right. \\
&\left.+2c_{002}x_3\right)-\left(\frac{\partial a_1}{\partial x_1}\partial x_j+\frac{\partial a_2}{\partial x_2}\partial x_j+\frac{\partial a_3}{\partial x_3}\partial x_j\right) f=0
\end{aligned}$$

We will use the properties of polynomials to list the coefficient equations in order to solve the (3). For the cross terms and square terms coefficient, we have

$$\begin{aligned}&\Delta a_j-\frac{\partial^2 a_j}{\partial t^2}+2\frac{\partial^2 a_j}{\partial x_i^2}-2\left(\frac{\partial^2 a_1}{\partial x_j\partial x_1}+\frac{\partial^2 a_2}{\partial x_j\partial x_2}+\frac{\partial^2 a_3}{\partial x_j\partial x_3}\right)\\&+a_j\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}\right)+a_1\frac{\partial a_1}{\partial x_j}+a_2\frac{\partial a_2}{\partial x_j}+a_3\frac{\partial a_3}{\partial x_j}=0\end{aligned} \tag{4}$$

First, we consider $j=1$.

The constant coefficient equation is

$$\begin{aligned}&2a_1\left(c_{200}+c_{020}+c_{002}\right)+\Delta a_1-\frac{\partial^2 a_1}{\partial t^2}+2\frac{\partial^2 a_1}{\partial x_i^2}-2\left(\frac{\partial^2 a_1}{\partial x_1\partial x_1}+\frac{\partial^2 a_2}{\partial x_1\partial x_2}+\frac{\partial^2 a_3}{\partial x_1\partial x_3}\right)\\&+a_1\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}\right)+a_1\frac{\partial a_1}{\partial x_1}+a_2\frac{\partial a_2}{\partial x_1}+a_3\frac{\partial a_3}{\partial x_1}+\frac{\partial a_1}{\partial x_1}c_{100}\\&+\frac{\partial a_2}{\partial x_2}c_{010}+\frac{\partial a_3}{\partial x_3}c_{001}-\left(\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}\right)c_{100}=0\end{aligned}$$

The coefficient equation of x_1 is

$$\frac{\partial a_1}{\partial x_2}c_{110}+\frac{\partial a_1}{\partial x_3}c_{101}-2\frac{\partial a_2}{\partial x_2}c_{200}-2\frac{\partial a_3}{\partial x_3}c_{200}=0 \tag{5}$$

The coefficient equation of x_2 is

$$2\frac{\partial a_1}{\partial x_2}c_{020}+\frac{\partial a_1}{\partial x_3}c_{011}-\frac{\partial a_2}{\partial x_2}c_{110}-\frac{\partial a_3}{\partial x_3}c_{110}=0 \tag{6}$$

The coefficient equation of x_3 is

$$\frac{\partial a_1}{\partial x_2}c_{011}+2\frac{\partial a_1}{\partial x_3}c_{002}-\frac{\partial a_2}{\partial x_2}c_{101}-\frac{\partial a_3}{\partial x_3}c_{101}=0 \tag{7}$$

Because of the (4), the coefficient equation of constant can be

$$2a_1\left(c_{200}+c_{020}+c_{002}\right)+\frac{\partial a_1}{\partial x_2}c_{010}+\frac{\partial a_1}{\partial x_3}c_{001}-\frac{\partial a_2}{\partial x_2}c_{100}-\frac{\partial a_3}{\partial x_3}c_{100}=0 \tag{8}$$

$(6)\times c_{110}\times c_{101}-(7)\times c_{200}\times c_{101}-(8)\times c_{200}\times c_{110}$ we have

$$\begin{aligned}&\frac{\partial a_1}{\partial x_2}\left(c_{110}c_{110}c_{101}-2c_{020}c_{200}c_{101}-c_{011}c_{200}c_{110}\right)\\&+\frac{\partial a_1}{\partial x_3}\left(c_{101}c_{110}c_{101}-c_{011}c_{200}c_{101}-2c_{020}c_{011}c_{200}\right)=0\end{aligned} \tag{9}$$

Deformation by (6), we have

$$\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}=2\frac{\partial a_1}{\partial x_2}c_{020}+\frac{\partial a_1}{\partial x_3}c_{011}\Big/c_{110} \tag{10}$$

Simulaneous (8) and (10), we have

$$2a_1\left(c_{200}+c_{020}+c_{002}\right)+\frac{\partial a_1}{\partial x_2}\left(c_{010}-2\frac{c_{020}c_{100}}{c_{110}}\right)+\frac{\partial a_1}{\partial x_3}\left(c_{001}-\frac{c_{011}c_{100}}{c_{110}}\right)=0 \tag{11}$$

First, for (9) we can use mathematica to get

$$a_1 = C_1\left[x_1\right]\left[\frac{-\left(c_{101}c_{110}c_{101} - c_{011}c_{200}c_{101} - 2c_{020}c_{011}c_{200}\right)x_2 + \left(c_{110}c_{110}c_{101} - 2c_{020}c_{200}c_{101} - c_{011}c_{200}c_{110}\right)x_3}{c_{110}c_{110}c_{101} - 2c_{020}c_{200}c_{101} - c_{011}c_{200}c_{110}}\right]$$

where C_1 is a constant, $[\ ]$ denotes the arbitrary combination of functions represented as independent variables in square brackets. For example, $[x]$ is represented as $x\sin x$ or $e^x \ln x \cos x$ and so on.

Next, from (11) we can obtain

$$a_1 = C_2 e^{\frac{2(c_{200}+c_{020}+c_{002})x_2}{c_{010}-2c_{020}c_{100}/c_{110}}}\left[x_1\right]\left[\frac{-\left(c_{001} - c_{011}c_{100}/c_{110}\right)x_2 + \left(c_{010} - 2c_{020}c_{100}/c_{110}\right)x_3}{c_{010} - 2c_{020}c_{100}/c_{110}}\right]$$

where C_2 is a constant.

We can observe the above a_1 and the general properties of two surfaces, a_1 is irrelevant to the x_2 and x_3, so $a_1 = a_1(x_1)$.

Because of $a_1 = a_1(x_1)$, we take a_1 into the (11) can be obtain

$$2a_1\left(c_{200} + c_{020} + c_{002}\right) = 0$$

By two surfaces we can obtain

$$a_1 = 0$$

Similarly, we can prove that $j = 2,3$, we have

$$a_2, a_3 = 0$$

In summary, when the Equation (2) is acting on the quadric, we have

$$\begin{cases} A_1 = \partial_{x_1} \\ A_2 = \partial_{x_2} \\ A_3 = \partial_{x_3} \end{cases}$$

3. Polynomial Solutions

3.1. First Order Polynomial Solution

Below we construct a polynomial solution. First, the constant must satisfy the equation so that all constant are the solutions of the Equation (1) and (2). Then we define the solution of a polynomial form on a surface by

$$A_i = a_i x_1 + b_i x_2 + c_i x_3 + d_i$$

where $i = 1,2,3$, A_i is satisfied the (1) because of not contain time *t*. Then we just need to bring A_i into (2). We have

$$\Delta A_j - \partial_j\left(\mathrm{div}A\right) + \sum_{i=1}^{3}\left(\partial_j A_i A_i - A_j \partial_i A_i\right) = 0 \tag{12}$$

Equation (12) is composed of three equations. First we consider the case of $j = 1$. So the constant coefficient equation is

$$b_2 d_2 - b_2 d_1 + a_3 d_3 - c_3 d_1 = 0$$

The coefficient equation of x_1 is

$$a_2 b_2 - a_1 b_2 + a_3 a_3 - a_1 c_3 = 0$$

The coefficient equation of x_2 is

$$b_2b_2 - b_1b_2 + a_3b_3 - b_1c_3 = 0$$

The coefficient equation of x_3 is

$$b_2c_2 - b_2c_1 + a_3c_3 - c_1c_3 = 0$$

When $j = 2$, the relationship of the coefficients are

$$\begin{cases} b_1d_1 - a_1d_2 + c_3d_3 - c_3d_2 = 0 \\ a_1b_1 - a_1a_2 + a_3c_3 - a_2c_3 = 0 \\ b_1b_1 - a_1b_2 + b_3c_3 - b_2c_3 = 0 \\ b_1c_1 - a_1c_2 + c_3c_3 - c_2c_3 = 0 \end{cases}$$

When $j = 3$, the relationship of the coefficients are

$$\begin{cases} c_1d_1 - a_1d_3 + c_2d_2 - b_2d_3 = 0 \\ a_1c_1 - a_1a_3 + a_2c_2 - a_2b_3 = 0 \\ b_1c_1 - a_1b_3 + b_2c_2 - b_2b_3 = 0 \\ c_1c_1 - a_1c_3 + c_2c_2 - b_2c_3 = 0 \end{cases}$$

There exist 12 equations. By solving the above equations, we can obtain

$$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = c_1 = c_2 = c_3$$

$$d_1 = d_2 = d_3$$

Therefore

$$A_i = ax_1 + ax_2 + ax_3 + b \tag{13}$$

where $a, b \in R$ $\quad i = 1, 2, 3$.

In summary, the solution of the polynomial form of Yang-Mills equation is expressed in the form of (13).

3.2. The Quadratic Polynomial Solution

In this section, we mainly discuss the solution of the quadratic polynomial form of the Yang-Mills equation on the two surfaces. We define by

$$\begin{cases} A_1 = \sum\limits_{\alpha_1+\alpha_2+\alpha_3\le 2} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\ A_2 = \sum\limits_{\beta_1+\beta_2+\beta_3\le 2} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \\ A_3 = \sum\limits_{\gamma_1+\gamma_2+\gamma_3\le 2} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \end{cases}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}$ $\quad i = 1, 2, 3$, $a_{\alpha_1\alpha_2\alpha_3}, b_{\beta_1\beta_2\beta_3}, c_{\gamma_1\gamma_2\gamma_3} \in R$ are coefficients. So A_1, A_2, A_3 must satisfy the Equation (1), therefore, it just needs to take A_1, A_2, A_3 into (12), we have

$$\Delta A_j - \partial_{x_j x_1}\left(\sum_{\alpha_1+\alpha_2+\alpha_3\le 2} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right) - \partial_{x_j x_2}\left(\sum_{\beta_1+\beta_2+\beta_3\le 2} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)$$

$$-\partial_{x_j x_3}\left(\sum_{\gamma_1+\gamma_2+\gamma_3\le 2} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}\right) + \left[\partial_{x_j}\left(\sum_{\alpha_1+\alpha_2+\alpha_3\le 2} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right)\right] \cdot \sum_{\alpha_1+\alpha_2+\alpha_3\le 2} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

$$-A_j \cdot \left[\partial_{x_1}\left(\sum_{\alpha_1+\alpha_2+\alpha_3\le 2} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right)\right] + \left[\partial_{x_j}\left(\sum_{\beta_1+\beta_2+\beta_3\le 2} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)\right] \cdot \sum_{\beta_1+\beta_2+\beta_3\le 2} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$$

$$-A_j \cdot \left[\partial_{x_2} \left(\sum_{\beta_1+\beta_2+\beta_3 \le 2} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \right) \right] + \left[\partial_{x_j} \left(\sum_{\gamma_1+\gamma_2+\gamma_3 \le 2} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \right) \right] \cdot \sum_{\gamma_1+\gamma_2+\gamma_3 \le 2} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}$$
$$-A_j \cdot \left[\partial_{x_3} \left(\sum_{\gamma_1+\gamma_2+\gamma_3 \le 2} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \right) \right] = 0$$

There exist 30 equations and 30 unknowns. Solving the equations we can obtain the following results

$$a_{200} = a_{020} = a_{002} = a_{110} = a_{101} = a_{011} = 0$$
$$b_{200} = b_{020} = b_{002} = b_{110} = b_{101} = b_{011} = 0$$
$$c_{200} = c_{020} = c_{002} = c_{110} = c_{101} = c_{011} = 0$$
$$a_{100} = a_{010} = a_{011} = b_{100} = b_{010} = b_{001} = c_{100} = c_{010} = c_{001} \in \mathbb{R}$$
$$a_{000} = b_{000} = c_{000} \in \mathbb{R}$$

So the solution of the equation can be written

$$A_i = ax_1 + ax_2 + ax_3 + b \tag{14}$$

where $a, b \in \mathbb{R}$ $i = 1,2,3$.

In summary, the solution of the quadratic polynomial form of Yang-Mills equation is (14). It obvious that (13) is equal to (14). So we conjecture that the solution of n-degree polynomial on n-sub surface is also (14). In the next section, we will proof the hypothesis.

3.3. Solution of N-Degree Polynomial

In this section, we mainly use mathematical induction to prove the hypothesis. We define that by

$$\begin{cases} A_1 = \sum\limits_{\alpha_1+\alpha_2+\alpha_3 \le n} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\ A_2 = \sum\limits_{\beta_1+\beta_2+\beta_3 \le n} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \\ A_3 = \sum\limits_{\gamma_1+\gamma_2+\gamma_3 \le n} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \end{cases} \tag{15}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}$ $i = 1,2,3$, $a_\alpha, b_\beta, c_\gamma \in R$ are coefficients.

In the front two sections, it is easy for us to conclude that when $n = 1,2$ the solutions are the same. So we will use mathematical induction to prove that when $n \ge 2$ the solution is also (14).

First, we assume that when $n(n \ge 2)$ the solution of the equation is

$$A_i = ax_1 + ax_2 + ax_3 + b$$

where $a, b \in \mathbb{R}$ $i = 1,2,3$.

Now when $n+1$, we have

$$\Delta A_j - \partial_{x_j x_1} \left(\sum_{\alpha_1+\alpha_2+\alpha_3 \le n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \right) - \partial_{x_j x_2} \left(\sum_{\beta_1+\beta_2+\beta_3 \le n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \right)$$
$$-\partial_{x_j x_3} \left(\sum_{\gamma_1+\gamma_2+\gamma_3 \le n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \right) + \left[\partial_{x_j} \left(\sum_{\alpha_1+\alpha_2+\alpha_3 \le 2} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \right) \right] \cdot \sum_{\alpha_1+\alpha_2+\alpha_3 \le n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

$$-A_j \cdot \left[\partial_{x_1}\left(\sum_{\alpha_1+\alpha_2+\alpha_3\le n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right)\right] + \left[\partial_{x_j}\left(\sum_{\beta_1+\beta_2+\beta_3\le 2} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)\right] \cdot \sum_{\beta_1+\beta_2+\beta_3\le n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$$
$$-A_j \cdot \left[\partial_{x_2}\left(\sum_{\beta_1+\beta_2+\beta_3\le n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)\right] + \left[\partial_{x_j}\left(\sum_{\gamma_1+\gamma_2+\gamma_3\le 2} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}\right)\right] \cdot \sum_{\gamma_1+\gamma_2+\gamma_3\le n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}$$
$$-A_j \cdot \left[\partial_{x_3}\left(\sum_{\gamma_1+\gamma_2+\gamma_3\le n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}\right)\right] = 0$$

To further simplify (15), we have

$$\begin{cases} A_1 = \sum_{\alpha_1+\alpha_2+\alpha_3\le n} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} + \sum_{\alpha_1+\alpha_2+\alpha_3= n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\ A_2 = \sum_{\beta_1+\beta_2+\beta_3\le n} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} + \sum_{\beta_1+\beta_2+\beta_3= n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \\ A_3 = \sum_{\gamma_1+\gamma_2+\gamma_3\le n} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} + \sum_{\gamma_1+\gamma_2+\gamma_3= n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \end{cases}$$

To bring into the equation, we have

$$\Delta J_j - \partial_{x_j x_1}\left(\sum_{\alpha_1+\alpha_2+\alpha_3= n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right) - \partial_{x_j x_2}\left(\sum_{\beta_1+\beta_2+\beta_3= n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)$$
$$-\partial_{x_j x_3}\left(\sum_{\gamma_1+\gamma_2+\gamma_3= n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}\right)$$
$$+\left[\partial_{x_j}\left(\sum_{\alpha_1+\alpha_2+\alpha_3= n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right)\right] \cdot \sum_{\alpha_1+\alpha_2+\alpha_3= n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$
$$-A_j \cdot \left[\partial_{x_1}\left(\sum_{\alpha_1+\alpha_2+\alpha_3= n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right)\right]$$
$$+\left[\partial_{x_j}\left(\sum_{\beta_1+\beta_2+\beta_3= n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)\right] \cdot \sum_{\beta_1+\beta_2+\beta_3= n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$$
$$-A_j \cdot \left[\partial_{x_2}\left(\sum_{\beta_1+\beta_2+\beta_3= n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}\right)\right]$$
$$+\left[\partial_{x_j}\left(\sum_{\gamma_1+\gamma_2+\gamma_3= n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}\right)\right] \cdot \sum_{\gamma_1+\gamma_2+\gamma_3= n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}$$
$$-A_j \cdot \left[\partial_{x_3}\left(\sum_{\gamma_1+\gamma_2+\gamma_3= n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}\right)\right] = 0$$

where J_j is

$$\begin{cases} J_1 = \sum_{\alpha_1+\alpha_2+\alpha_3= n+1} a_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\ J_2 = \sum_{\beta_1+\beta_2+\beta_3= n+1} b_{\beta_1\beta_2\beta_3} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \\ J_3 = \sum_{\gamma_1+\gamma_2+\gamma_3= n+1} c_{\gamma_1\gamma_2\gamma_3} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \end{cases}$$

On the number of x in the above equation is either less than n, or more than $n+1$. When the number of x is less than n, the solution of the equation is

$$A_i = ax_1 + ax_2 + ax_3 + b$$

And the number of more than $n+1$ of the items in the n-sub surfaces is always equal to zero.

4. Conclusion

In summary, we can get the solution of the polynomial type of Yang-Mills equation by mathematical induction is

$$A_i = ax_1 + ax_2 + ax_3 + b$$

where $a, b \in \mathbb{R}$ $i = 1, 2, 3$.

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