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Excluded Minority of P_8 for $GF(4)$-Representability

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Abstract

One of the mainly interesting things of matroid theory is the representability of a matroid. Finding the set of all excluded minors for the representability is the solution of the representability. In 2000, Geelen, Gerards and Kapoor proved that $\left\{U_{2,6}, U_{4,6}, F_7^-, \left(F_7^-\right)^*, P_6, P_8\right\}$ is the complete set of $GF(4)$-representability. In this paper, we show that P_8 is an excluded minor for $GF(4)$-representability.

Keywords

Excluded Minor

1. Introduction

Matroid theory dates from the 1930's when Whitney first used the term matroid in his basic paper [1]. Matroid theory is a common generalization of linear independence of graphs and matrices. One of the mainly interesting things of matroid theory is the representability of a matroid. One problem of representability is to find the fields over which the given matroid is representable. The other problem is to find the excluded minors for which the matroid is representable over the given field. It was found the complete set of excluded minors for two or three element fields [2]-[5]. In 1984, Kahn and Seymour conjectured that $\left\{U_{2,6}, U_{4,6}, F_7^-, \left(F_7^-\right)^*, P_6\right\}$ is the complete set of excluded minors for $GF(4)$-representability. Oxley showed that the conjecture is wrong by showing P_8 is also excluded minor for $GF(4)$-representability in his brief note [6]. Geelen, Gerarads and Kapoor proved that it is enough to add P_8 to the list of Kahn and Seymour [7]. It is not an easy problem to find the excluded minors for $GF(q)$-representability when q is more than 4. Instead, we have the conjecture by Rota that the number of excluded minors are finite for any prime powers q [8]. In this paper, we study the properties of minor and show that P_8 is excluded minor for $GF(4)$-representability deliberately.

2. Preliminaries

2.1. Matroid

Matroid theory has exactly the same relationship to linear algebra as does point set topology to the theory of real numbers. That is, point set topology postulate the properties of the open sets of real line and matroid axiomatize the character of the independent set in vector space:

Definition 2.1. A matroid M is a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

(I_1) $\varnothing \in \mathcal{I}$.

(I_2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

(I_3) If X, Y are in $\mathcal{I}$ and $|X| < |Y|$, then there exists $x \in Y - X$ such that $X \cup x \in \mathcal{I}$.

By the definition, for a finite vector space V and for the collection of linearly independent subsets $\mathcal{I}$ of vectors of V, $(V, \mathcal{I})$ is a *matroid*. If $M = (E, \mathcal{I})$ is a matroid, M is called a matroid on E. Also E which is denoted by $E(M)$ is called *ground set* of M and $\mathcal{I} = \mathcal{I}(M)$ is called the set of *independent set* in M. A subset of E that is not in $\mathcal{I}$ is called *dependent set*. On the other point of view, matroid can be defined by abstraction of the properties of the cycles of a graph. Let G be a graph and let $E(G)$ be the set of all edges of G. Also let $\mathcal{C}$ be the set of all cycles in G. Then $\mathcal{C}$ has the following properties:

(C_1) $\varnothing \notin \mathcal{C}$.

(C_2) If C_1 and C_2 are in $\mathcal{C}$ and $C_1 \subset C_2$, then $C_1 = C_2$.

(C_3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Now, let $\mathcal{C}$ be a subset of the power set 2^E of a finite set E. If $\mathcal{C}$ satisfies the conditions (C_1), (C_2) and (C_3), then $\mathcal{C}$ is called the set of *circuits* of a matroid on E. Let $\mathcal{C}$ be the set of circuits of a matroid M. Then, the set $\mathcal{I}$ of all subsets of E which contain no member of $\mathcal{C}$ satisfies the independent conditions (I_1), (I_2) and (I_3). Also for a matroid $M = (E, \mathcal{I})$, the set $\mathcal{C}(M)$ of all minimal dependent set M satisfies the three circuits conditions. Thus, the matroid defined by circuits is the same as the one defined by independent sets. We need two other definitions of a matroid.

Definition 2.2. Let E be a finite set and r be a function from 2^E to the set of non-negative integers and satisfies the following conditions:

(R_1) If $X \subseteq E$, then $0 \le r(X) \le |X|$.

(R_2) $X \subseteq Y \subseteq E$, then $r(X) \le r(Y)$.

(R_3) If X and Y are subsets of E, then $r(X \cup Y) + r(X \cap Y) \le r(X) + r(Y)$. Then, r is called the *rank function* of a matroid M on E. Let M be the matroid $(E, \mathcal{I})$ and suppose that $X \subset E$. Let $\mathcal{I}|_X$ be $\{I \subset X \mid I \in \mathcal{I}\}$. Then, it is easy to see that the pair $(X, \mathcal{I}|_X)$ is a matroid. We call this matroid the *restriction* of M to X or the *deletion* of $E - X$ from M. It is denoted by $M|_X$ or $M \setminus E - X$. We define the rank $r(X)$ of X to be the size of a maximal independent set of $M|_X$. This function which is called the *rank function* of M satisfies the conditions (R_1), (R_2) and (R_3). This function is denoted by r_M and $r(E(M))$ will be denoted by $r(M)$. On the other hand, if r is a rank function of a matroid M, the set $\mathcal{I}=\{X\subset E \mid r(X)=|X|\}$ is the set of independent set of M. Thus, the definition of a matroid by a rank function is equivalent to the definition by independent sets. One of the definitions of matroid is the one by closure operator. Throughout this thesis, by E means a finite set.

Definition 2.3. Let cl be a function from 2^E to 2^E satisfying the following;

(CL_1) If $X\subset E$, then $X\subset cl(X)$.

(CL_2) If $X\subset Y\subset E$, then $cl(X)\subset cl(Y)$.

(CL_3) If $X\subset E$, then $cl(cl(X))=cl(X)$.

(CL_4) If $X\subset E$, $x\in E$, and $y\in cl(X\cup x)-cl(X)$, then $x\in cl(X\cup y)$.

Then cl is called the *closure operator* of a matroid M on E. Let M be a matroid on E with the rank function r. Define cl to be the function from 2^E to 2^E by $cl(X)=\{x\in E \mid r(X\cup x)=r(X)\}$ for all $X\subset E$. Then, we can see that cl satisfies (CL_1)-(CL_4). On the other hand, if cl is the closure operator of a matroid M on E, then $\mathcal{I}=\{X\subset E \mid x\notin cl(X-x) \text{ for all } x\in X\}$ satisfies the independent axioms and $M=(E,\mathcal{I})$ is the matroid having closure operator cl. Thus, the four definitions of matroid defined by the above are equivalent. We can see that there are a lot of equivalent definition of matroid. We have to introduce one more definition of matroid. For a matroid M, the maximal independent set in M is called a *basis* or *base* of M. It is easy to see that every basis element has the same cardinal number. Let $\mathcal{B}(M)$ be the set of all base element of M. Then, $\mathcal{B}(M)$ has the properties;

(B_1) $\mathcal{B}(M)$ is non-empty.

(B_2) If B_1 and B_2 are in $\mathcal{B}(M)$ and $x\in B_1-B_2$ then there is $y\in B_2-B_1$ such that $(B_1-x)\cup y\in\mathcal{B}(M)$.

Conversely, let $\mathcal{B}$ be a collection of subsets of E satisfying the axioms (B_1) and (B_2). And let $\mathcal{I}=\{I\subset E \mid I\subset B \text{ for some } B\in\mathcal{B}\}$. Then $(E,\mathcal{I})$ is a matroid having $\mathcal{B}$ as its collection of bases. If M is a matroid and $X\subset E(M)$, then we call $cl(X)$ the *closure* or *span* of X in M, and we write this as $cl_M(X)$. If $X=cl(X)$, then X is called a *flat* or a *closed set* of M. A *hyperplane* of M is a flat of rank $r(M)-1$. A subset X of $E(M)$ is a *spanning set* of M if $cl(X)=E(M)$. Let M be a matroid and $\mathcal{B}^*(M)$ be $\{E(M)-B \mid B\in\mathcal{B}(M)\}$. Then, $\mathcal{B}^*(M)$ satisfies the following axiom which is equivalent to (B_2):

(B_2)* If B_1 and B_2 are in $\mathcal{B}^*(M)$ and $x\in B_2-B_1$, then there is an element $y\in B_1-B_2$ such that $(B_1-y)\cup x\in\mathcal{B}^*(M)$. The matroid M^* having the set of all basis element $\mathcal{B}^*(M)$ is called the *dual* of M. Thus $\mathcal{B}(M^*)=\mathcal{B}^*(M)$ and $(M^*)^*=M$. Also $E(M^*)=E(M)$. The bases of M^* are called *cobases* of M. Similarly, the circuits, hyperplanes, independent sets and spanning sets of M^* are called *cocircuits*, *cohyperplanes*, *coindependent sets*, and *cospanning sets* of M. The next result gives some elementary relationships between these sets.

Proposition 2.4. *Let M be a matroid on a set E and suppose* $X\subset E$. *Then*

1) *X is independent if and only if* $E-X$ *is cospanning.*

2) *X is spanning if and only if* $E-X$ *is coindependent.*

3) *X is a hyperplane if and only if* $E-X$ *is cocircuit.*

4) *X is a circuit if and only if* $E-X$ *is a cohyperplane.*

Proof. 1) Let *X* be an independent set in *M*. Then, $X \subset B$ for some basis element *B* of *M*. Thus $E-B \subset E-X$ and $cl^*(E-B) = E \subset cl^*(E-X)$, where cl^* is the closure in M^*. If $E-X$ is cospanning, then $E-X \supset B^*$ for some basis element B^* of M^*. It means that $X \subset E-B^*$ and *X* is independent set. 2) is deduced by applying 1) to M^*. 3) is obtained by the following equivalent statements; a) *X* is a hyperplane of *M*. b) *X* is a non-spanning set of *M* but $X \cup y$ is spanning for all $y \notin X$. c) $E-X$ is dependent in M^* but $(E-X)-y$ is independent in M^* for all $y \in E-X$. d) $E-X$ is a cocircuit of *M*. 4) is the one obtained by applying 3) to M^*.

□

Let's remind of the finite fields. If *F* is a finite field, then *F* has exactly p^k-elements for some prime *p* and some positive integer *k*. Indeed, for all such *p* and *k*, there is a unique field $GF(p^k)$ having p^k-elements. This field is called the Galois field of order p^k. When $k=1$, $GF(p^k)$ coincides with $\mathbb{Z}_p$, the ring of integers modulo *p*. When $k>1$, $GF(p^k)$ can be constructed as follows. Let $h(\omega)$ be a polynomial of degree *k* with coefficients in $\mathbb{Z}_p$ and suppose that the polynomial is irreducible. Consider the set *S* of all polynomials in ω that have degree at most $k-1$ and have coefficients in $\mathbb{Z}_p$. There are exactly *p* choices for each of the *k*-coefficient of a member of *S*. Hence, $|S| = p^k$. If we take $p = 2$ and $k = 2$ we get the field *GF*(4). Moreover, under addition and multiplication, both of which are performed modulo $h(\omega)$, *S* forms a field namely $GF(p^k)$. In case of $GF(4)$, we can take the irreducible polynomal to
$h(\omega) = \omega^2 + \omega + 1$.

Let *A* be a matrix over a field *F*. Then, the collection of independent column vectors $\mathcal{I}$ of *A* satisfies the independent axioms of matroid. Thus, $(ColA, \mathcal{I})$ is a matroid and this matroid is denoted by $M[A]$, where $ColA$ is the set of all column vectors of *A*.

Now, let *G* be a graph. Then $M(G)$ is the matroid on the edge set $E(G)$ with the set of all cycles of *G* as circuit. This matroid is called the *cycle matroid* of *G*. Two matroids M_1 and M_2 are isomorphic, denoted by $M_1 \cong M_2$, if there is a bijection ψ from $E(M_1)$ to $E(M_2)$ such that, for all $X \subset E(M_1)$, $\psi(X)$ is independent in M_2 if and only if *X* is independent in M_1. A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. If *M* is isomorphic to $M[A]$ for a matrix *A* over *F*, then *M* is called *F-representable*. In the sequel, by *F* we mean a finite field.

We call an element *e* a *loop* of a matroid *M* if $\{e\}$ is a circuit of *M*. Moreover, if *f* and *g* are element of $E(M)$ such that $\{f, g\}$ is a circuit, then *f* and *g* are said to be *parallel* in *M*. A parallel class of *M* is a maximal subset *X* of $E(M)$ such that any two distinct members *X* are parallel and no member of *X* is a loop. A parallel class is trivial if it contains just one element. If *M* has no loops and no non-trivial parallel classes, it is called a *simple matroid* or a *combinatorial geometry*.

2.2. Uniform Matroid $U_{m,n}$

Let *m* and *n* be non-negative integers such that $m \le n$. Let *E* be a set of cardinality *n* and $\mathcal{I}$ be all subsets of E of cardinality less than or equal to m. This is a matroid on E, called the *uniform matroid* of rank m and denoted by $U_{m,n}$. By definition, the set of basis $\mathcal{B}(U_{m,n})$ of $U_{m,n}$ is $\{X \subset E \mid |X| = m\}$, and the set of circuits $\mathcal{C}(U_{m,n})$ is $\{X \subset E \mid |X| = m+1\}$.

2.3. Affine Matroid

Now, we are going to define affine matroid. A set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subset F^m$ is affinely dependent if $k \geq 1$ and there are elements $a_1, a_2, \cdots, a_k$ of F, not all zero, such that $\sum_{i=1}^{k} a_i \mathbf{v}_i = 0$ and $\sum_{i=1}^{k} a_i = 0$. It is easy to show that affine dependence of $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subset F^m$ is equivalent to each of the followings;

(Ad_1) $\{(1, \mathbf{v}_1), \cdots, (1, \mathbf{v}_k)\} \subset F^{m+1}$ is linearly dependent, where $(1, \mathbf{v}_i)$ is the $(m+1)$-tuple of elements of F.

(Ad_2) $\{\mathbf{v}_k - \mathbf{v}_1, \cdots, \mathbf{v}_2 - \mathbf{v}_1\} \subset F^m$ is linearly dependent.

A set $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} \subset F^m$ is affinely independent if it is not affinely dependent.

Suppose that $E = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subset F^m$. Let A be the $(m+1) \times n$ matrix over F, the i-th column of with is $(1, \mathbf{v}_i)^{\mathrm{T}}$. The matroid $M[A]$ is called an *affine matroid* over F. In particular, if $F = GF(q)$, and $E = F^m$, then the affine matroid is denoted by $AG(m, q)$. In general, if M is an affine matroid over $\mathbb{R}$ of rank $(m+1)$, where $m \leq 3$, then a subset X of $E(M)$ is dependent in M if, in the representation of X by points in $\mathbb{R}^m$, there are two identical points, or three collinear points, or four coplanar points, or five points in space. Hence the flats of M of ranks one, two, and three are represented geometrically by points, lines, and planar, respectively.

We extend the use of diagram of affine matroid to represent arbitrary matroids of rank at most four. Generally, such diagrams are governed by the following rules. All loops are marked in a single inset. Parallel elements are represented by touching points. If three elements form a circuit, the corresponding points are collinear. Likewise, if four elements form a circuit, the corresponding points are coplanar. In such a diagram, the lines need not be straight and the planes may be twisted. Certain lines with fewer than three points on them will be marked as part of the indication of a plane, or as construction lines. We call such a diagram a *geometric representation* for the matroid.

Now we will define the projective geometry. Let V be a vector space over F. For each $v, w \in V - \{0\}$, $v \sim w$ if v and w lie on the same 1-dimensional subspace of V. Then, $\sim$ is an equivalence relation on V and $V - \{0\}/\sim$ is called the *projective space* of V or *projective geometry* and will be denoted by $PG(V)$. For a matroid M, delete all the loops from M and then, for each non-trivial parallel class X, delete all but one distinguished element of X, the matroid we obtain is called the *simple matroid* associated with M and is denoted by $\tilde{M}$. Evidently the construction of $PG(V)$ from V is analogous to the construction of the simple matroid $\tilde{M}$ from a matroid M. It is clear that a matroid M is F-representable if and only if its associated simple matriod is F-representable. Hence, when we discuss representability questions, it is enough to concentrate on simple matroids.

2.4. Projective Geometrices

If $V = F^{n+1}$, then $PG(V)$ has dimension n and it is denoted by $PG(n,F)$. In particular, when F is $GF(q)$, it will be written $PG(n,q)$ for $PG(n,F)$. Let's find the geometric representation of $PG(2,2)$. For each $v \in \mathbb{Z}_2^3 - \{0\}$, there are no non-zero elements except v on the 1-dimensional subspace of $\mathbb{Z}_2^3$ through v (Figure 1). Thus

$$PG(2,2) = \mathbb{Z}_2^3 - \{0\} = \{1=(1,0,0), 2=(1,1,0), 3=(0,1,0), 4=(0,1,1), \\ 5=(0,0,1), 6=(1,0,1), 7=(1,1,1)\}.$$

It is easy to see $\{1,2,3\},\{3,4,5\}$ and $\{1,5,6\}$ lie on the same line (plane) of $PG(2,2)$ ($\mathbb{Z}_2^3$). Also, $\{1,4,7\},\{2,5,7\}$ and $\{3,6,7\}$ are circuits of $PG(2,2)$. Furthermore, $\{2,4,6\}$ is a circuit, because $(1,1,0)+(0,1,1)=(1,0,1)$. Thus the geometric representation of $PG(2,2)$ is Figure 2.

$PG(2,2)$ is called the *Fano matroid* and will be denoted by F_7. In F_7, $\{2,4,6\}$ is a circuit and a hyperplane. The matroid N obtained from F_7 by relaxing the circuit hyperplane $\{2,4,6\}$ is called *non-Fano matroid* and is denoted by F_7^- (Figure 3).

2.5. Duals of Representable Matroids

Give a $m \times n$ matrix A by elementary row operations and interchanging two columns

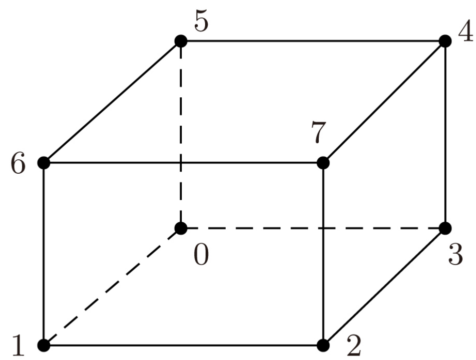


Figure 1. Vector space $\mathbb{Z}_2^3$.

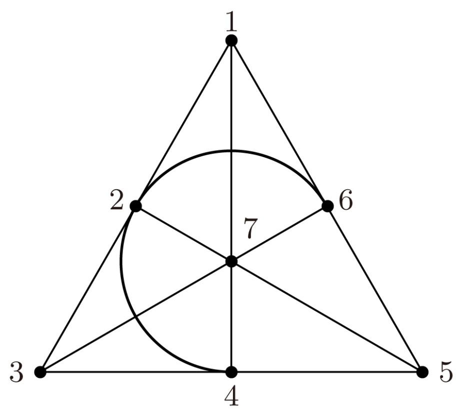


Figure 2. Geometric representation of $PG(2,2) = F_7$.

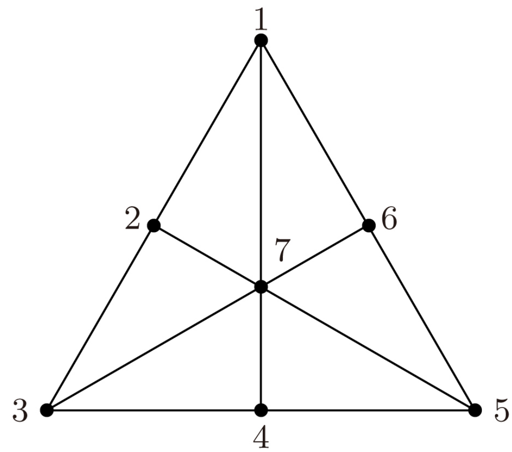


Figure 3. Geometric representation of F_7^-.

or deleting a zero row. A can be transformed to a form $[I_r \mid D]$, where I_r is $r \times r$ identity matrix and D is $r \times (n-r)$ matrix. It is clear that $M = M[A]$ is isomorphic to $M[I_r \mid D]$. We can show that the dual matroid M^* of M is $M\left[-D^{\mathrm{T}} \mid I_{n-r}\right]$ ([9]). Hence, the dual of F-representable matroid is F-representable. For example, let

$$\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
A = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] = [I_3 \mid D]
\end{array}$$

be a matrix over $\mathbb{Z}_2$. Then, we can see that $M[A]$ is isomorphic to F_7.

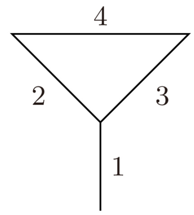
$$\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[-D^{\mathrm{T}} \mid I_4\right] = \left[\begin{array}{ccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]
\end{array}$$

and we can see that the set of circuits of $M\left[-D^{\mathrm{T}} \mid I_4\right]$ is

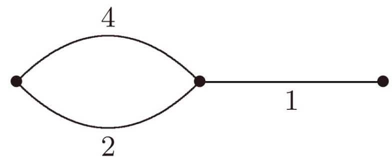
$$\{\{1,5,6,7\},\{2,4,6,7\},\{3,4,5,7\},\{1,2,3,7\},\{1,2,4,5\},\{1,3,4,6\},\{2,3,5,6\}\}.$$

3. Minors

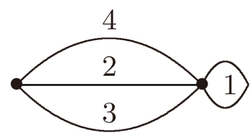
In this section, we define minors which are important to representability. For the definition of minor, we have to define contraction which is the dual of the operation of deletion. We can see that contraction for matroids generalizes the operation of contraction for graphs. Let M be a matroid on E and T be a subset of E. Then $M / T = \left(M^* \setminus T\right)^*$ is called the *contraction* of T from M and also denoted by $M \cdot (E - T)$. For easy understanding, let us see what it means in graphic matroids. Let G be a graph



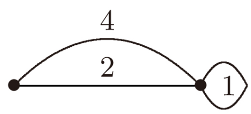
and $T = \{3\}$. Then, $G/3$ is the graph



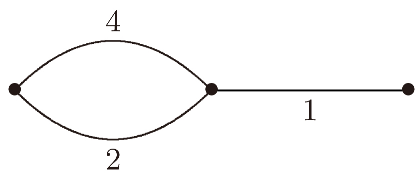
and G^* is



Also $G^* \setminus 3$ is



and $\left(G^* \setminus 3\right)^*$ is



which is the same as $G/3$. Thus,

$$\begin{aligned} M(G)/3 &= \left(M^*(G)\setminus 3\right)^* \cong \left(M\left(G^*\right)\setminus 3\right)^* = \left(M\left(G^* \setminus 3\right)\right)^* \\ &= M^*\left(G^* \setminus 3\right) = M\left(\left(G^* \setminus 3\right)^*\right) = M\left(G/3\right) \end{aligned}$$

and we see that the contraction of a graphic matroid is the same as the matroid of the contracted graph, where we used $M^*(G) \cong M\left(G^*\right)$ for a planar graph G.

Now let M^* be the dual of a matroid M. Then the rank function r^* of M^* is given by $r^*(X) = |X| + r(E - X) - r(E)$ ([9]). If $T \subset E$, the rank function of $M \setminus T$ is the restriction of r_M to the subset of $E - T$, that is, for all $X \subset E - T$, $r_{M\setminus T}(X) = r_M(X)$.

Proposition 3.1. *If* $T \subset E$, *then for all* $X \subset E - T$, $r_{M/T}(X) = r_M(X \cup T) - r_M(T)$.

Proof. By definition, $r_{M/T}(X) = r_{(M^* \backslash T)^*}(X)$. Thus

$$\begin{aligned} r_{M/T}(X) &= |X| + r_{M^* \backslash T}(E - T - X) - r_{M^* \backslash T}(E - T) \\ &= |X| + r^*(E - (T \cup X)) - r^*(E - T) \\ &= |X| + \left[|E - (T \cup X)| + r_M(T \cup X) - r_M(E)\right] - \left[|E - T| + r_M(T) - r_M(E)\right] \\ &= r_M(T \cup X) - r_M(T) \end{aligned}$$

because $|E - (T \cup X)| = E - |T| - |X|$ and $E - |T| = |E - T|$. □

Proposition 3.2. *Let* B_T *be a basis for* $M|_T$. *Then*

$$\begin{aligned} \mathcal{I}(M/T) &= \left\{ I \subset E - T \mid I \cup B_T \in \mathcal{I}(M) \right\} \\ &= \left\{ I \subset E - T \mid \text{there exists a basis } B \text{ of } M|_T \text{ such that } B \cup I \in \mathcal{I}(M) \right\}. \end{aligned}$$

Proof. For the convenience, let's denote the equality by $\mathcal{A} = \mathcal{B} = \mathcal{C}$. It is clear that $\mathcal{B} \subset \mathcal{C}$. To show that $\mathcal{C} \subset \mathcal{A}$, let $I \in \mathcal{C}$. Then, $B \cup I \in \mathcal{I}$ for some basis B of $M|_T$. Cleary $I \cup B$ is a basis of $I \cup T$. So $r_M(I \cup B) = r_M(I \cup T)$. Thus

$$r_{M/T}(I) = r_M(I \cup T) - r_M(T) = r_M(I \cup B) - r_M(T) = |I \cup B| - |B| = |I| + |B| - |B| = |I|,$$

and it was proved that $I \in \mathcal{I}(M/T)$. Now if we show that $\mathcal{A} \subset \mathcal{B}$, the proof is completed. Let $I \in \mathcal{I}(M/T)$. Then

$$|I| = r_{M/T}(I) = r_M(I \cup T) - r_M(T) = r_M(I \cup B_T) - |B_T|,$$

since B_T is a basis of $M|_T$. Hence $r_M(I \cup B_T) = |B_T| + |I| = |I \cup B_T|$ and $I \in \mathcal{B}$. □

Proposition 3.3. *If* $T \subset E(M)$, *then*

1) $M \backslash T = (M^* / T)^*$,
2) $M^* / T = (M \backslash T)^*$, *and*
3) $M^* \backslash T = (M / T)^*$.

Proof. 1) $M^* / T = \left((M^*)^* \backslash T\right)^* = (M \backslash T)^*$. Thus $M \backslash T = (M^* / T)^*$.

2) $M^* / T = \left((M^*)^* \backslash T\right)^* = (M \backslash T)^*$. 3) is obtained if we replace M by M^* in the left-hand side of 1). □

Now, let A be a matroid over F and T be a subset of the set E of column levels of A. We shall denote by $A \backslash T$ the matrix obtained from A by deleting all the columns whose labels are in T. Clearly, $M[A] \backslash T = M[A \backslash T]$. Moreover, by the following, we can see that the class of F-representable matroids is minor closed.

Proposition 3.4. *Every contraction of an F-representable matroid is F-representable.*

Proof. The duals of F-representable matroid are F-representable. Since $M[A]/T = \left(M^*[A] \backslash T\right)^*$, we proved that a contraction of F-representable matroid is F-representable. □

Now suppose that e is the label of a non-zero column of A. Then, by pivoting on a non-zero entry of e, we can transform A into a matrix A' in which the column labelled by e has single non-zero entry. In this case, A'/e will denote the matrix obtained from A' by deleting the row and column containing the unique non-zero entry in *e*. Then, we have the following property.

Proposition 3.5. $M[A]/e = M[A']/e = M[A'/e]$.

Proof. It is enough to show that the second equation is true, because the first equation is clear. By using row and column swaps if necessary, A' can be considered as the matrix in which the unique nonzero entry of e is in row 1 and column 1. Let I be a k-element subset of the ground set of $M[A']$ such that $e \notin I$. Then the set of columns labelled by $I \cup e$ is linearly independent if and only if the matrix B which has columns $e \cup I$ and the 1st column of it is the column corresponding to e has rank $k+1$. This is equivalent to the matrix deleted row 1 and column 1 of B has rank k and this is equivalent to the columns of A'/e labelled by I is linearly independent. Thus, $\mathcal{I}(M[A'/e]) = \mathcal{I}(M[A']/e)$.

□

4. Representability of P_8

Now, we shall describe the construction of representations for matroids. Two matrices A_1 and A_2 are equivalent if $M[A_1]$ and $M[A_2]$ are isomorphic. It is easy to see that if $M[A]$ is a rank-r matroid, then A is equivalent to a standard matrix $[I_r \mid D]$, where I_r is the $r \times r$ identity matrix. Given such a matrix, let its columns be labelled in order, $e_1, e_2, \cdots, e_r$. Let B be the basis $\{e_1, e_2, \cdots, e_r\}$ of $M[A]$. For all i in $\{1, 2, \cdots, r\}$, the unique non-zero entry in column i of $[I_r \mid D]$ is in row i. Thus it is natural to label the rows of $[I_r \mid D]$ by $e_1, e_2, \cdots, e_r$. Hence, D has its rows labelled by $e_1, e_2, \cdots, e_r$ and its columns labelled by $e_{r+1}, e_{r+2}, \cdots, e_n$. For all $k \in \{r+1, r+2, \cdots, n\}$, there exists a unique circuit $C(e_k, B)$ contained in $e_k \cup B$. In fact,

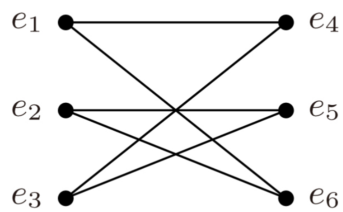
$C(e_k, B) = e_k \cup \{e_i \mid e_i \in B \text{ and } D \text{ has a non-zero entry in row } e_i \text{ and column } e_k\}$.

$C(e_k, B)$ is called the B-fundamental circuit of e_k. Let $D^\#$ be the matrix obtained from D by replacing each non-zero entry of D by 1. Then the columns of $D^\#$ are precisely the incidence vectors of the sets $C(e_k, B) - e_k$. This matrix $D^\#$ is called the *B-fundamental-circuit incidence matrix* of $M[A]$. Now let $M = M[A]$ be a rank-r matroid and B be a basis $\{e_1, e_2, \cdots, e_r\}$ for *M*. Let X be the B-fundamental-circuit incidence matrix of *M*. And let columns of X be labelled by $e_{r+1}, e_{r+2}, \cdots, e_n$. Then, $X = D^\#$. Thus the task of finding an *F*-representation for *M* can be viewed as being one of finding the specific elements of *F* that correspond to the non-zero elements of $D^\#$. We can see that most of the entries of D can be predetermined by the following Proposition 4.1. Before stating it, we shall require some preliminaries.

Let the rows of $D^\#$ be indexed by $e_1, e_2, \cdots, e_r$ and its columns by $e_{r+1}, e_{r+2}, \cdots, e_n$. Let $G(D^\#)$ denote the associated simple bipartite graph, that is, $G(D^\#)$ has vertex classes $\{e_1, e_2, \cdots, e_r\}$ and $\{e_{r+1}, e_{r+2}, \cdots, e_n\}$ and two vertices e_i and e_j are adjacent if and only if the entry in row e_i and column e_j of $D^\#$ is 1. For example, if

$$D^\# = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array},$$

then $G(D^{\#})$ is



We have a nice way for representing matroid;

Proposition 4.1. *Let the* $r \times n$ *matrix* $[I_r \mid D_1]$ *be an F-representation for the matroid M. Let* $\{b_1, b_2, \cdots, b_k\}$ *be a basis of the cycle matroid of* $G(D_1^{\#})$. *Then* $k = n - \omega(G(D_1^{\#}))$, *where* $\omega(G(D_1^{\#}))$ *is the number of connected component of* $G(D_1^{\#})$. *Moreover, if* $(\theta_1, \theta_2, \cdots, \theta_k)$ *is an ordered k-tuple of non-zero elements of F, then M has an F-representation* $[I_r \mid D_2]$ *that is equivalent to* $[I_r \mid D_1]$ *such that, for each i in* $\{1, 2, \cdots, k\}$, *where the entry of* D_2 *corresponding to* b_i *is* θ_i ([9] [10]).

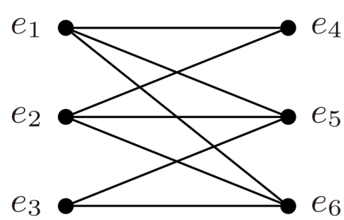
By the above proposition, we can find the fields on which given matroid is representable.

Example 4.2. P_6 is the matroid whose geometric representation is the following Figure 4.

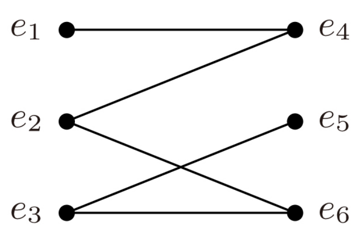
Let $[I_3 \mid D]$ be a representation over $GF(q)$ of P_6. Then

$$D^{\#} = X = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Thus, the associated simple bipartite graph $G(D^{\#})$ is



and a basis of $G(D^{\#})$ is



Therefore, $[I_3 \mid D]$ is equivalent to the form of

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & a & c \\ 0 & 1 & 0 & 1 & b & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}. \end{array}$$

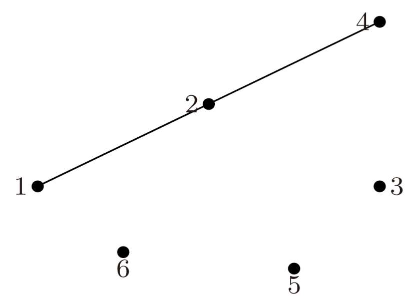


Figure 4. Geometric representation of P_6.

$$\det(1,5,6) = b-1 \neq 0,\quad \det(2,5,6) = c-a \neq 0,\quad \det(3,5,6) = a-bc \neq 0,$$

$$\det(3,4,5) = b-a \neq 0,\quad \det(3,4,6) = c-1 \neq 0,\quad \det(4,5,6) = b-a+c-1 \neq 0.$$

We want to find the fields $GF(q)$ in which the negative equations satisfy. It is easy to see that the equations have no solution if $GF(q)$ is equal to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In case of $GF(4) = \{0,1,\omega,\omega+1\}$, let's check if the equations have a common solution. There are three cases;

1) $a = 1$.

If $b = c$, then $b-a+c-1 = 2b-2 = 0$. Thus, $b \neq c$ and $bc = \omega(\omega+1) = 1 = a$. Hence, we don't have a solution.

2) $a = \omega$.

In this case, $b = c = \omega+1$ and $bc = \omega = a$. Thus, there are no solution.

3) $a = \omega+1$.

In this case, $b = c = \omega$ and $bc = \omega^2 = \omega+1 = a$. Thus we have no solution.

In $\mathbb{Z}_5$, if $a = 1, b = 2, c = 4$, then $bc = 8 = 3 \neq 1 = a$ and $b-a+c-1 = 2-1+4-1 = 4 \neq 0$. Therefore we showed that P_6 is representable over F if and only if $|F| \geq 5$.

Example 4.3. *If*

$$A = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_3 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array}\right]$$

is the matrix over F with $charF \neq 2$, then it is easy to see that $M[A] = F_7^-$ by **Figure 3**. Also, if $[I_3 \mid D]$ is a matrix over F and $M[I_3 \mid D] = F_7^-$, then

$$D^\# = X = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}. \end{array}$$

By taking a basis of $M\left[G\left(D^{\#}\right)\right]$, $[I_3 \mid D]$ is equivalent to a matrix

$$[I_3 \mid D_1] = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & c & 1 \\ 0 & 0 & 1 & a & b & 0 & 1 \end{bmatrix} \end{array}.$$

Since $\det(1,4,7) = 1 - a = 0, a = 1$. Also, because $\det(2,5,7) = b - 1 = 0$ and $\det(3,6,7) = 1 - c = 0$, $b = c = 1$. Thus $D_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

Since $\det(4,5,6) = 2 \neq 0$, $charF \neq 2$. Therefore, we showed that F_7^- is representable over F if and only if $charF \neq 2$. P_7 is the matroid of the matrix

$$A = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

over $\mathbb{Z}_3$. It's geometric representation is the following Figure 5.

Lemma 4.4. *P_7 is representable over a field F if and only if $|F| \geq 3$.*

Proof. If $B = \{1,2,3\}$, then the B-fundamental circuit incidence matrix of P_7 is

$$X = \begin{array}{c|cccc} & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 & 0 \end{array}.$$

By taking a basis of $M\left[G(X)\right]$, A is equivalent to a matrix $[I_3 \mid D]$, where

$$D = \begin{array}{c|cccc} & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & c \\ 3 & a & 1 & b & 0 \end{array}.$$

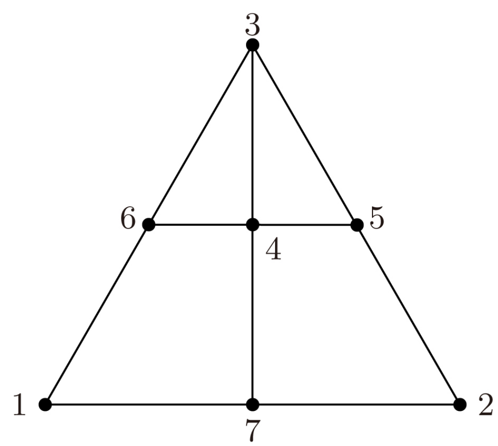


Figure 5. Geometric representation of P_7.

Because $\det(3,4,7) = c-1 = 0$ and $\det(4,5,6) = b+1-a = 0, c = 1$ and $b = a-1$. As $b \neq 0, a \neq 1$. Therefore

$$\begin{array}{c c} & \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ D = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ a & 1 & a-1 & 0 \end{bmatrix}, \end{array}$$

where $a \notin \{0,1\}$. Therefore, we proved that P_7 is representable over *F* if and only if $|F| \geq 3$. □

Non F-representable matroid for which every proper minor is *F*-representable is called the *excluded* or *forbidden minor* for *F*-representability. Because a matroid *M* is *F*-representable if and only if all its minors are F-representable (Proposition 3.5), finding the complete set of excluded minors for *F*-representability is the solution for the *F*-representability. Since the duals of F-representable matroid are *F*-representable, the dual of an excluded minor for *F*-representability is an excluded minor for *F*-representability.

To find an excluded minor for $\mathbb{Z}_2$-representability, we need the following property:

Proposition 4.5. *Let F be a field and k be an integer exceeding* 1*. Then uniform matroid* $U_{2,k}$ *is F-representable if and only if* $|F| \geq k-1$.

Proof. Let $U_{2,k} \cong M[A]$, where *A* is a $2 \times k$ matrix. We can consider *A* as a matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{k-2} \end{pmatrix},$$

where $\alpha_i \ (1 \leq i \leq k-2)$ are non-zero different elements of *F*. Thus, it should be $|F|-1 \geq k-2$ and so $|F| \geq k-1$. Conversely, if $|F| \geq k-1$, then $|F|-1 \geq k-2$ and we can choose non-zero different $(k-2)$-elements of *F*.

□

From the above proposition, we can see that $U_{2,q+2}$ and $(U_{2,q+2})^* = U_{q,q+2}$ are excluded minors for *GF*(*q*)-representability. In 1958, Tutte showed that $U_{2,4}$ is the only excluded minor for $\mathbb{Z}_2$-representability ([2]). The problem of finding the complete set of excluded minors for $\mathbb{Z}_3$-representability was solved by Bixby and Seymour in 1979 ([3] [4]). The set is $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$.

By Proposition 4.5, it is easy to see that $U_{2,6}$ and $U_{4,6}$ are excluded minors for *GF*(4)-representability. In Examples 4.2 and 4.3, we can see that P_6 and F_7^- are not *GF*(4)-representable. It is not difficult to see that every proper minor of them is *GF*(4)-representable. Thus P_6 and F_7^- are excluded minors for *GF*(4)-representability. Clearly P_6 is self-dual.

Let

$$\begin{array}{c c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ A = & \left[\begin{array}{c|cccc} & 0 & 1 & 1 & -1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & -1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

be the matrix over $\mathbb{Z}_3$. Then, P_8 is the matroid $M[A]$.

Lemma 4.6. *P_8 is representable over a field F if and only if $charF \neq 2$.*

Proof. Let $[I_r \mid D]$ be an *F*-representation for P_8. If $B = \{1,2,3,4\}$, then the *B*-fundamental circuit incidence matrix for P_8 is

$$D^{\#} = X = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

By choosing a basis for $M\left[G\left(D^{\#}\right)\right]$, we can consider

$$D = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & d \\ 1 & 0 & 1 & 1 \\ a & 1 & 0 & e \\ b & 1 & c & 0 \end{bmatrix} \end{array}.$$

Because $\det(1,4,5,8) = e - a = 0$ and $\det(2,3,6,7) = c - 1 = 0, a = e$ and $c = 1$. Substituting these to the matrix *D*, we have

$$D = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & d \\ 1 & 0 & 1 & 1 \\ a & 1 & 0 & a \\ b & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

From the circuits $\{1,5,6,7\}$, $\{2,5,6,8\}$, $\{3,5,7,8\}$ and $\{4,6,7,8\}$, we get the equations

$$a = b - 1, ad + ab - bd = 0, b + d - db = 0 \text{ and } a + 1 = d.$$

From the first and fourth equation, we get $b = d$. Substituting b for d in the second and third equation, we get $b(2a - b) = 0$ and $b(2 - b) = 0$. As $b \neq 0$, it follows that $b = 2$ and $a = 1$. Because $0 \neq b = 2$, $charF \neq 2$. Thus

$$D = \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

In fact, we can show that $M[I_4 \mid D] = P_8$. □

For a matroid *M*, an automorphism is a permutation σ of $E(M)$ such that $r(X) = r(\sigma(X))$ for all $X \subset E(M)$. The set of automorphisms of *M* forms a group under composition. This automorphism is transitive if, for every two elements *x* and *y* of $E(M)$, there is an automorphism that maps *x* to *y*.

Lemma 4.7. *The automorphism group of P_8 acts transitively on P_8.*

Proof. We can see that the geometric representation of P_8 is the following Figure 6 because P_8 has only 10 4-circuits $\{1,2,3,8\}$, $\{1,2,4,7\}$, $\{1,3,4,6\}$, $\{2,3,4,5\}$, $\{1,4,5,8\}$, $\{2,3,6,7\}$, $\{1,5,6,7\}$, $\{2,5,6,8\}$, $\{3,5,7,8\}$ and $\{4,6,7,8\}$. From the geometric representation of P_8, it is easy to see that the permutations $\sigma_1 = (1,8,4,5)(2,7,3,6)$ and $\sigma_2 = (1,2,4,3)(5,6,8,7)$ are both automorphisms of P_8. For example, $\sigma_1 P_8$ and $\sigma_2 P_8$ are the following Figure 7 and Figure 8.

Thus the automorphism group of P_8 is $\langle \sigma_1, \sigma_2 \rangle$.

Any two elements in $\{1,8,4,5\}$ and $\{2,3,6,7\}$ can be mapped to each other by an automorphism in $\langle \sigma_1 \rangle$. Similarly, any two elements in $\{1,2,4,3\}$ and $\{5,6,8,7\}$ can be mapped to each other by an automorphism in $\langle \sigma_2 \rangle$. For the remaining two elements of P_8, they are mapped each other by the following; $\sigma_2 \sigma_1^{-1}(1) = \sigma_2(5) = 6$, $\sigma_2^{-1}\sigma_1(2) = \sigma_2^{-1}(7) = 8$, $\sigma_2\sigma_1(1) = \sigma_2(8) = 7$, $\sigma_2\sigma_1(2) = \sigma_2(7) = 5$,

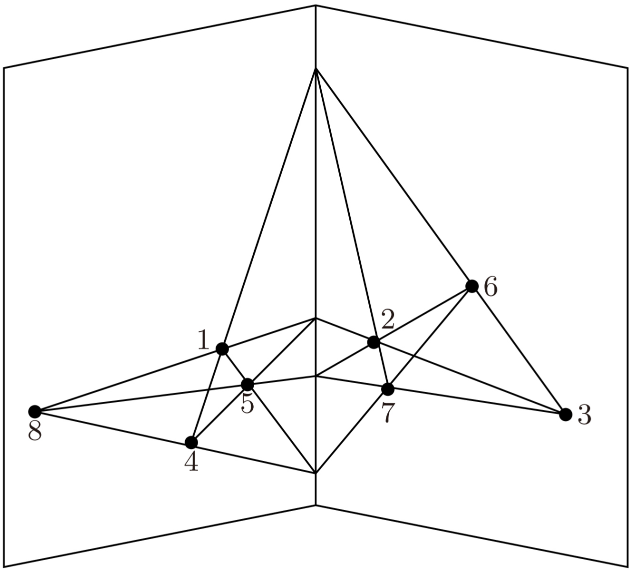


Figure 6. Geometric representation of P_8.

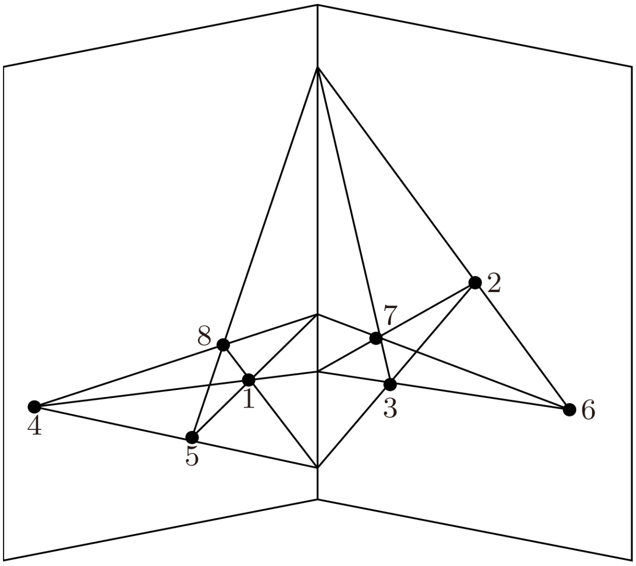


Figure 7. Geometric representation of $\sigma_1 P_8$.

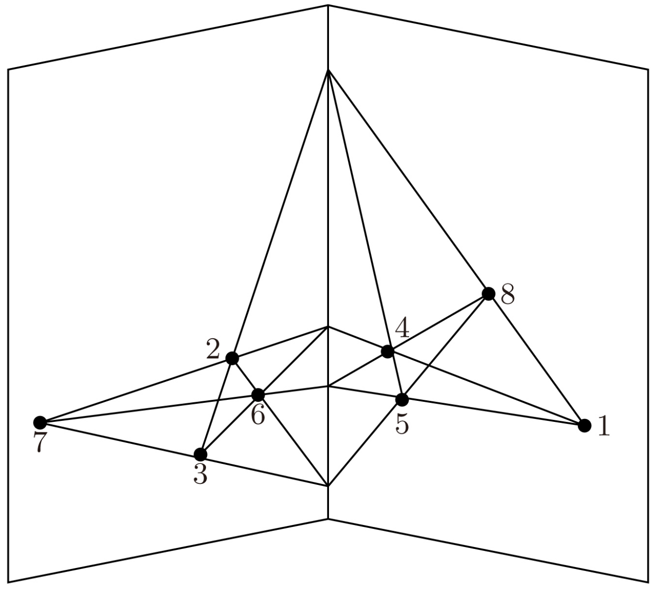


Figure 8. Geometric representation of $\sigma_2 P_8$.

$\sigma_2^{-1}\sigma_1^{-1}(3)=\sigma_2^{-1}(7)=8$, $\sigma_2\sigma_1^{-1}(4)=\sigma_2(8)=7$, $\sigma_2\sigma_1^{-1}(3)=\sigma_2(7)=5$, $\sigma_2\sigma_1(4)=\sigma_2(5)=6$.

Thus, the automorphism group of P_8 acts transitively on P_8.

□

Now, we get the following result, which is the purpose of this paper.

Theorem 4.8. P_8 *is an excluded minor for* $GF(4)$*-representability.*

Proof. By Lemma 4.6, P_8 is not $GF(4)$ representable. Because the automorphism group of P_8 acts transitively on P_8 by Lemma 4.7, for any element e of $E(P_8)$, we have $P_8/e \cong P_8/1$.

By Proposition 3.5, $P_8/1 = M[A]/1 = M[A/1]$. Because

$$A/1 = \left[\begin{array}{c|cccc} & 1 & 0 & 1 & 1 \\ I_3 & 1 & 1 & 0 & 1 \\ & -1 & 1 & 1 & 0 \end{array}\right],$$

$P_8/1 \cong P_7$. But, since P_7 is representable over $GF(4)$ by Lemma 4.4, every contraction of P_8 is $GF(4)$-representable. By Proposition 3.3(1), for each element $e \in E(P_8)$, $P_8 \backslash e = (P_8^*/e)^*$. Because P_8 is self-dual, $P_8^* \cong P_8$. Thus $P_8 \backslash e = (P_8^*/e)^* \cong (P_8/e)^* \cong P_7^*$. Hence, every deletion of P_8 is $GF(4)$-representable. Therefore, every proper minor of P_8 is $GF(4)$-representable and we proved that P_8 is an excluded minor for $GF(4)$-representability. □

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