

The Harmonic Oscillator with Random Damping in Non-Markovian Thermal Bath

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Abstract

In this paper, we define the harmonic oscillator with random damping in non-Markovian thermal bath. This model represents new version of the random oscillators. In this side, we derive the overdamped harmonic oscillator with multiplicative colored noise and translate it into the additive colored noise by changing the variables. The overdamped harmonic oscillator is stochastic differential equation driving by colored noise. We derive the change in the total entropy production (CTEP) of the model and calculate the mean and variance. We show the fluctuation theorem (FT) which is invalid at any order in the time correlation. The problem of the deriving of the CTEP is studied in two different examples of the harmonic potential. Finally, we give the conclusion and plan for future works.

Keywords

Random Damping, Total Entropy, Non-Markovian Bath, Fluctuation Theorem, Additive Colored Noise

1. Introduction

In the 1980s, studies of linear and non linear oscillators were extended to the case of colored noise driving force. Many applications of the random damping in Markovian thermal bath include water waves influenced by turbulent wind field, the Ginzburg-Landau equation with a convective term, mean flow passing through a region under study, open flows of liquids, dendritic growth, chemical waves and motion of vortices [1]-[7] respectively. The effect of correlations in the random driving force on the stationary probability density and its moments were studied [8]-[12]. The exact formula is found for the first moment and the system of equations for second moments for harmonic oscillator with random mass [13]. The analytical expressions are derived for the sta-

tionary probability density of the particle's energy [14]. Both theoretical approaches were formulated in the context of the standard Langevin equation [15] [16], where the friction force was proportional to the speed with a constant friction coefficient and additive Gaussian white noise. The non-equilibrium process efforts are commonly formulated in the form of stochastic thermodynamic culminates into fluctuation relations connecting extensive thermodynamic variables such as work, free energy, and entropy [17]-[21]. The violation of the Markovian approximation of the environment leads to generation of additional entropy [21]. The ideas behind the entropy production are studied and some insights are given about relevance [22]. The statistical properties of stochastic entropy production associated with the non stationary transport of heat through system coupled to a time dependent nanisothermal heat bath [23]. When the harmonic oscillator with external noise in non-Markovian thermal bath, the cumulants of order two and three contain the natural effects of the non-Markovian bath through the noise correlation time, consequently the non Gaussian characteristic of the total entropy change drives to a breakdown of the usual fluctuation theorems [24]. The purpose of this paper is discussing the change in total entropy production for the harmonic oscillator with random damping in non-Markovian thermal bath and studying the fluctuation theorem (FT) at any order in time correlation when the harmonic potentials are represented the time dependent driving force or the time dependent dragging force where this force is arbitrary time dependent. We derive in this paper the stochastic differential equation (SDE) driving by the multiplicative colored noise and translate it into additive colored noise by changing variables from x to y . In this side, in order to compatible the additive colored noise system with potential, we change variables in harmonic potentials (time dependent driving force and time dependent dragging force). We calculate the mean, variance and the distribution function for the change in total entropy production in new formulas of the harmonic potentials. We show that in our model, the fluctuation theorem is invalid at any order in the noise correlation time. Finally, we present our conclusion and we give the future works. This paper can be divided by six sections. In Section 2, we define the new model based on the SDE driving by additive colored noise in the overdamped approximation, and we find the Fokker Planck equation which it is associated of the SDE. In Section 3, we change variables in Equation (1) from x into y , this represent first example. In this example, we find the change in total entropy, mean and variance. In Section 4, we change variables in Equation (2), this represent second example. In this example, we also compute the change in total entropy, mean and variance. In Section 5, we show that the FT is invalid at any order in the time correlation. Finally, we introduce the conclusions and future works.

2. Stochastic Differential Equation (SDE) Driving by Additive Colored Noise in Overdamped Approximation

In this section, we define the harmonic oscillator with random damping in non-Markovian thermal bath. We derive the stochastic differential equation (SDE) driving by the multiplicative colored noise and translate it into the additive colored noise by changing variables in overdamped approximation and its stochastic treatment. Our model can be defined as,

$$\dot{x} = v_t, m \frac{dv}{dt} = -\frac{dU}{dx} - \frac{\lambda}{\tau} \int_0^t \exp\left[-\frac{t-s}{\tau}\right] (1 + \xi_s) v_s ds = -kx + f_t - \frac{\lambda}{\tau} \int_0^t \exp\left[-\frac{t-s}{\tau}\right] (1 + \xi_s) v_s ds, \quad (1)$$

where ξ_t is Ornstein-Uhlenbeck noise (special type of colored noise), λ is friction constant, τ is correlation time, $U(x, t)$ is the harmonic potential, f_t is arbitrary time depend force, $x(t)$ is particle's position and v_t is the velocity. The ξ_t is Gaussian distribution with zero mean and correlation function is,

$$\langle \xi_t \xi_s \rangle = \frac{D}{\tau} \exp\left[-\frac{t-s}{\tau}\right], \quad (2)$$

where $D = k_B T \lambda$ such that k_B is Boltzmann constant and T is heat temperature. We assume that the following ,

$$\phi_t = -\frac{\lambda}{\tau} \int_0^t \exp\left[-\frac{t-s}{\tau}\right] (1 + \xi_s) v_s ds, \quad (3)$$

and read the Equation (1) as,

$$m \frac{dv}{dt} = -kx + f_t + \phi_t. \quad (4)$$

In overdamped approximation , the Equation (1) become,

$$\phi_t = kx - f_t, \tag{5}$$

taking the time derivative of Equation (3) we get,

$$\dot{\phi}_t = -\frac{\lambda}{\tau}(1 + \xi_t)v_t - \frac{\phi}{\tau}. \tag{6}$$

Substituting Equation (5) in (6), one can obtain,

$$\dot{x} = \frac{\tau \dot{f}_t + f_t - kx}{(\tau k + \lambda) \left(1 + \frac{\lambda \xi_t}{\tau k + \lambda}\right)}, \tag{7}$$

let $b_t = \tau \dot{f}_t + f_t$ and $a = \frac{\lambda}{\tau k + \lambda}$, the Equation (7) read as,

$$\dot{x} = a \frac{b_t - kx}{1 + a \xi_t}. \tag{8}$$

By using power series at first order in the noise ξ_t , the above equation become,

$$\dot{x} = a \frac{(b_t - kx)}{\lambda} - a^2 \frac{(b_t - kx) \xi_t}{\lambda}, \tag{9}$$

let $f(x) = a \frac{(b_t - kx)}{\lambda}$ and $g(x) = -a^2 \frac{b_t - kx}{\lambda}$ where $g(x) \neq \text{zero}$. Equation (9) is SDE driving by multiplicative colored noise. To translate Equation (9) from multiplicative colored noise into additive ,we must divided Equation (9) by $g(x)$, one can obtain,

$$\frac{\dot{x}}{g(x)} = -\frac{1}{a} + \xi_t, \tag{10}$$

let,

$$\dot{y} = \frac{\dot{x}}{g(x)}, \tag{11}$$

then the translation [25] is defined as,

$$y = \int \frac{dz}{g(z)}, \tag{12}$$

then the Equation (10) is,

$$\dot{y} = -\frac{1}{a} + \xi_t. \tag{13}$$

Equation (13) represent SDE driving by additive colored noise. The Fokker Planck equation [25] is defined,

$$\frac{dP}{dt} = \frac{1}{a} \frac{dP}{dY} + D \left(1 - \exp\left[-\frac{t}{\tau}\right]\right) \frac{d^2P}{dY^2}, \tag{14}$$

where the initial condition $P(y, 0/y_0) = \delta(y - y_0)$, we solve the Fokker Planck equation by Fourier transformation [26] as,

$$F \left[P(y, t/y_0) \right] = \frac{1}{2\Pi} \int_0^\infty \exp[-ry] P(y, t/y_0) dy = \bar{P}(y, t/y_0), \tag{15}$$

we take the time derivative into above equation, we have,

$$\frac{d\bar{P}(r, t/y_0)}{dt} = \frac{1}{2\Pi} \int_0^\infty \exp[-ry] \left[\frac{1}{a} \frac{dP}{dY} + D \left(1 - \exp\left[-\frac{t}{\tau}\right]\right) \frac{d^2P}{dY^2} \right] dy. \tag{16}$$

Assume that $\frac{d}{dy} = ir$, then we have,

$$\bar{P}(r, t/y_0) = \exp\left[-M_t r^2 + \left(\frac{it}{a} - y_0\right)r\right], \quad (17)$$

where $M_t = Dt + D\tau\left(\exp\left[-\frac{t}{\tau}\right] - 1\right)$ then the transition probability is,

$$P(y, t/y_0) = \frac{\exp\left[\frac{\left[y - y_0 + \frac{it}{a}\right]^2}{4M_t}\right]}{2\sqrt{\pi M_t}}. \quad (18)$$

To obtain the initial distribution function we must assume that $\frac{dP}{dt} = 0$ at zero order in time correlation that mean (τ converge to zero), then we get,

$$\hat{P}(y_0) = \frac{\exp\left[-\frac{y_0}{aD}\right]}{aD}, \quad (19)$$

where the initial distribution is exponential distribution. Then the marginal probability of the particle's position is,

$$\hat{P}(y) = \frac{\exp\left[-\frac{y}{aD}\right]}{aD}, \quad (20)$$

also the distribution of y is exponential distribution, and note that y and y_0 have same exponential distribution.

3. The First Example

In this section, we change variables in the time dependent driving force from x into y which is defined as

$$U(x, t) = \frac{k}{2}x^2 - xf_t, \quad (21)$$

where f_t is arbitrary time depend force and under the new formula of the harmonic potential, we calculate the change in the total entropy production (CTEP), mean and the variance. From Equation (12), we have,

$$x = \frac{1}{k}\left[b_t - \exp\left[-\frac{a^2ky}{\lambda}\right]\right]. \quad (22)$$

Substituting Equation (22) in Equation (21), one can obtain,

$$U(y, t) = \frac{1}{2k}\left[b_t - \exp\left[-\frac{a^2ky}{\lambda}\right]\right]^2 - \frac{1}{k}\left[b_t - \exp\left[-\frac{a^2ky}{\lambda}\right]\right]f_t. \quad (23)$$

Equation (23) represent first new formula of harmonic potential in y . The change in new harmonic potential can be defined as,

$$\Delta U(y, t) = \frac{1}{2k}\left[\exp\left[-\frac{2a^2ky}{\lambda}\right] - \exp\left[-\frac{2a^2ky_0}{\lambda}\right]\right] - \frac{\tau \dot{f}_t \exp\left[-\frac{a^2ky}{\lambda}\right]}{k} + \frac{(\tau \dot{f}_t)^2 - f_t^2}{2k}, \quad (24)$$

where we assume that $f(0) = \dot{f}(0) = 0$. The based on the stochastic thermodynamic approach [17] [27] [28], the first law thermodynamic like can be defined as,

$$Q = W - \Delta U(y, t), \quad (25)$$

where the work can be computed [29] as,

$$W = \int_0^t \frac{\partial U(y, s)}{\partial s} ds = \frac{\tau}{k} \int_0^t \left[b_s - \exp\left[-\frac{a^2 ky}{\lambda}\right] \right] \ddot{f}_s ds - \frac{1}{k} \int_0^t f_s \dot{b}_s ds. \quad (26)$$

The mean of the work is calculated as,

$$\langle W \rangle = \frac{\tau}{k} \int_0^t \left[b_s - \left\langle \exp\left[-\frac{a^2 ky}{\lambda}\right] \right\rangle \right] \ddot{f}_s ds - \frac{1}{k} \int_0^t f_s \dot{b}_s ds, \quad (27)$$

where the quantity $\left\langle \exp\left[-\frac{a^2 ky}{\lambda}\right] \right\rangle$ can be found as,

$$\left\langle \exp\left[-\frac{a^2 ky}{\lambda}\right] \right\rangle = \int_0^\infty \exp\left[-\frac{a^2 ky}{\lambda}\right] \hat{P}(y) dy = \frac{\lambda}{\lambda + a^3 kD}. \quad (28)$$

Putting Equation (28) inside Equation (27), we get,

$$\langle W \rangle = \frac{\tau}{k} \int_0^t \left[b_s - \frac{\lambda}{\lambda + a^3 kD} \right] \ddot{f}_s ds - \frac{1}{k} \int_0^t f_s \dot{b}_s ds, \quad (29)$$

the variance of the work can be calculated as,

$$\text{var}(W) = \frac{\tau^2}{k^2} \int_0^t \text{var}\left(\exp\left[-\frac{a^2 ky}{\lambda}\right]\right) \ddot{f}_s^2 ds, \quad (30)$$

where the quantity $\text{var}\left(\exp\left[-\frac{a^2 ky}{\lambda}\right]\right)$ is found as,

$$\text{var}\left(\exp\left[-\frac{a^2 ky}{\lambda}\right]\right) = \frac{\lambda}{\lambda + 2a^3 kD} - \frac{\lambda^2}{(\lambda + a^3 D)^2}. \quad (31)$$

Substituting Equation (31) inside Equation (30). one can obtain,

$$\text{var}(W) = \frac{\tau^2}{k^2} \left[\frac{\lambda}{\lambda + 2a^3 kD} - \frac{\lambda^2}{(\lambda + a^3 D)^2} \right] \int_0^t \ddot{f}_s^2 ds. \quad (32)$$

The change in the environment entropy ΔS_E is obtain as,

$$\begin{aligned} \Delta S_E &= \frac{W - \Delta U(y, t)}{T} \\ &= \frac{W}{T} + \frac{\exp\left[-\frac{2a^2 ky_0}{\lambda}\right] - \exp\left[-\frac{2a^2 ky}{\lambda}\right]}{2kT} + \frac{\tau \dot{f}_t \exp\left[-\frac{a^2 ky}{\lambda}\right]}{kT} - \frac{(\tau \dot{f}_t)^2 - f_t^2}{2kT}. \end{aligned} \quad (33)$$

The change in entropy of the system ΔS is given as,

$$\Delta S = -\ln \frac{\hat{P}(y)}{\hat{P}(y_0)} = \frac{y}{aD} - \frac{y_0}{aD}. \quad (34)$$

Now, we can find the CTEP ΔS_{tot} as,

$$\begin{aligned} \Delta S_{tot} &= \Delta S_E + \Delta S \\ &= \frac{W}{T} + \frac{\exp\left[-\frac{2a^2ky_0}{\lambda}\right] - \exp\left[-\frac{2a^2ky}{\lambda}\right]}{2kT} \\ &\quad + \frac{\tau \dot{f}_t \exp\left[-\frac{a^2ky}{\lambda}\right]}{kT} + \frac{y}{aD} - \frac{y_0}{aD} + \gamma_t, \end{aligned} \quad (35)$$

where $\gamma_t = -\frac{((\tau \dot{f}_t)^2 - f_t^2)}{2kT}$ and the mean of the ΔS_{tot} is,

$$\begin{aligned} \langle \Delta S_{tot} \rangle &= \langle \Delta S_E \rangle + \langle \Delta S \rangle \\ &= \frac{\langle W \rangle}{T} + \frac{\left\langle \exp\left[-\frac{2a^2ky_0}{\lambda}\right] \right\rangle - \left\langle \exp\left[-\frac{2a^2ky}{\lambda}\right] \right\rangle}{2kT} \\ &\quad + \frac{\tau \dot{f}_t \left\langle \exp\left[-\frac{a^2ky}{\lambda}\right] \right\rangle}{kT} + \frac{\langle y \rangle}{aD} - \frac{\langle y_0 \rangle}{aD} + \gamma_t, \end{aligned} \quad (36)$$

we must calculate the following quantities $\left\langle \exp\left[-\frac{2a^2ky_0}{\lambda}\right] \right\rangle$, $\langle y \rangle$ and $\langle y_0 \rangle$ as,

$$\left\langle \exp\left[-\frac{2a^2ky_0}{\lambda}\right] \right\rangle = \int_0^\infty \exp\left[-\frac{2a^2ky_0}{\lambda}\right] \hat{P}(y_0) dy_0 = \frac{\lambda}{\lambda + 2a^3kD}, \quad (37)$$

where $\left\langle \exp\left[-\frac{2a^2ky_0}{\lambda}\right] \right\rangle = \left\langle \exp\left[-\frac{2a^2ky}{\lambda}\right] \right\rangle$,

$$\langle y \rangle = \int_0^\infty y \hat{P}(y) dy = \frac{1}{aD}, \quad (38)$$

and,

$$\langle y_0 \rangle = \int_0^\infty y_0 \hat{P}(y_0) dy_0 = \frac{1}{aD}, \quad (39)$$

note here $\langle y_0 \rangle = \langle y \rangle$. Putting Equation (29) and an above quantities's values inside Equation (36), we get,

$$\langle \Delta S_{tot} \rangle = \frac{\tau}{kT} \int_0^t \left[b_s - \frac{\lambda}{\lambda + a^3kD} \right] \ddot{f}_s ds - \frac{1}{k} \int_0^t f_s \dot{b}_s ds + \frac{\tau \dot{f}_t}{kT} \frac{\lambda}{\lambda + a^3kD} + \gamma_t, \quad (40)$$

before we find the variance of the CTEP, we make some the following assumptions, let $C_1 = \exp\left[-\frac{2a^2ky_0}{\lambda}\right]$

with it coefficient $c_1 = \frac{1}{2kT}$, $C_2 = \exp\left[-\frac{2a^2ky}{\lambda}\right]$ with it coefficient $c_2 = -\frac{1}{2kT}$, $C_3 = \exp\left[-\frac{a^2ky}{\lambda}\right]$ with it

coefficient $c_3 = \frac{\tau \dot{f}_t}{kT}$, $C_4 = y$ with it coefficient $c_4 = \alpha_t$ and $C_5 = y_0$ with it coefficient $c_5 = -\frac{1}{aD}$,

$$\text{var}(S_{tot}) = \frac{1}{T^2} \text{var}(W) + \sum_{i=1}^5 c_i^2 \text{var}(C_i) + 2 \sum_{i=1}^5 \sum_{j=i+1}^5 c_i c_j \text{Cov}(C_i, C_j), \quad (41)$$

we must calculate the following quantities $var\left(C_1 = \exp\left[-\frac{2a^2ky_0}{\lambda}\right]\right)$, $var\left(C_2 = \exp\left[-\frac{2a^2ky}{\lambda}\right]\right)$,

$var\left(C_3 = \exp\left[-\frac{a^2ky}{\lambda}\right]\right)$, $var(C_4 = y)$ and $var(C_5 = y_0)$ as,

$$var(C_1) = \frac{\lambda}{\lambda + 4a^3kD} - \frac{\lambda^2}{(\lambda + 2a^3kD)^2}, \quad (42)$$

$$var(C_2) = \frac{\lambda}{\lambda + 4a^3kD} - \frac{\lambda^2}{(\lambda + 2a^3kD)^2}, \quad (43)$$

note here $var(C_1) = var(C_2)$,

$$var(C_3) = \frac{\lambda}{\lambda + 2a^3kD} - \frac{\lambda^2}{(\lambda + a^3kD)^2}, \quad (44)$$

$$var(C_4) = \frac{1}{aD}, \quad (45)$$

and,

$$var(C_5) = \frac{1}{aD}. \quad (46)$$

Putting Equation (32) and an above quantities's values in Equation (41), one can obtain,

$$\begin{aligned} var(\Delta S_{tot}) = & \frac{1}{T^2} \left[\frac{\tau^2}{k^2} \left[\frac{\lambda}{\lambda + 2a^3kD} - \frac{\lambda^2}{(\lambda + a^3D)^2} \right] \int_0^t \dot{f}_s^2 ds \right] + \frac{1}{4(kT)^2} \left[\frac{2\lambda}{\lambda + 4a^3kD} - \frac{2\lambda^2}{(\lambda + 2a^3kD)^2} \right] \\ & + \frac{(\tau \dot{f}_t)^2}{(kT)^2} \left[\frac{\lambda}{\lambda + 2a^3kD} - \frac{\lambda^2}{(\lambda + a^3kD)^2} \right] + \frac{1}{(aD)^2} + 2 \sum_{i=1}^5 \sum_{j=i+1}^5 c_i c_j Cov(C_i, C_j). \end{aligned} \quad (47)$$

At zero order in time correlation τ , the change in total entropy production in here read as,

$$\Delta S_{tot} = \frac{W}{T} + \frac{\exp\left[\frac{2ky_0}{\lambda}\right] - \exp\left[-\frac{2ky}{\lambda}\right]}{2kT} + \frac{y - y_0}{aD}, \quad (48)$$

where $W = -\frac{1}{k} \int_0^t f_s \dot{b}_s ds$ then the mean of ΔS_{tot} is,

$$\langle \Delta S_{tot} \rangle = -\frac{1}{k} \frac{\int_0^t f_s \dot{b}_s ds}{T}, \quad (49)$$

and variance,

$$var(\Delta S_{tot}) = \frac{1}{4(kT)^2} \left[\frac{2\lambda}{\lambda + 4kD} - \frac{2\lambda^2}{(\lambda + 2kD)^2} \right] + \frac{2}{(D)^4} + 2 \sum_{i=1}^5 \sum_{j=i+1}^5 c_i c_j Cov(C_i, C_j), \quad (50)$$

where $c_3 = 0$ and $var(W) = 0$. Here we note that: First, the $\hat{P}(y)$ and $\hat{P}(y_0)$ are exponential distributions but they in [24] are Gaussian. The second, at zero order in time correlation we not found linear relation between the mean and variance of the change in total entropy, while it exist in [24].

4. The Second Example

In this section, we change variables in the time dependent driving force from x into y which is defined as

$$U(x,t) = \frac{k}{2} \left(x - \frac{f_t}{k} \right)^2, \tag{51}$$

where f_t is arbitrary time depend force and under the new formula of the harmonic potential , we calculate the change in the total entropy production (CTEP), mean and the variance. Substituting Equation (22) in Equation (51), one can obtain,

$$U(y,t) = \frac{1}{2k} \left[\tau \dot{f}_t - \exp \left[-\frac{a^2 ky}{\lambda} \right] \right]^2, \tag{52}$$

Equation (52) represent second new formula of harmonic potential in y .

$$\Delta U(y,t) = \frac{1}{2k} \left[\tau \dot{f}_t - \exp \left[-\frac{a^2 ky}{\lambda} \right] \right]^2 - \frac{1}{2k} \exp \left[-\frac{2a^2 ky_0}{\lambda} \right], \tag{53}$$

where we assume that $\dot{f}(0) = 0$. The work can be computed as,

$$\bar{W} = \frac{1}{k} \int_0^t \left[\tau^2 \dot{f}_s \ddot{f}_s - \tau \ddot{f}_s \exp \left[-\frac{a^2 ky}{\lambda} \right] \right] ds. \tag{54}$$

The mean of the work is calculated as,

$$\langle \bar{W} \rangle = \frac{1}{k} \int_0^t \left[\tau^2 \dot{f}_s \ddot{f}_s - \tau \ddot{f}_s \left[\frac{\lambda}{\lambda + a^3 kD} \right] \right] ds, \tag{55}$$

and the variance of the work is,

$$var(\bar{W}) = \frac{\tau^2}{k^2} \int_0^t \dot{f}_s^2 \left[\frac{\lambda}{\lambda + 2a^3 kD} - \frac{\lambda^2}{(\lambda + a^3 kD)^2} \right] ds. \tag{56}$$

We note that, the mean of the work in first and second example are different, while, the variance is equal. The change in the environment entropy $\Delta \bar{S}_E$ is defined as,

$$\Delta \bar{S}_E = \frac{\bar{W}}{T} + \frac{\exp \left[-\frac{2a^2 ky_0}{\lambda} \right] - \exp \left[-\frac{2a^2 ky}{\lambda} \right]}{2kT} + \frac{\tau \dot{f}_t \exp \left[-\frac{a^2 ky}{\lambda} \right]}{kT} + v_t, \tag{57}$$

where $v_t = -\frac{(\tau \dot{f}_t)^2}{2kT}$. Note that, $S_E - \bar{S}_E = \frac{W - \bar{W}}{T} + \frac{f_t^2}{2kT}$, that mean of the change in entropy of the environment different in two examples. The mean of the change in entropy of the environment is calculated as,

$$\langle \Delta \bar{S}_E \rangle = \frac{1}{kT} \int_0^t \left[\tau^2 \dot{f}_s \ddot{f}_s - \tau \ddot{f}_s \left[\frac{\lambda}{\lambda + a^3 kD} \right] \right] ds + \frac{\tau \dot{f}_t \left[\frac{\lambda}{\lambda + a^3 kD} \right]}{kT} + v_t. \tag{58}$$

The variance of the $\Delta \bar{S}_E$ is,

$$var(\Delta \bar{S}_E) = \frac{var(\bar{W})}{T^2} + \sum_{i=1}^3 c_i^2 var(C_i) + 2 \sum_{i=1}^3 \sum_{j=i+1}^3 c_i c_j Cov(C_i, C_j), \tag{59}$$

since the variance of the ΔS_E is,

$$var(\Delta S_E) = \frac{var(W)}{T^2} + \sum_{i=1}^3 c_i^2 var(C_i) + 2 \sum_{i=1}^3 \sum_{j=i+1}^3 c_i c_j Cov(C_i, C_j), \tag{60}$$

then we note, the variance of the change in entropy of the environment is same in tow examples, while the mean is different. Now, we can calculate $\Delta \bar{S}_{tot}$ as,

$$\Delta\bar{S}_{tot} = \frac{\bar{W}}{T} + \frac{\exp\left[-\frac{2a^2ky_0}{\lambda}\right] - \exp\left[-\frac{2a^2ky}{\lambda}\right]}{2kT} + \frac{\tau\dot{f}_t \exp\left[-\frac{a^2ky}{\lambda}\right]}{kT} + \frac{y-y_0}{aD} + \nu_t. \quad (61)$$

The mean of the $\Delta\bar{S}_{tot}$ is,

$$\langle\Delta\bar{S}_{tot}\rangle = \frac{1}{kT} \int_0^t \left[\tau^2 \dot{f}_s \ddot{f}_s - \tau \ddot{f}_s \left[\frac{\lambda}{\lambda + a^3kD} \right] \right] ds. \quad (62)$$

The variance of the $\Delta\bar{S}_{tot}$ is,

$$\begin{aligned} var(\Delta\bar{S}_{tot}) = & \frac{1}{T^2} \left[\frac{\tau^2}{k^2} \int_0^t \left[\frac{\lambda}{\lambda + 2a^3kD} - \frac{\lambda^2}{(\lambda + a^3kD)^2} \right] \dot{f}_s^2 ds \right] \\ & - \frac{1}{4(kT)^2} \left[\frac{\lambda(aD)^2}{\lambda + 4a^3kD} \right] + \frac{1}{4(kT)^2} \left[\frac{2\lambda}{\lambda + 4a^3kD} - \frac{2\lambda^2}{(\lambda + 2a^3kD)^2} \right] \\ & + \frac{(\tau\dot{f}_t)^2}{(kT)^2} \left[\frac{\lambda}{\lambda + 2a^3kD} - \frac{\lambda^2}{(\lambda + a^3kD)^2} \right] + \frac{2}{(aD)^4} + 2 \sum_{i=1}^5 \sum_{j=i+1}^5 c_i c_j Cov(C_i, C_j). \end{aligned} \quad (63)$$

At zero order in time correlation , the CTEP $\Delta\bar{S}_{tot}$ is,

$$\Delta\bar{S}_{tot} = \frac{\exp\left[-\frac{2ky_0}{\lambda}\right] - \exp\left[-\frac{2ky}{\lambda}\right]}{2kT} + \frac{y-y_0}{D}, \quad (64)$$

the mean is,

$$\langle\Delta\bar{S}_{tot}\rangle = 0, \quad (65)$$

and variance is,

$$var(\Delta\bar{S}_{tot}) = \frac{1}{4(kT)^2} \left[\frac{2\lambda}{\lambda + 4kD} - \frac{2\lambda^2}{(\lambda + 2kD)^2} \right] + \frac{2}{(D)^4} + 2 \sum_{i=1}^5 \sum_{j=i+1}^5 c_i c_j Cov(C_i, C_j), \quad (66)$$

where $W = 0$ and $c_3 = 0$. Here note that: At zero order in time correlation, in first example

$\langle\Delta S_{tot}\rangle = -\frac{1}{k} \int_0^t f_s \dot{b}_s ds$ while in second example $\langle\Delta\bar{S}_{tot}\rangle = 0$, also the $var(\Delta S_{tot})$ and $var(\Delta\bar{S}_{tot})$ in two examples are different. From Equations (59) and (60), we conclude that the variance of the CTEP is the same in two examples and at any order in time correlation and also we can not find any linear relation between the mean and variance of the CTEP at any order in time correlation , while ref. [24] is shown that the entropy variance is same in his two examples at first order in time correlation and it found the relation between the mean and variance of the CTEP at first order in time correlation.

5. Fluctuation Theorem

In this section, we show the fluctuation theorem (FT) is invalid at any order in time correlation whether the distribution function of the change in the total entropy production (CTEP) is Gaussian or non Gaussian. We study the distribution function of the CTEP with respect first example , because any example chosen no problem. We base on the relation between the moments and cumulants to find the distribution function of the CTEP which is defined as,

$$\mathcal{X}_{\Delta S_{tot}} = \exp[H(\kappa)] = \exp\left[i\kappa_1 u - \frac{\kappa_2 u^2}{2} + \dots \right], \quad (67)$$

where $\chi_{\Delta S_{tot}}$ is generating function and $H(\kappa) = i\kappa_1 u - \frac{\kappa_2 u^2}{2} + \dots$ is cumulant function, such that $\kappa_1 = \langle \Delta S_{tot} \rangle$, $\kappa_2 = var(\Delta S_{tot})$, ... are first cumulant, second cumulant, third cumulant and so on. We calculate the distribution function of the CTEP in two perspectives. The first perspective, at second order of approximation in u , the Equation (62) becomes,

$$\chi_{\Delta S_{tot}} = \exp \left[i\kappa_1 u - \frac{\kappa_2 u^2}{2} \right], \quad (68)$$

from above equation, we find that the $\chi_{\Delta S_{tot}}$ is generating function of the Gaussian distribution function, that mean ΔS_{tot} has Gaussian distribution with mean $\langle \Delta S_{tot} \rangle$ and variance $var(\Delta S_{tot})$. The FT is invalid as

$$\frac{P(\Delta S_{tot})}{P(-\Delta S_{tot})} = 1. \quad (69)$$

The other perspective, at any order in u , this perspective is studied in [24], it show that the distribution function of the CTEP is non Gaussian and the FT is invalid as

$$\frac{P(\Delta S_{tot})}{P(-\Delta S_{tot})} \neq \exp[\Delta S_{tot}]. \quad (70)$$

Based on Equation (70), we note that ,at any order in time correlation, the FT in our work is invalid, while at zero order in time correlation, the FT in [24] becomes valid.

6. Conclusion and Future Work

In this letter, we defined the harmonic oscillator with random in non-Markovian thermal bath and we derived the SDE driving by multiplicative colored noise. By changing variables, we translated SDE from multiplicative colored noise into additive colored noise to become the calculations easier. Under the new formulas of the harmonic potential in the two examples, we derived the change in the total entropy production (CTEP) of the our model and calculated the mean and the variance. By comparing our results in the two examples, we found the variances of the CTEP are the same while the means are different. At zero order in the time correlation, in first example the mean of the CTEP equal zero while in other example the mean is nonzero, also we find the variances in the two examples are different. In the two examples we can not obtain on the linear relation between the variance and the mean at any order in time correlation, while [24] it obtained this relation at zero order in the time correlation. The FT in our work is invalid at any order in the time correlation while in [24] the FT is valid at zero order in the time correlation. Finally , we will study the harmonic oscillator with random frequency in Markovian and non-Markovian thermal bath. This problem will be done in future.

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