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Projections and Reflections in Vector Space

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Abstract

We study projections onto a subspace and reflections with respect to a subspace in an arbitrary vector space with an inner product. We give necessary and sufficient conditions for two such transformations to commute. We then generalize the result to affine subspaces and transformations.

Keywords

Projection, Reflection, Commute, Inner Product, Affine Subspace

1. Introduction

Two lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$ are considered. When is the reflection over ℓ_1 followed by the reflection over ℓ_2 the same as the reflection over ℓ_2 followed by the reflection over ℓ_1? It is easy to see that it is the case if and only if $\ell_1 \perp \ell_2$ or $\ell_1 = \ell_2$.

When considering subspaces of $\mathbb{R}^3$, we can ask similar questions for lines, for planes or for the mixed case of one line and one plane. Instead of addressing those cases one by one, we generalize the situation of arbitrary two linear subspaces of a vector space with an inner product.

2. Projection

Supposing that U is a vector space equipped with an inner product, $V \subset U$ is a linear subspace of U. Given a vector $u \in U$, we know from linear algebra [1] [2] that u can be decomposed uniquely as $u = p_V(u) + u'$ where $p_V(u) \in V$ is the projection of the vector u onto V and $u' \perp V$, *i.e.* $U = V \oplus V^\perp$.

Here are some elementary properties of the projection p_V:

1) p_V is linear.
2) $u \in V$ if and only if $p_V(u) = u$.
3) $u \in V^\perp$ if and only if $p_V(u) = 0$.
4) $V \cap W \subset p_V(W)$.

5) If V_1 and V_2 are subspaces of U, then $p_{V_1 \oplus V_2}(u) = p_{V_1}(u) + p_{V_2}(u)$, for all $u \in U$.
6) If V_1, V_2 and W are subspaces of U, then $p_{V_1 \oplus V_2}(W) \subset p_{V_1}(W) \oplus p_{V_2}(W)$.
7) If V_1, V_2 and W are subspaces of U, then $p_W(V_1 \oplus V_2) = p_W(V_1) \oplus p_W(V_2)$.

Lemma 2.1. *Supposing that U is a linear space and V, W are two linear subspaces of U, if* $p_W(V) = p_V(W)$ *then* $p_W(V) = p_V(W) = V \cap W$.

Proof. We first show that $p_V(W) = V \cap W$. Since $p_V(W) \subset V$ and $p_V(W) = p_W(V) \subset W$, we have $p_V(W) \subset V \cap W$. On the other hand, if $u \in V \cap W$, then $u \in V$, hence $u = p_V(u) \in p_V(W)$ and thus $V \cap W \subset p_V(W)$. As a result, $p_V(W) = V \cap W$. The proof of $p_W(V) = V \cap W$ is similar. ■

Suppose U is a vector space and V, W are two subspaces of U. Intersecting the identity $U = (V \cap W) \oplus (V \cap W)^{\perp}$ with V and W, we get $V = (V \cap W) \oplus \left(V \cap (V \cap W)^{\perp}\right)$ and $W = (V \cap W) \oplus \left(W \cap (V \cap W)^{\perp}\right)$. It is obvious that these two sums are orthogonal.

Denote $V' = V \cap (V \cap W)^{\perp}$ and $W' = W \cap (V \cap W)^{\perp}$. Using these notations, $V = (V \cap W) \oplus V'$ and $W = (V \cap W) \oplus W'$.

Lemma 2.2. $p_{V'}(W') = p_{W'}(V') = 0$ *if and only if* $p_W(V) = p_V(W)$.

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$$
\begin{aligned}
p_V(W) &= p_{(V \cap W) \oplus V'}(W) = p_{(V \cap W) \oplus V'}\left((V \cap W) \oplus W'\right) \\
&\subset p_{V \cap W}(V \cap W) \oplus p_{V'}(V \cap W) \oplus p_{V \cap W}(W') \oplus p_{V'}(W') \\
&= (V \cap W) \oplus \{0\} \oplus \{0\} \oplus p_{V'}(W') \\
&= (V \cap W) \oplus p_{V'}(W')
\end{aligned}
$$

($\Rightarrow$) If $p_{V'}(W') = 0$, then $p_V(W) \subset V \cap W$. On the other hand, by the fourth property of projection above, $V \cap W \subset p_V(W)$. Similarly, $p_W(V) = V \cap W$. Thus, $p_V(W) = p_W(V)$.

($\Leftarrow$) By Lemma 2.1, $p_V(W) = V \cap W$. For $w' \in W'$,

$$p_V(w') = p_{V \cap W}(w') + p_{V'}(w') = p_{V'}(w')$$

$p_V(w') \in p_V(W) = V \cap W$ and $p_{V'}(w') \in V'$, but $(V \cap W) \cap V' = 0$, we must have $p_{V'}(w') = 0$, *i.e.* $p_{V'}(W') = 0$. Similarly, $p_{W'}(V') = 0$. ■

Theorem 2.3. *Supposing that U is a vector space and V, W are two subspaces of U, then* $p_V \circ p_W = p_W \circ p_V$ *if and only if* $p_W(V) = p_V(W)$.

Proof. ($\Rightarrow$) Assume that $p_W\left(p_V(u)\right) = p_V\left(p_W(u)\right)$ for all $u \in U$. In particular,

$$p_W(v) = p_W\left(p_V(v)\right) = p_V\left(p_W(v)\right) \in p_V(W) \text{ for all } v \in V.$$

Thus, $p_W(V) \subset p_V(W)$. Similarly, $p_V(W) \subset p_W(V)$.

($\Leftarrow$) Assume $p_W(V) = p_V(W)$. By Lemma 2.2, $p_{V'}(W') = p_{W'}(V') = 0$.

$$
\begin{aligned}
p_W\left(p_V(u)\right) &= p_W\left(p_{V \cap W}(u) + p_{V'}(u)\right) \\
&= p_W\left(p_{V \cap W}(u)\right) + p_W\left(p_{V'}(u)\right) \\
&= p_W\left(p_{V \cap W}(u)\right) + p_{V \cap W}\left(p_{V'}(u)\right) + p_{W'}\left(p_{V'}(u)\right) \\
&= p_{V \cap W}(u).
\end{aligned}
$$

Similarly, $p_V\left(p_W(u)\right) = p_{V \cap W}(u)$. ■

3. Reflection over a Subspace

Supposing that U is a vector space equipped with an inner product, $V \subset U$ is a subspace of U. We define the refection of $u \in U$ with respect to V as

$$r_V(u) = 2p_V(u) - u.$$

The above formula can be easily derived from the observation that $p_V(u) = \frac{1}{2}\left(u + r_V(u)\right)$. Note that if $u \in V$, then $r_V(u) = u$.

Lemma 3.1. *Supposing that U is a vector space and V, W are two vector subspaces of U, then* $r_V \circ r_W = r_W \circ r_V$ *if and only if* $p_V \circ p_W = p_W \circ p_V$.

Proof.

$$r_W\left(r_V\left(u\right)\right) = 2p_W\left(r_V\left(u\right)\right) - r_V\left(u\right) = 2p_W\left(2p_V\left(u\right) - u\right) - \left(2p_V\left(u\right) - u\right)$$
$$= 4p_W\left(p_V\left(u\right)\right) - 2p_W\left(u\right) - 2p_V\left(u\right) + u.$$

Similarly, $r_V\left(r_W\left(u\right)\right) = 4p_V\left(p_W\left(u\right)\right) - 2p_V\left(u\right) - 2p_W\left(u\right) + u$. Hence,

$$r_W \circ r_V = r_V \circ r_W \text{ if and only if } p_W \circ p_V = p_V \circ p_W.$$

Theorem 3.2. *Supposing that U is a vector space and V, W are two subspaces of U, then* $r_V \circ r_W = r_W \circ r_V$ *if and only if* $p_V\left(W\right) = p_W\left(V\right)$.

Poor. By Lemma 3.1, $r_V \circ r_W$ if and only if $p_V \circ p_W = p_W \circ p_V$. By Theorem 2.3, $p_V \circ p_W = p_W \circ p_V$ if and only if $p_V\left(W\right) = p_W\left(V\right)$. ■

4. Projection onto a Translated Subspace

Define the projection of $u \in U$ onto a translated subspace $\hat{V} = V + v_0$ as

$$p_{\hat{V}}\left(u\right) = p_V\left(u - v_0\right) + v_0 = p_V\left(u\right) - p_V\left(v_0\right) + v_0.$$

$p_{\hat{V}}$ is well defined: supposing $V + v_0 = V + v_0'$, then $v_0 - v_0' \in V$. Hence $p_V\left(v_0\right) - p_V\left(v_0'\right) = p_V\left(v_0 - v_0'\right) = v_0 - v_0'$ and thus

$$p_V\left(u\right) - p_V\left(v_0\right) + v_0 = p_V\left(u\right) - p_V\left(v_0'\right) + v_0'.$$

Theorem 4.1. $p_{\hat{V}} \circ p_{\hat{W}} = p_{\hat{W}} \circ p_{\hat{V}}$ *if and only if* $\hat{V} \cap \hat{W} \neq \phi$ *and* $p_V\left(W\right) = p_W\left(V\right)$.

Proof.

$$p_{\hat{V}}\left(p_{\hat{W}}\left(u\right)\right) = p_V\left(p_{\hat{W}}\left(u\right)\right) - p_V\left(v_0\right) + v_0$$
$$= p_V\left(p_W\left(u\right) - p_W\left(w_0\right) + w_0\right) - p_V\left(v_0\right) + v_0$$
$$= p_V\left(p_W\left(u\right)\right) - p_V\left(p_W\left(w_0\right)\right) + p_V\left(w_0\right) - p_V\left(v_0\right) + v_0.$$

Similarly, $p_{\hat{W}}\left(p_{\hat{V}}\left(u\right)\right) = p_W\left(p_V\left(u\right)\right) - p_W\left(p_V\left(v_0\right)\right) + p_W\left(v_0\right) - p_W\left(w_0\right) + w_0$.

Thus, $p_{\hat{V}}\left(p_{\hat{W}}\left(u\right)\right) = p_{\hat{W}}\left(p_{\hat{V}}\left(u\right)\right)$ if and only if

$$\begin{cases} p_V\left(p_W\left(u\right)\right) = p_W\left(p_V\left(u\right)\right) \\ -p_V\left(p_W\left(w_0\right)\right) + p_V\left(w_0\right) - p_V\left(v_0\right) + v_0 = -p_W\left(p_V\left(v_0\right)\right) + p_W\left(v_0\right) - p_W\left(w_0\right) + w_0 \end{cases}$$

(⇒) By Theorem 2.3, the first equation implies $p_V\left(W\right) = p_W\left(V\right)$. The second equation simply means that $\hat{V} \cap \hat{W} \neq \phi$.

(⇐) By Theorem 2.3, the first equation is satisifed. To show the second equation, since $\hat{V} \cap \hat{W} \neq \phi$, we have $\tilde{v} + v_0 = \tilde{w} + w_0$, for some $\tilde{v} \in V$ and $\tilde{w} \in W$, or $v_0 = \tilde{w} + w_0 - \tilde{v}$:

$$-p_W\left(p_V\left(v_0\right)\right) + p_W\left(v_0\right) + p_V\left(v_0\right) - v_0$$
$$= -p_W\left(p_V\left(\tilde{w} + w_0 - \tilde{v}\right)\right) + p_W\left(\tilde{w} + w_0 - \tilde{v}\right) + p_V\left(\tilde{w} + w_0 - \tilde{v}\right) - \tilde{w} - w_0 + \tilde{v}$$
$$= -p_V\left(\tilde{w}\right) - p_W\left(p_V\left(w_0\right)\right) + p_W\left(\tilde{v}\right) + \tilde{w} + p_W\left(w_0\right) - p_W\left(\tilde{v}\right) + p_V\left(\tilde{w}\right) + p_V\left(w_0\right) - \tilde{v} - \tilde{w} - w_0 + \tilde{v}$$
$$= -p_V\left(p_W\left(w_0\right)\right) + p_W\left(w_0\right) + p_V\left(w_0\right) - w_0$$

which is the second equation.

5. Reflection over a Translated Subspace

We next discuss the reflection over a translated subspace. Let $V \subset U$ be a subspace. A translated subspace is $\hat{V} = V + v_0$ for some $v_0 \in U$. We define the reflection of $u \in U$ over $\hat{V}$ as

$$r_{\hat{V}}(u)=r_V(u-v_0)+v_0=r_V(u)-r_V(v_0)+v_0.$$

$r_{\hat{V}}$ is well-defined: supposing $V+v_0=V+v_0'$, then $v_0-v_0'\in V$ and hence $r_V(v_0)-r_V(v_0')=r_V(v_0-v_0')=v_0-v_0'$. As a result,

$$r_V(u)-r_V(v_0)+v_0=r_V(u)-r_V(v_0')+v_0'.$$

Supposing $\hat{W}=W+w_0$ for some $w_0\in W$ is another translated subspace.

$$\begin{aligned} r_{\hat{W}}\left(r_{\hat{V}}(u)\right) &= r_W\left(r_{\hat{V}}(u)\right)-r_W(w_0)+w_0 \\ &= r_W\left(r_V(u)-r_V(v_0)+v_0\right)-r_W(w_0)+w_0 \\ &= r_W\left(r_V(u)\right)-r_W\left(r_V(v_0)\right)+r_W(v_0)-r_W(w_0)+w_0. \end{aligned}$$

Similarly, $r_{\hat{V}}\left(r_{\hat{W}}(u)\right)=r_V\left(r_W(u)\right)-r_V\left(r_W(w_0)\right)+r_V(w_0)-r_V(v_0)+v_0$.

Theorem 5.1. $r_{\hat{W}}\circ r_{\hat{V}}=r_{\hat{V}}\circ r_{\hat{W}}$ *if and only if* $\hat{V}\cap\hat{W}\neq\phi$ *and* $p_V(W)=p_W(V)$.

Proof. $r_{\hat{W}}\left(r_{\hat{V}}(u)\right)=r_{\hat{V}}\left(r_{\hat{W}}(u)\right)$ if and only if

$$\begin{cases} r_W\left(r_V(u)\right)=r_V\left(r_W(u)\right) \\ -p_W\left(p_V(v_0)\right)+p_W(v_0)-p_W(w_0)+w_0=-p_V\left(p_W(w_0)\right)+p_V(w_0)-p_V(v_0)+v_0 \end{cases}$$

(⇒) By Theorem 3.2, $r_V\circ r_W=r_W\circ r_V$ implies $p_V(W)=p_W(V)$. The second equation simply means $\hat{V}\cap\hat{W}\neq\phi$.

(⇐) We express $r_{\hat{W}}\left(r_{\hat{V}}(u)\right)$ and $r_{\hat{V}}\left(r_{\hat{W}}(u)\right)$ in terms of projections:

$$r_{\hat{W}}\left(r_{\hat{V}}(u)\right)=r_W\left(r_V(u)\right)-4p_W\left(p_V(v_0)\right)+4p_W(v_0)+2p_V(v_0)-2v_0-2p_W(w_0)+2w_0,$$

$$r_{\hat{V}}\left(r_{\hat{W}}(u)\right)=r_V\left(r_W(u)\right)-4p_V\left(p_W(w_0)\right)+4p_V(w_0)+2p_W(w_0)-2w_0-2p_V(v_0)+2v_0.$$

By Theorem 3.2, $p_V(W)=p_W(V)$ implies $r_V\circ r_W=r_W\circ r_V$. By Lemma 3.1, we also have $p_V\circ p_W=p_W\circ p_V$. To show $r_{\hat{W}}\circ r_{\hat{V}}=r_{\hat{V}}\circ r_{\hat{W}}$, it suffices to verify the second equation

$$-p_W\left(p_V(v_0)\right)+p_W(v_0)+p_V(v_0)-v_0=-p_V\left(p_W(w_0)\right)+p_V(w_0)+p_W(w_0)-w_0.$$

Since $\hat{V}\cap\hat{W}\neq\phi$, we must have $\tilde{v}+v_0=\tilde{w}+w_0$ for some $\tilde{v}\in V$ and $\tilde{w}\in W$, or $v_0=\tilde{w}+w_0-\tilde{v}$:

$$\begin{aligned} & -p_W\left(p_V(v_0)\right)+p_W(v_0)+p_V(v_0)-v_0 \\ &= -p_W\left(p_V(\tilde{w}+w_0-\tilde{v})\right)+p_W(\tilde{w}+w_0-\tilde{v})+p_V(\tilde{w}+w_0-\tilde{v})-\tilde{w}-w_0+\tilde{v} \\ &= -p_W\left(p_V(\tilde{w})\right)-p_W\left(p_V(w_0)\right)+p_W(\tilde{v})+\tilde{w}+p_W(w_0)-p_W(\tilde{v}) \\ &\quad +p_V(\tilde{w})+p_V(w_0)-\tilde{v}-\tilde{w}-w_0+\tilde{v} \\ &= -p_W\left(p_V(\tilde{w})\right)-p_W\left(p_V(w_0)\right)+p_W(w_0)+p_V(\tilde{w})+p_V(w_0)-w_0 \\ &= -p_V\left(p_W(\tilde{w})\right)-p_V\left(p_W(w_0)\right)+p_W(w_0)+p_V(\tilde{w})+p_V(w_0)-w_0 \\ &= -p_V(\tilde{w})-p_V\left(p_W(w_0)\right)+p_W(w_0)+p_V(\tilde{w})+p_V(w_0)-w_0 \\ &= -p_V\left(p_W(w_0)\right)+p_V(w_0)+p_W(w_0)-w_0 \end{aligned}$$

6. Mixed Transformations

Theorem 6.1. $p_{\hat{V}}\circ r_{\hat{W}}=r_{\hat{W}}\circ p_{\hat{V}}$ *if and only if* $\hat{V}\cap\hat{W}\neq\phi$ *and* $p_V(W)=p_W(V)$.
Theorem 6.2. $p_{\hat{V}}\circ r_{\hat{W}}=r_{\hat{V}}\circ p_{\hat{W}}$ *if and only if* $\hat{V}\cap\hat{W}\neq\phi$ *and* $V=W$.
Theorem 6.3. $p_{\hat{V}}\circ r_{\hat{W}}=p_{\hat{W}}\circ r_{\hat{V}}$ *if and only if* $\hat{V}\cap\hat{W}\neq\phi$ *and* $V=W$.

7. Generalizations

If we denote Σ_n, the permutation group of order *n*, then

Theorem 7.1.

$$p_{V_{\sigma(1)}} \circ \cdots \circ p_{V_{\sigma(n)}} = p_{V_{\tau(1)}} \circ \cdots \circ p_{V_{\tau(n)}} \text{ for all } \sigma, \tau \in \Sigma_n$$

if and only if

$$p_{V_{\sigma(1)}} \cdots p_{V_{\sigma(n-1)}}\left(V_{\sigma(n)}\right) = p_{V_{\tau(1)}} \cdots p_{V_{\tau(n-1)}}\left(V_{\tau(n)}\right) \text{ for all } \sigma, \tau \in \Sigma_n.$$

Theorem 7.2.

$$r_{V_{\sigma(1)}} \circ \cdots \circ r_{V_{\sigma(n)}} = r_{V_{\tau(1)}} \circ \cdots \circ r_{V_{\tau(n)}} \text{ for all } \sigma, \tau \in \Sigma_n$$

if and only if

$$p_{V_{\sigma(1)}} \cdots p_{V_{\sigma(n-1)}}\left(V_{\sigma(n)}\right) = p_{V_{\tau(1)}} \cdots p_{V_{\tau(n-1)}}\left(V_{\tau(n)}\right) \text{ for all } \sigma, \tau \in \Sigma_n.$$

Theorem 7.3.

$$p_{\hat{V}_{\sigma(1)}} \circ \cdots \circ p_{\hat{V}_{\sigma(n)}} = p_{\hat{V}_{\tau(1)}} \circ \cdots \circ p_{\hat{V}_{\tau(n)}} \text{ for all } \sigma, \tau \in \Sigma_n$$

if and only if $\hat{V}_i \cap \hat{V}_j \neq \phi$ *for all* i, j *and*

$$p_{V_{\sigma(1)}} \cdots p_{V_{\sigma(n-1)}}\left(V_{\sigma(n)}\right) = p_{V_{\tau(1)}} \cdots p_{V_{\tau(n-1)}}\left(V_{\tau(n)}\right) \text{ for all } \sigma, \tau \in \Sigma_n.$$

Theorem 7.4.

$$r_{\hat{V}_{\sigma(1)}} \circ \cdots \circ r_{\hat{V}_{\sigma(n)}} = r_{\hat{V}_{\tau(1)}} \circ \cdots \circ r_{\hat{V}_{\tau(n)}} \text{ for all } \sigma, \tau \in \Sigma_n$$

if and only if $\hat{V}_i \cap \hat{V}_j \neq \phi$ *for all* i, j *and*

$$p_{V_{\sigma(1)}} \cdots p_{V_{\sigma(n-1)}}\left(V_{\sigma(n)}\right) = p_{V_{\tau(1)}} \cdots p_{V_{\tau(n-1)}}\left(V_{\tau(n)}\right) \text{ for all } \sigma, \tau \in \Sigma_n.$$

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