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Quasinilpotent Part of w-Hyponormal Operators

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Abstract

For a w-hyponormal operator T acting on a separable complex Hilbert space H, we prove that: 1) the quasi-nilpotent part $\mathrm{H}_0(T-\lambda I)$ is equal to $\ker(T-\lambda I)$; 2) T has Bishop's property β; 3) if $\sigma_w(T)=\{0\}$, then it is a compact normal operator; 4) If T is an algebraically w-hyponormal operator, then it is polaroid and reguloid. Among other things, we prove that if T^n and T^{n^*} are w-hyponormal, then T is normal.

Keywords

Aluthge Transformation, w-Hyponormal Operators, Polaroid Operators, Reguloid Operators, SVEP, Property β, Quasinilpotent Part

Subject Areas: **Functional Analysis, Mathematical Analysis**

1. Introduction

Let H be a complex Hilbert space and let $\mathrm{B}(\mathrm{H})$ be the algebra of all bounded linear operators acting on H. If $T\in\mathrm{B}(\mathrm{H})$ we shall write $\ker(T)$ and $\Re(T)$ for the null spaceand range of T, respectively. Also, let $\alpha(T):=\dim\ker(T)$, $\beta(T):=\operatorname{codim}\Re(T)$, and let $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ denote the spectrum, approximate point spectrum andpoint spectrum of T, respectively. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in\mathrm{H}$ and also T is said to be strictly positive (denoted by $T>0$) if T is positive and invertible. An operator T is called p-hyponormal if $|T|^{2p}\geq|T^*|^{2p}$ for every $0<p\leq 1$. It is easily to see that every p-hyponormal is q-hyponormal for $p\geq q>0$ by Löwner-Heinz theorem "$A\geq B\geq 0$ ensures $A^\alpha\geq B^\alpha$ for any $\alpha\in[0,1]$". Let T be a p-hyponormal operator whose polar decomposition is $T=U|T|$. Aluthge [1] introduced the operator $\tilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which called the Aluthge transformation, and also showed the following result.

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Proposition 1.1. *Let* $T=U|T|\in B(H)$ *be the polar decomposition of a* p*-hyponormal for* $0<p<1$ *and* U *is unitary. Then the following assertions hold*:

1) $\tilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $\left(p+\frac{1}{2}\right)$-hyponormal if $0<p<\frac{1}{2}$.

2) $\tilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is 1-hyponormal if $\frac{1}{2}\le p<1$.

As a natural generalization of Aluthge transformation Ito [2] introduced the operator $\tilde{T}_{s,t}=|T|^{s}U|T|^{t}$ for $s>0$ and $t>0$. Recall [3], an operator $T\in B(H)$ is said to be w-hyponormal if $|\tilde{T}|\ge|T|\ge|\tilde{T}^{*}|$. We remark that w-hyponormal operator is defined by using Aluthge transformation $\tilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. w-hyponormal was defined by Aluthge and Wang [3] and the following theorem is shown in [3].

Theorem 1.2. *Let* $T\in B(H)$.

1) If T is a p-hyponormal operator for $p>0$, then T is w-hyponormal.

2) If T is w-hyponormal operator, then $|T^{2}|\ge|T|^{2}$ and $|T^{*}|^{2}\ge|T^{*2}|$ hold.

3) If T is w-hyponormal operator, then T^{-1} is also w-hyponormal.

Let $\lambda\in C$. The quasinilpotent part of $T-\lambda I$ is defined as

$$H_{0}(T-\lambda I)=\left\{x\in H:\lim_{n\to\infty}\left\|(T-\lambda I)^{n}x\right\|^{\frac{1}{n}}=0\right\}.$$

In general, $\ker(T-\lambda I)\subset H_{0}(T-\lambda I)$ and $H_{0}(T-\lambda I)$ is not closed. However, it is known that if T is hyponormal, then $H_{0}(T-\lambda I)=\ker(T-\lambda I)\subset\ker(T-\lambda I)^{*}$.

In this paper, we characterize the quasinilpotent part of w-hyponormal. This is a generalization of the hyponormal operator case.

2. Basic Properties of w-Hyponormal Operators

In this section we prove basic properties of w-hyponormal operators. These properties are induced by the following famous inequalities.

Lemma 2.1. (*Hansen inequality*). *If* $A, B\in B(H)$ *satisfy* $A\ge 0$ *and* $\|B\|\le 1$, *then* $(B^{*}AB)^{\alpha}\ge B^{*}A^{\alpha}B$ *for all* $\alpha\in(0,1]$.

Theorem 2.2. *Let* $T\in B(H)$ *be a* w*-hyponormal operator and* M *be its invariant subspace. Then the restriction* $T|_{M}$ *of T to* M *is also a* w*-hyponormal operator.*

Proof. Decompose T as

$$T=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}\quad\text{on}\quad H=M\oplus M^{\perp}.$$

Let $Q=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ be the orthogonal projection onto M. Since $A=TQ|_{M}$ we have $A^{*}A=QT^{*}TQ$. By Hansen's inequality we have

$$\begin{pmatrix}(A^{*}A)^{p} & 0\\ 0 & 0\end{pmatrix}=(QT^{*}TQ)^{p}\ge Q(T^{*}T)^{p}Q$$

while $AA^{*}=TQT^{*}=QTQT^{*}Q$. So we have

$$(AA^{*})^{p}=(TQT^{*})^{p}=Q(TQT^{*})^{p}Q\le Q(TT^{*})^{p}Q\quad\text{for all}\quad p\in(0,1].$$

Since T is W-hyponormal then $\tilde{T}$ is semi-hyponormal and hence $\tilde{A}=\tilde{T}|_{M}$ is semi-hyponormal by ([4], Lemma 4). Hence

$$\left|\tilde{A}\right| \geq \left|\tilde{A}^*\right|.$$

Now

$$\left|\tilde{A}\right| = \left|\tilde{T}\right|_M \geq \left|T\right|_M = \left|A\right|$$

also

$$\left|\tilde{A}^*\right| = \left|\tilde{T}^*\right|_M \leq \left|T\right|_M = \left|A\right|.$$

Therefore, A is w-hyponormal.

As a generalization of w-hyponormal operators, Ito [2] introduced a new class of operators as follows:

Definition 2.1. *For each* $s > 0$ *and* $t > 0$, *an operator* T *belongs to class* $wA(s,t)$ *if an operator* T *satisfies*

$$\left(\left|T^*\right|^t \left|T\right|^{2s} \left|T^*\right|^t\right)\frac{t}{t+s} \geq \left|T^*\right|^{2t} \tag{2.1}$$

and

$$\left|T\right|^{2s} \geq \left(\left|T\right|^s \left|T^*\right|^{2t} \left|T\right|^s\right)^{\frac{s}{s+t}}. \tag{2.2}$$

The following theorem on $\tilde{T}_{s,t}$ is a generalization of Proposition 1.1.

Theorem 2.3. *Let* $T = U\left|T\right|$ *be the polar decomposition of a* w*-hyponormal operator. Then* $\tilde{T}_{s,t}$ *is* $\dfrac{\min\{s,t\}}{s+t}$*-hyponormal for* $s \geq \dfrac{1}{2}$ *and* $t \geq \dfrac{1}{2}$.

In order to give the proof of Theorem 2.3, we need the following lemma from [2].

Lemma 2.4. *Let* $A \geq 0$ *and* $T = U\left|T\right|$ *be the polar decomposition of* T. *Then for each* $\alpha > 0$ *and* $\beta > 0$, *the following assertion holds*:

$$\left(U\left|T\right|^{\beta} A\left|T\right|^{\beta} U^*\right)^{\alpha} = U\left(\left|T\right|^{\beta} A\left|T\right|^{\beta}\right)^{\alpha} U^*.$$

Proof of Theorem 2.3. Suppose that T is w-hyponormal, then T belongs to class $wA(s,t)$ for each $s \geq \dfrac{1}{2}$ and $t \geq \dfrac{1}{2}$. Hence

$$\begin{aligned}
\left(\tilde{T}^*_{s,t}\tilde{T}_{s,t}\right)^{\frac{\min\{s,t\}}{s+t}} = \left(\left|T\right|^t U^* \left|T\right|^{2s} U\left|T\right|^t\right)^{\frac{\min\{s,t\}}{s+t}} &= \left(U^*U\left|T\right|^t U^* \left|T\right|^{2s} U\left|T\right|^t U^*U\right)^{\frac{\min\{s,t\}}{s+t}} \\
&= U^*\left(U\left|T\right|^t U^* \left|T\right|^{2s} U\left|T\right|^t U^*\right)^{\frac{\min\{s,t\}}{s+t}} U \quad (\text{By Lemma 2.4}) \\
&= U^*\left(\left|T^*\right|^t U^* \left|T\right|^{2s} U\left|T^*\right|^t\right)^{\frac{\min\{s,t\}}{s+t}} U \\
&\geq U^*\left|T^*\right|^{2\min\{s,t\}} U.
\end{aligned}$$

Thus

$$\left|\tilde{T}_{s,t}\right|^{\frac{\min\{s,t\}}{s+t}} \geq \left|T\right|^{2\min\{s,t\}} \tag{2.3}$$

and the last inequality holds by Equation (2.2) and Löwner-Heinz theorem.

On the other hand

$$\left(\tilde{T}_{s,t}\tilde{T}^*_{s,t}\right)^{\frac{\min\{s,t\}}{s+t}} = \left(\left|T\right|^s U\left|T\right|^{2t} U^* \left|T\right|^s\right)^{\frac{\min\{s,t\}}{s+t}} = \left(\left|T\right|^s \left|T^*\right|^{2t} \left|T\right|^s\right)^{\frac{\min\{s,t\}}{s+t}}.$$

Hence

$$\left|\tilde{T}_{s,t}^{*}\right|^{\frac{\min\{s,t\}}{s+t}}\le|T|^{2\min\{s,t\}} \tag{2.4}$$

and the last inequality holds by Equation (2.1) and Löwner-Heinz theorem.

Therefore Equations (2.3) and (2.4) ensure

$$\left|\tilde{T}_{s,t}\right|^{\frac{\min\{s,t\}}{s+t}}\ge|T|^{2\min\{s,t\}}\ge\left|\tilde{T}_{s,t}^{*}\right|^{\frac{\min\{s,t\}}{s+t}}.$$

That is, $\tilde{T}_{s,t}$ is $\dfrac{\min\{s,t\}}{s+t}$-hyponormal.

Theorem 2.5. *Let* $T=U|T|$ *be the polar decomposition of* w*-hyponormal operator. Then*

$$\left\|\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}-\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right\|\le\phi\left(\frac{1}{p}\right)\left\|\tilde{T}_{s,t}\right\|^{2(1-p)}\min\left\{\frac{p}{\pi}\int_{\sigma(\tilde{T}_{s,t})}r^{2p-1}\mathrm{d}r\mathrm{d}\theta,\frac{1}{\pi^{p}}\left(\int_{\sigma(\tilde{T}_{s,t})}r\mathrm{d}r\mathrm{d}\theta\right)^{p}\right\}.$$

Moreover, if T is invertible w-hyponormal, then

$$\left\|\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}-\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right\|\le\left\|\tilde{T}_{s,t}\right\|^{2}\frac{1}{\pi}\int_{\sigma(\tilde{T}_{s,t})}r^{-1}\mathrm{d}r\mathrm{d}\theta.$$

If we use $\int_{\sigma(\tilde{T}_{s,t})}r^{-1}\mathrm{d}r\mathrm{d}\theta\le\left\|\tilde{T}_{s,t}^{-1}\right\|^{2}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)$, we have also

$$\left\|\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}-\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right\|\le\left(\left\|\tilde{T}_{s,t}\right\|\left\|\tilde{T}_{s,t}^{-1}\right\|\right)^{2}\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)^{p}$$

where $p=\dfrac{\min\{s,t\}}{s+t}$ and $\phi(p)=\begin{cases}p, & \text{if } p\in\mathbb{N};\\ p+2, & \text{otherwise.}\end{cases}$

Proof. Let $p=\dfrac{\min\{s,t\}}{s+t}$. Since $\tilde{T}_{s,t}$ is p-hyponormal operator By Lemma 2 and Proposition 1 of [5]

$$\begin{aligned}\left\|\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}-\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right\|&\le\phi(1/p)\left\|\left(\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}\right)^{p}\right\|^{\frac{1}{p}-1}\left\|\left(\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}\right)^{p}-\left(\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right)^{p}\right\|\\&\le\phi(1/p)\left\|\tilde{T}_{s,t}\right\|^{2p\left(\frac{1}{p}-1\right)}\min\left\{\frac{p}{\pi}\int_{\sigma(\tilde{T}_{s,t})}r^{2p-1}\mathrm{d}r\mathrm{d}\theta,\frac{1}{\pi^{p}}\left(\int_{\sigma(\tilde{T}_{s,t})}r\mathrm{d}r\mathrm{d}\theta\right)^{p}\right\}\\&=\phi(1/p)\left\|\tilde{T}_{s,t}\right\|^{2(1-p)}\min\left\{\frac{p}{\pi}\int_{\sigma(\tilde{T}_{s,t})}r^{2p-1}\mathrm{d}r\mathrm{d}\theta,\frac{1}{\pi^{p}}\left(\int_{\sigma(\tilde{T}_{s,t})}r\mathrm{d}r\mathrm{d}\theta\right)^{p}\right\}.\end{aligned}$$

Next, we assume that $\tilde{T}_{s,t}$ is invertible. Since every p-hyponormal operator is q-hyponormal operator if $0<q\le p$, by above

$$\left\|\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}-\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right\|\le\phi(1/q)\left\|\tilde{T}_{s,t}\right\|^{2(1-q)}\frac{q}{\pi}\int_{\sigma(\tilde{T}_{s,t})}r^{2q-1}\mathrm{d}r\mathrm{d}\theta=\left(\frac{1}{q}+2\right)\frac{q}{\pi}\int_{\sigma(\tilde{T}_{s,t})}r^{2q-1}\mathrm{d}r\mathrm{d}\theta.$$

Letting $q\downarrow 0$, we have the result.

Let $\Re(\sigma(T))$ denotes the set of all rational functions on $\sigma(T)$. The operator T *is said to be* n*-multicyclic if there are* n vectors $x_1,\cdots,x_n\in H$, called generating vectors, such that $\vee\left\{g(T)x_i:i=1,\cdots,n,g\in\Re(\sigma(T))\right\}=H$.

Theorem 2.6. *If* T *is* w*-hyponormal operator. Then*

$$\left\|\left(\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}\right)^{p}-\left(\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right)^{p}\right\|\le\left(\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)\right)^{p}$$

where $p=\dfrac{\min\{s,t\}}{s+t}$.

Proof. Since $\tilde{T}_{s,t}$ is $\dfrac{\min\{s,t\}}{s+t}$-hyponormal operator, let x be an arbitrary unit vector in H. We define

$$H_0=\vee\left\{g\left(\tilde{T}_{s,t}\right)x:g\in\mathfrak{R}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)\right\}.$$

Since H_0 *is an invariant subspace for* $\tilde{T}_{s,t}$, *Lemma* 4 *of* [4] *implies that* $T'=\tilde{T}_{s,t}\big|_{H_0}$ *is a* (1-*multicyclic*) p-*hyponormal operator*. *If* $\lambda\in\rho\left(\tilde{T}_{s,t}\right)$, *then for any* $y\in H_0$, $\left(\tilde{T}_{s,t}-\lambda\right)^{-1}y\in H_0$. *Therefore*, $\lambda\in\rho\left(T'\right)$. *Hence*, $\sigma\left(T'\right)\subset\sigma\left(\tilde{T}_{s,t}\right)$. *By Berger-Shaw's Theorem* [4],

$$tr\left(\left\{\left(T'^{*}T'\right)^{p}-\left(T'T'^{*}\right)^{p}\right\}^{\frac{1}{p}}\right)\le\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(T'\right)\right)\le\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right).$$

And the maximal eigenvalues of positive trace class operator $\left\{\left(T'^{*}T'\right)^{p}-\left(T'T'^{*}\right)^{p}\right\}^{\frac{1}{p}}$ is equal to or less than $\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)$. Thus, the maximal eigenvalue of $\left(T'^{*}T'\right)^{p}-\left(T'T'^{*}\right)^{p}$ is equal to or less than $\left\{\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)\right\}^{p}$. Therefore,

$$\left\|\left(T'^{*}T'\right)^{p}-\left(T'T'^{*}\right)^{p}\right\|\le\left\{\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)\right\}^{p}.$$

Let P be the projection onto H_0. Then, by Lemma 4 of [4],

$$\begin{aligned}\left\{\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)\right\}^{p}&\ge\left\langle\left\{\left(T'^{*}T'\right)^{p}-\left(T'T'^{*}\right)^{p}\right\}x,x\right\rangle\\&\ge\left\langle\left\{P\left(\tilde{T}^{*}_{s,t}\tilde{T}_{s,t}\right)^{p}P-P\left(\tilde{T}_{s,t}\tilde{T}^{*}_{s,t}\right)^{p}P\right\}x,x\right\rangle\\&=\left\langle\left(\tilde{T}^{*}_{s,t}\tilde{T}_{s,t}\right)\left\{0^{p}-\left(\tilde{T}_{s,t}\tilde{T}^{*}_{s,t}\right)^{p}\right\}x,x\right\rangle.\end{aligned}$$

Since $x\in H$ is arbitrary unit vector,

$$\left\|\left(\tilde{T}^{*}_{s,t}\tilde{T}_{s,t}\right)^{p}-\left(\tilde{T}_{s,t}\tilde{T}^{*}_{s,t}\right)^{p}\right\|\le\left\{\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(\tilde{T}_{s,t}\right)\right)\right\}^{p}.$$

Corollary 2.7. *Let* T *be* w-*hyponormal operator. Then*

$$\left\|\left|\tilde{T}\right|-\left|\tilde{T}^{*}\right|\right\|\le\frac{1}{\pi}\operatorname{Area}\left(\sigma\left(T\right)\right).$$

Moreover, if $\operatorname{Area}\left(\sigma\left(T\right)\right)=0$, then T is normal.

Theorem 2.8. *Let* T *be a* w-*hyponormal operator*. *If* M *is an invariant subspace of* T and $T|_M$ is an injective normal operator, then M reduces T.

Proof. Decompose T into

$$T=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}\quad\text{on}\quad \mathrm{H}=M\oplus M^{\perp}$$

and let $A=T|_M$ be injective normal operator. Let Q be the orthogonal projection of H onto M. Since $\ker\left(A\right)=\ker\left(A^{*}\right)=\{0\}$, we have $M=\overline{\mathfrak{R}\left(A\right)}$.

Then

$$\begin{pmatrix} |A|^2 & 0 \\ 0 & 0 \end{pmatrix} = Q|T|^2 Q \le Q|T^2|Q \le \left|Q|T^2|^2 Q\right|^{\frac{1}{2}} = \begin{pmatrix} |A^2| & 0 \\ 0 & 0 \end{pmatrix}$$

by Hansen's inequality. Since A is normal we can write

$$|T^2| = \begin{pmatrix} |A|^2 & S \\ S^* & D \end{pmatrix}.$$

Then

$$\begin{pmatrix} |A|^4 & 0 \\ 0 & 0 \end{pmatrix} = QT^*T^*TTQ = Q|T^2|^2 Q = \begin{pmatrix} |A|^4 + |C|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and $S = 0$. Hence

$$\begin{pmatrix} |A|^4 & 0 \\ 0 & D^2 \end{pmatrix} = |T^2|^2 = T^*T^*TT = \begin{pmatrix} |A^2|^2 & A^{*2}(AB+BC) \\ (AB+BC)^* A^2 & (AB+BC)^*(AB+BC) + |C^2|^2 \end{pmatrix}.$$

Since A is an injective normal operator, $AB + BC = 0$ and $D = |C^2|$.

$$0 \le |T^2| - |T|^2 = \begin{pmatrix} |A^2| - |A|^2 & -A^*B \\ -B^*A & -|B|^2 \end{pmatrix}$$

thus $B = 0$.

Theorem 2.9. *If T and T^* are w-hyponormal operators, then T is normal.*

In order to give the proof of Theorem 2.9, we need the following lemma from [6].

Lemma 2.10. *Let $A \ge 0$ and $B \ge 0$. If*

$$B^{\frac{1}{2}} A B^{\frac{1}{2}} \ge B^2 \tag{2.5}$$

and

$$A^{\frac{1}{2}} B A^{\frac{1}{2}} \ge A^2 \tag{2.6}$$

then $A = B$.

Proof of Theorem 2.9. Since T is w-hyponormal then we have from ([7], Corollary 1.2) that

$$|T| \ge \left(|T|^{\frac{1}{2}} |T^*| |T|^{\frac{1}{2}}\right)^{\frac{1}{2}} \quad \text{and} \quad |T^*| \le \left(|T^*|^{\frac{1}{2}} |T| |T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{2.7}$$

Similarly, since T^* is w-hyponormal, we have

$$|T^*| \ge \left(|T^*|^{\frac{1}{2}} |T| |T^*|^{\frac{1}{2}}\right)^{\frac{1}{2}} \quad \text{and} \quad |T| \le \left(|T|^{\frac{1}{2}} |T^*| |T|^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{2.8}$$

From Equations (2.7) and (2.8) and Lemma 2.10 we conclude $|T| = |T^*|$. Therefore, T is normal.

In the following result, 1) and 2) are due to [2], 3) and 4) to [8].

Lemma 2.11. *Let $T \in B(H)$.*

1) For each $s > 0$ and $t > 0$. If T belongs to class $wA(s,t)$, then T belongs to class $wA(\alpha,\beta)$ for each $\alpha \ge s$ and $\beta \ge t$.

2) T is a class $wA\left(\frac{1}{2},\frac{1}{2}\right)$ operator if and only if T is a w-hyponormal operator.

3) Let T be a w-hyponormal operator. Then T^n is also w-hyponormal for all positive integer n.

4) Let T be a class $wA(s,t)$ operator for $s \in [0,1]$ and $t \in (0,1]$.

Then T^n belongs to class $wA\left(\frac{s}{n},\frac{t}{n}\right)$ for all positive integer n.

Let $\mathrm{Hol}(\sigma(T))$ be the space of all functions that analytic inan open neighborhoods of $\sigma(T)$. Following [9]. Wesay that $T\in B(H)$ has the single-valued extension property (SVEP) at point $\lambda\in\mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f:U_\lambda\to H$ which satisfies the equation $(T-\mu)f(\mu)=0$ is the constant function $f\equiv 0$. It is well-known that $T\in B(H)$ has SVEP at every point of the resolvent $\rho(T):=C\setminus\sigma(T)$. Moreover, from the identity Theoremfor analytic function it easily follows that $T\in B(H)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In ([10], Proposition 1.8), Laursen proved that if T is of finite ascent, then T has SVEP.

Definition 2.2. [11] *An operator T is said to have Bishop's property (β) at $\lambda\in C$ if for every open neighborhood G of λ, the function $f_n\in\mathrm{Hol}(G)$ with $(T-\lambda)f_n(\mu)\to 0$ uniformly on every compact subset of G implies that $f_n(\mu)\to 0$ uniformly on every compact subset of G, where $\mathrm{Hol}(G)$ means the space of all analytic functions on G. When T has Bishop's property (β) at each $\lambda\in C$, simply say that T has property (β).*

Lemma 2.12. [12] Let G be open subset of complex plane $\mathbb{C}$ and let $f_n\in\mathrm{Hol}(G)$ be functions such that $\mu f_n(\mu)\to 0$ uniformly on every compact subset of G, then $f_n(\mu)\to 0$ uniformly on every compact subset of G.

Remark: *The relations between T and its transformation $\tilde{T}$ are*

$$\tilde{T}|T|^{\frac{1}{2}}=|T|^{\frac{1}{2}}U|T|=|T|^{\frac{1}{2}}T \tag{2.9}$$

and

$$U|T|^{\frac{1}{2}}\tilde{T}=U|T|U|T|^{\frac{1}{2}}=TU|T|^{\frac{1}{2}}. \tag{2.10}$$

It is shown in [13] that every p-hyponormal operator has Bishop's property (β).

Theorem 2.13. *Let $T\in B(H)$ be w-hyponormal. Then T has the property (β). Hence T has SVEP.*

Proof. Since $\tilde{T}$ is semi-hyponormal by ([3], Theorem 2.4), it is suffices to show that T has property (β) if and only if $\tilde{T}$ has property (β). Suppose that $\tilde{T}$ has property β. Let G be an open neighborhood of λ and let $f_n\in\mathrm{Hol}(G)$ be functions such that $(\mu-T)f_n(\mu)\to 0$ uniformly on every compact subset of G. By Equation (2.9), $(\tilde{T}-\mu)|T|^{\frac{1}{2}}f_n(\mu)=|T|^{\frac{1}{2}}(T-\mu)f_n(\mu)\to 0$ uniformly on every compact subset of G. Hence $Tf_n(\mu)=U|T|f_n(\mu)\to 0$ uniformly on every compact subset of G, and T having property β follows by Lemma 2.12. Suppose that T has property (β). Let G be an open neighborhood of λ and let $f_n\in\mathrm{Hol}(G)$ be functions such that $(\mu-\tilde{T})f_n(\mu)\to 0$ uniformly on every compact subset of G. By Equation (2.10), since $(\mu-T)\left(|T|^{\frac{1}{2}}f(\mu)\right)=U|T|^{\frac{1}{2}}(\mu-\tilde{T})f_n(\mu)\to 0$ uniformly on every compact subset of G. Hence $Tf_n(\mu)=U|T|f_n(\mu)\to 0$ uniformly on every compact subset of G for T has property (β), so that $\mu f_n(\mu)\to 0$ uniformly on every compact subset of G, and $\tilde{T}$ has property (β) follows by Lemma 2.12.

Theorem 2.14. *Let T be w-hyponormal. Then $H_0(T-\lambda I)=\ker(T-\lambda I)$ for $\lambda\in C$.*

Proof. Let $F\subset C$ be closed set. Define the global spectral subspace by

$$X_T(F)=\{x\in H|\exists\text{ analytic } f(z):(T-zI)f(z)=x\text{ on } C\setminus F\}.$$

It is known that $H_0(T-\lambda I)=X_T(\{\lambda\})$ by ([14], Theorem 2.20). As T has Bishop's property (β) by Theorem 2.13, $X_T(F)$ is closed and $\sigma\left(T|_{X_T(F)}\right)\subset F$ by ([15], Proposition 1.2.19). Hence $H_0(T-\lambda I)$ is closed and $T|_{H_0(T-\lambda I)}$ is w-hyponormal by Theorem 2.2. Since $\sigma\left(T|_{H_0(T-\lambda I)}\right)\subset\{\lambda\}$, $T|_{H_0(T-\lambda I)}$ is normal by Corollary 2.7. If $\sigma\left(T|_{H_0(T-\lambda I)}\right)=\varnothing$, then $H_0(T-\lambda I)=\{0\}$ and $\ker(T-\lambda I)=\{0\}$ If $\sigma\left(T|_{H_0(T-\lambda I)}\right)=\{\lambda\}$, then $T|_{H_0(T-\lambda I)}=\lambda I$ and $H_0(T-\lambda I)\subset\ker(T-\lambda I)$.

Remark 2.15. *If* $\lambda \neq 0$, *then* $\mathrm{H}_0(T-\lambda I) = \ker(T-\lambda I) \subset \ker(T-\lambda I)^*$. *Moreover, if* $\lambda \in \sigma(T)\setminus\{0\}$ *is an* isolated point then $\mathrm{H}_0(T-\lambda I) = \ker(T-\lambda I) \subset \ker(T-\lambda I)^*$.

Example 2.16. Let A and B be $n \times n$ matrices and satisfy $A \geq B \geq 0$. Let $H = \bigoplus_{j=-\infty}^{\infty} H_j$, where $H_j = C^n$ for every $j \in \mathbb{Z}$. Let U be the bilateral shift on H, that is $(Ux)_n = x_{n-1}$, where $x = (\cdots, x_{-1}, x_0, x_1, \cdots) \in H$. Let $\{P_j\}$ be

$$P_j = \begin{cases} B & \text{if } j \leq 0 \\ A \in & \text{if } j \geq 1. \end{cases}$$

We define $(Px)_j = P_j x_j$ for $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$ and let $T = UP$. Then T is w-hyponormal and so $H_0(T-\lambda) = \mathrm{Ker}(T-\lambda)$.

Proposition 2.17. [3] *Let* T *be* w*-hyponormal. Then* $(T-\lambda I)x = 0$ *implies* $(T-\lambda I)^* x = 0$.

3. Variants of Weyl's Theorems

An operator $T \in \mathrm{B}(\mathrm{H})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T)$$

T is called *Weyl* if it is Fredholm of index 0, and Browder if it is Fredholm "of finite ascent and descent". Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\Re(T^q) = \Re(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. The essential spectrum $\sigma_F(T)$, the Weyl spectrum $w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T) = \{\lambda \in \mathrm{C} : T - \lambda \text{ is not Fredholm}\}$$

$$\sigma_W(T) = \{\lambda \in \mathrm{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathrm{C} : T - \lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup acc\sigma(T)$$

where we write $accK$ for the accumulation points of $K \subseteq \mathrm{C}$. Following [16], we say that *Weyl's theorem* holds for T if $\sigma(T)\setminus\sigma_W(T) = E_0(T)$, where $E_0(T)$ is the set of all eigenvalues λ of finite multiplicity isolated in $\sigma(T)$. And *Browder's theorem* holds for T if $\sigma(T)\setminus\sigma_W(T) = \pi_0(T)$, where $\pi_0(T)$ is the set of all poles of T of finite rank.

Theorem 3.1. *If* T *is* w*-hyponormal operator with* $\sigma_W(T) = \{0\}$, *then it is a compact normal operator.*

Proof. Since Weyls theorem holds for T by ([17], Theorem 3.4), each element in $\sigma(T)\setminus\sigma_W(T) = \sigma(T)\setminus\{0\}$ is an eigenvalue of T with finite multiplicity, and is isolated in $\sigma(T)$. This implies that $\sigma(T)\setminus\{0\}$ is a finite set or a countable infinite set with 0 as its only accumulation point. Put $\sigma(T)\setminus\{0\} = \{\lambda_n\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\{|\lambda_n|\}$ is a non-increasing sequence. Since T is normaloid, we have $|\lambda_1| = \|T\|$. By ([3], Theorem 3.2), $(T-\lambda_1)x = 0$ implies $(T-\lambda_1)^* x = 0$. In fact,

$$\left\| \left| \|T\|^2 - T^*T \right|^{\frac{1}{2}} x \right\| = \|T\|^2 \|x\|^2 - \|Tx\|^2 = \|T\|^2 \|x\|^2 - \|\lambda_1 x\|^2 = 0$$

$\lambda_1 T^* x = T^* T x = \|T\|^2 x = |\lambda_1|^2 x$ and $T^* x = \overline{\lambda_1} x$. Hence $\ker(T-\lambda_1)$ is a reducing subspace of T. Let P_1 be the orthogonal projection onto $\ker(T-\lambda_1)$. Then $T = \lambda_1 \oplus T_1$ on $\mathrm{H} = \Re(P_1) \oplus \Re(I-P_1)$. Since T_1 is w-hyponormal operator and $\sigma_p(T) = \sigma_p(T_1) \cup \{\lambda_1\}$, we have $\lambda_2 \in \sigma_p(T_1)$. By the same argument as above, $\ker(T-\lambda_2) = \ker(T_1-\lambda_2)$ is a finite dimensional reducing subspace of T which is included in $\Re(I-P_1)$.

Put P_2 be the orthogonal projection onto $\ker(T-\lambda_2)$. Then $T=\lambda_1 P_1 \oplus \lambda_2 P_2 \oplus T_2$ on $H=\Re(P_1)\oplus\Re(P_2)\oplus\Re(I-P_1-P_2)$. By repeating above argument, each $\ker(T-\lambda_n)$ is a reducing subspace of T and

$$\left\|T-\oplus_{k=1}^{n}\lambda_k P_k\right\|=\|T_n\|=|\lambda_{k+1}|\to 0 \text{ as } n\to\infty .$$

Here P_k is the orthogonal projection onto $\ker(T-\lambda_k)$ and $T=\left(\oplus_{k=1}^{n}\lambda_k P_k\right)\oplus T_n$ on $H=\left(\oplus_{k=1}^{n}\Re(P_k)\right)\oplus\Re\left(I-\sum_{k=1}^{n}P_k\right)$. Hence $T=\oplus_{k=1}^{\infty}\lambda_k P_k$ is compact and normal because each P_k is a finite rank orthogonal projection which satisfies $P_kP_l=0$ whenever $k\neq l$ by ([3], Corollary 3.4) and $\lambda_n\to 0$ as $n\to\infty$.

Definition 3.1. *An operator $T\in B(H)$ is called algebraically w-hyponormal operator if there exists a nonconstant complex polynomial p such that $p(T)$ is w-hyponormal operator.*

In general, the following implications hold: class w-hyponormal $\Rightarrow$ algebraically w-hyponormal.

The following facts follow from the above definition and some well known facts about class w-hyponormal.

1) If $T\in B(H)$ is algebraically w-hyponormal then so is $T-\lambda I$ for each $\lambda\in C$.

2) If $T\in B(H)$ is algebraically w-hyponormal and M is a closed T-invariant subspace of H then $T|_M$ is algebraically w-hyponormal.

Lemma 3.2. Let $T\in B(H)$ belong to class w-hyponormal. Let $\lambda\in C$. Assume that $\sigma(T)=\{\lambda\}$. Then $T=\lambda I$.

Proof. We consider two cases:

Case (I). $(\lambda=0)$: Since T is an w-hyponormal, T is normaloid. Therefore $T=0$.

Case (II). $(\lambda\neq 0)$: Here T is invertible, and since T is an w-hyponormal, we see that T^{-1} is also belongs class w-hyponormal. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1})=\left\{\frac{1}{\lambda}\right\}$, so $\|T\|\|T^{-1}\|=|\lambda|\left|\frac{1}{\lambda}\right|=1$. It follows that T is convexoid, so $W(T)=\{\lambda\}$. Therefore $T=\lambda$.

Proposition 3.3. *Let T be a quasinilpotent algebraically W-hyponormal operator. Then T is nilpotent.*

Proof. Assume that $p(T)$ is w-hyponormal operator for some nonconstant polynomial p. Since $\sigma(p(T))=p(\sigma(T))$ the operator $p(T)-p(0)$ is quasinilpotent. Thus Lemma 3.2 would implythat

$$cT^m(T-\lambda_1 I)\cdots(T-\lambda_n I)\equiv p(T)-p(0)=0$$

where $m\geq 1$. Since $T-\lambda_j I$ is invertible for every $\lambda_j\neq 0$, we must have $T^m=0$.

An operator $T\in B(H)$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator $T\in B(H)$ is called normaloid if $r(T)=\|T\|$, where $r(T)$ is the spectral radius of T. $X\in B(H)$ is called a quasiaffinity if it has trivial kernel and dense range. $S\in B(H)$ is said to be a quasiaffine transform of $T\in B(H)$ (notation: $S\prec T$) if there is a quasiaffinity $X\in B(H)$ such that $XS=TX$. If both $S\prec T$ and $T\prec S$ then we say that S and T are quasisimilar.

An operator $T\in B(H)$ is said to be polaroid if $iso\sigma(T)\subseteq\pi(T)$ where $iso\sigma(T)$ be the set of isolated points of the spectrum $\sigma(T)$ of T and $\pi(T)$ is the set of all poles of T. In general, if T is polaroid then it is isoloid. However, the converse is not true. Consider the following example. Let $T\in\ell^2(N)$ be defined by

$$T(x_1,x_2,\cdots)=\left(\frac{x_2}{2},\frac{x_3}{3},\cdots\right).$$

Then T is a compact quasinilpotent operator with $\dim\ker(T)=1$, and so T is isoloid. However, since T does not have finite ascent, T is not polaroid.

In [3] they showed that every w-hyponormal operator is isoloid. We can prove more:

Theorem 3.4. *Let T be an algebraically w-hyponormal operator. Then T is polaroid.*

Proof. Suppose T is an algebraically w-hyponormal operator. Then $p(T)$ is w-hyponormal for some nonconstant polynomial p. Let $\lambda\in iso(\sigma(T))$. Using the spectral projection $P:=\frac{1}{2i\pi}\int_{\partial D}(\mu-T)^{-1}\,d\mu$ where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T=\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix},\quad \text{and}\quad \sigma(T_1)=\{\lambda\}\quad \text{and}\quad \sigma(T_2)=\sigma(T)\setminus\{\lambda\}.$$

Since T_1 is algebraically w-hyponormal and $\sigma(T_1)=\{\lambda\}$. But $\sigma(T_1-\lambda I)=\{0\}$ it follows from Proposition 3.3 that $T_1-\lambda I$ is nilpotent. Therefore $T_1-\lambda$ has finite ascent and descent. On the other hand, since $T_2-\lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T-\lambda I$ has finite ascent anddescent. Therefore λ is a pole of the resolvent of T. Thus if $\lambda\in iso(\sigma(T))$ implies $\lambda\in\pi(T)$, and so $iso(\sigma(T))\subset\pi(T)$. Hence T is polaroid.

Corollary 3.5. *Let T be an algebraically w-hyponormal operator. Then T is isoloid.*

For $T\in B(H)$, $\lambda\in\sigma(T)$ is said to be a regular point if there exists $S\in B(H)$ such that $T-\lambda I=(T-\lambda I)S(T-\lambda I)$. T is is called reguloid if every isolated point of $\sigma(T)$ is a regular point. It is well known ([18], Theorems 4.6.4 and 8.4.4) that $T-\lambda I=(T-\lambda I)S(T-\lambda I)$ for some $S\in B(H)\Leftrightarrow T-\lambda I$ has a closed range.

Theorem 3.6. *Let T be an algebraically W-hyponormal operator. Then T is reguloid.*

Proof. Suppose T is an algebraically w-hyponormal operator. Then $p(T)$ is w-hyponormalfor some nonconstant polynomial p. Let $\lambda\in iso(\sigma(T))$. Using the spectral projection $P:=\frac{1}{2i\pi}\int_{\partial D}(\mu-T)^{-1}d\mu$ where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T=\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix},\quad \text{and}\quad \sigma(T_1)=\{\lambda\}\quad \text{and}\quad \sigma(T_2)=\sigma(T)\setminus\{\lambda\}$$

Since T_1 is algebraically w-hyponormal and $\sigma(T_1)=\{\lambda\}$. it follows from Lemma 3.2 that $T_1=\lambda I$. Therefore by ([17], Corollary 2.6),

$$H=E(H)\oplus E(H)^{\perp}=\ker(T-\lambda I)\oplus\ker(T-\lambda I)^{\perp}\tag{3.1}$$

Relative to decomposition 3.1, $T=\lambda I\oplus T_2$. Therefore $T-\lambda I=0\oplus T-\lambda I$ and hence

$$\operatorname{ran}(T-\lambda I)=(T-\lambda I)(H)=0\oplus(T_2-\lambda I)\left(\ker(T-\lambda I)^{\perp}\right)$$

since $T_2-\lambda I$ is invertible, $T-\lambda I$ has closed range.

For a bounded operator T and nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]}=T$). If for some n the range $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi- B-Fredholm operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [19]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-*B*-Fredholm operator is an upper or a lower semi-Fredholm operator. An operator $T\in B(X)$ is said to be a B-*Weyl operator* if it is a B-Fredholm operator of index zero. the semi- B-Fredholm spectrum $\sigma_{SBF}(T)$ and the B-Weyl spectrum σ_{BW} of T are defined by

$$\sigma_{SBF}(T):=\{\lambda\in\mathbb{C}:T-\lambda I\text{ is not a semi-}B\text{-Fredholm operator}\},$$

$$\sigma_{BW}:=\{\lambda\in\mathbb{C}:T-\lambda I\text{ is not a }B\text{-Weyl operator}\}.$$

Recall that an operator $T\in B(X)$ is a *Drazin invertible* if and only if it has a finite ascent and descent, which is alsoequivalent to the fact that $T=T_0\oplus T_1$, where T_0 is nilpotent operator and T_1 is invertible operator, see ([20], Proposition A). The Drazin spectrum is given by

$$\sigma_D(T):=\{\lambda\in\mathbb{C}:T-\lambda I\text{ is not Drazin invertible}\}$$

We observe that $\sigma_D(T)=\sigma(T)\setminus\pi(T)$, where $\pi(T)$ is the set of allpoles.

Define

$$E(T):=\{\lambda\in iso\sigma(T):0<\alpha(T-\lambda)\}$$

we also say that the *generalized Weyl's theorem* holds for T (in symbol, $T\in gW$) if

$$\sigma(T)\setminus\sigma_{BW}(T)=E(T)$$

and that the *generalized Browder's theorem* holds for T (in symbol, $T\in g\mathrm{B}$) if

$$\sigma(T)\setminus\sigma_{BW}(T)=\pi(T).$$

It is Known [21] [22] that

$$g\mathrm{W}\subseteq g\mathrm{B}\cup\mathrm{W} \quad\text{and that}\quad g\mathrm{B}\cup\mathrm{W}\subseteq\mathrm{B}.$$

Moreover, given $T\in g\mathrm{B}$, then it is clear $T\in g\mathrm{W}$ if and only if $E(T)=\pi(T)$, see [21] [22].

Let $SF_+(\mathrm{X})$ be the class of all *upper semi-Fredholm* operators, $SF_+^-(\mathrm{X})$ be the class of all $T\in SF_+(\mathrm{X})$ with $\operatorname{ind}(T)\le 0$, and for any $T\in\mathrm{B}(X)$ let

$$\sigma_{SF_+^-}(T):=\{\lambda\in\mathrm{C}:T-\lambda I\notin SF_+^-(\mathrm{X})\}.$$

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma_a(T)$. According to [23], we say that T satisfies a-*Weyl's theorem* (and we write $T\in a\mathrm{W}$) if

$$\sigma_{SF_+^-}(T)=\sigma_a(T)\setminus E_0^a(T)$$

and that a-Browder's theorem holds for T (in symbol, $T\in a\mathrm{B}$) if

$$\sigma_a(T)\setminus\sigma_{SF_+^-}(T)=\pi_0^a(T)$$

where $\pi_0^a(T)$ is the set of all left poles of finite rank.

Let $SBF_+(\mathrm{X})$ be the class of all *upper semi-B-Fredholm* operators, and $SBF_+^-(\mathrm{X})$ the class of all $T\in SBF_+(\mathrm{X})$ such that $\operatorname{ind}(T)\le 0$, and

$$\sigma_{SBF_+^-}(T):=\{\lambda\in\mathrm{C}:T-\lambda\notin SBF_+^-(\mathrm{X})\}.$$

Recall that an operator $T\in\mathrm{B}(T)$ satisfies the *generalized a-Weyl's* theorem (in symbol, $T\in ga\mathrm{W}$) if

$$\sigma_{SBF_+^-}(T)=\sigma_a(T)\setminus E^a(T)$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$.

Define a set $LD(\mathrm{X})$ by

$$LD(\mathrm{X}):=\left\{T\in\mathrm{B}(X):a(T)<\infty \text{ and } \mathrm{R}\left(T^{a(T)+1}\right)\text{ is closed}\right\}.$$

An operator $T\in\mathrm{B}(\mathrm{H})$ iscalled *left Drazin invertible* if $a(T)<\infty$ and $\mathrm{R}\left(T^{a(T)+1}\right)$ is closed (see [22], Definition 2.4). The left Drazin spectrum is given by

$$\sigma_{LD}(T):=\{\lambda\in\mathrm{C}:T-\lambda I\text{ is not left Drazin invertible}\}.$$

Recall ([22], Definition 2.5) that $\lambda\in\sigma_a(T)$ is a left pole of T if $T-\lambda I$ is left Drazin invertible operator and $\lambda\in\sigma_a(T)$ is a left pole of finite rank if λ is a left pole of T and $\alpha(T-\lambda)<\infty$. We will denote $\pi^a(T)$ the set of all left pole of T. We have $\sigma_{LD}(T)=\sigma_a(T)\setminus\pi^a(T)$. Note that if $\lambda\in\pi^a(T)$, then it is easily seen that $T-\lambda$ is an operator of topological uniform descent. Therefore, it follows from ([21], Theorem 2.5) that λ is isolated in $\sigma_a(T)$. Following [22] if $T\in\mathrm{B}(\mathrm{H})$ and $\lambda\in\mathrm{C}$ is anisolated in $\sigma_a(T)$, then $\lambda\in\pi^a(T)$ if and only if $\lambda\notin\sigma_{SBF_+^-}(T)$ and $\lambda\in\pi_0^a(T)$ if and only if $\lambda\notin\sigma_{SF_+^-}(T)$. We will say that *generalized* a-*Browder's theorem* holds for T (in symbol $T\in ga\mathrm{B}$) if

$$\sigma_{SBF_+^-}(T)=\sigma_a(T)\setminus\pi^a(T).$$

It is known [21]-[23] that

$$g\mathrm{W}\cup g\mathrm{B}\cup a\mathrm{W}\cup ga\mathrm{B}\subseteq ga\mathrm{W}\text{ and that }a\mathrm{B}\cup\mathrm{W}\subseteq a\mathrm{W}\text{ and that }\mathrm{B}\subseteq a\mathrm{B}.$$

Definition 3.2. ([23]) *An operator* $T\in\mathrm{B}(\mathrm{H})$ *is said to satisfy property* (w) *if*

$$\Delta_a(T)=\sigma_a(T)\setminus\sigma_{SF_+^-}(T)=E_0(T).$$

In [24], it is shown that the property (w) implies Weyls theorem. For $T\in\mathrm{B}(\mathrm{H})$, let

$\Delta^g(T)=\sigma(T)\setminus\sigma_{BW}(T)$ and $\Delta_a^g(T)=\sigma(T)\setminus\sigma_{SBF_+^-}(T)$. If T^* has the SVEP, then it is known from [15] that $\sigma(T)=\sigma_a(T)$ and from [25] we have $\sigma_{BW}(T)=\sigma_{SBF_+^-}(T)$. Thus $E(T)=E^a(T)$ and $\Delta^g(T)=\Delta_a^g(T)$.

Definition 3.3. ([26]) An operator $T\in B(X)$ is said to satisfy property (gw) if

$$\Delta_a^g(T)=E(T).$$

Theorem 3.7. *Let* $T\in B(H)$. *If* T *is a* w*-hyponormal. Then the following assertions are equivalent*:

1) generalized Weyl's theorem holds for T;
2) generalized Browder's theorem holds for T;
3) Weyl's theorem holds for T;
4) Browder's theorem holds for T.

Proof. Since w-hyponormal operators are polaroid. Hence the result follows now from ([27], Corollary 2.1).

Theorem 3.8. *Let* $T\in B(H)$. *If* T^* *is a* w*-hyponormal. Then the following assertions are equivalent*:

1) generalized a-Weyl's theorem holds for T;
2) generalized a-Browder's theorem holds for T;
3) a-Weyl's theorem holds for T;
4) a-Browder's theorem holds for T.

Proof. If T^* is a w-hyponormal, then T is a-polaroid and so $E^a(T)=\pi^a(T)$. Hence the result followsnow from ([27], Corollary 2.3).

Theorem 3.9. *Let* $T\in B(H)$. *If* T^* *is a* w*-hyponormal. Then the following assertions are equivalent*:

1) generalized a-Weyl's theorem holds for T;
2) generalized Weyl's theorem holds for T;
3) T satisfies property (gw);
4) generalized a-Browder's theorem holds for T;
5) a-Weyl's theorem holds for T;
6) a-Browder's theorem holds for T;
7) T satisfies property (w).

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). This equivalence follows from ([26], Theorem 2.7), since T^* has SVEP. (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). This equivalence follows from Theorem 3.8. (iii) $\Leftrightarrow$ (vii). Since T^* has SVEP and T is polaroid, then $E(T)=\pi^a(T)$. Therefore, the equivalence follows now from Theorem 2.5 of [26].

Recall that a bounded operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T)=0$. From the spectral mapping theorem it easily follows that the spectrum of analgebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n\in N$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic.

Theorem 3.10. *Suppose that* $T\in B(H)$, *and* $K\in B(X)$ *is an algebraic operator commuting*with T.

1) If T is algebraically w-hyponormal then property (gw) holds for T^*+K^*.
2) If T^* is algebraically w-hyponormal then property (gw) holds for $T+K$.

Proof. (i) If T is an algebraically w-hyponormal then T has SVEP and hence $T+K$ has SVEP by Theorem 2.14 of [28]. Moreover, T is polaroid so also $T+K$ is polaroid by Theorem 2.14 of [28]. By Theorem 2.10 of [26], then property (gw) holds for T^*+K^*.

(ii) If T^* is an algebraically w-hyponormal then T^* has SVEP and hence T^*+K^* has SVEP by Theorem 2.14 of [28]. Moreover, T^* is polaroid so also T^*+K^* is polaroid by Theorem 2.14 of [28]. By Theorem 2.10 of [26], then property (gw) holds for $T+K$.

4. Riesz Idempotent of w-Hyponormalc

Let $T\in B(H)$ and $\lambda\in\sigma(T)$ be an isolated of $\sigma(T)$. then there exists a closed disc D_λ centered λ which satisfies $D_\lambda\cap\sigma(T)=\{\lambda\}$. The operator

$$P=\frac{1}{2\pi i}\int_{\partial D_\lambda}(T-\lambda I)^{-1}\,d\lambda$$

is called the Riesz idempotent with respect to λ which has properties that

$$P^2 = P,\ PT = TP,\ \ker(T-\lambda I) \subset P\mathrm{H} \quad \text{and} \quad \sigma\left(T|_{P\mathrm{H}}\right) = \{\lambda\}.$$

In [29], Stampfli proved that if T is hyponormal and $\lambda \in \sigma(T)$ is isolated, then the Riesz idempotent P with respect to λ is self-adjoint and satisfies

$$P\mathrm{H} = \ker(T-\lambda I) = \ker(T-\lambda I)^*.$$

In this paper we extend these result to the case of w-hyponormal operator.

Theorem 4.1. *Let $T \in \mathrm{B}(\mathrm{H})$ be a w-hyponormal operator and λ be a non-zero isolated point of $\sigma(T)$.* Let D_λ denote the closed disc which centered λ such that $\mathrm{D}_\lambda \cap \sigma(T) = \{\lambda\}$. Then the Riesz idempotent λ satisfies that

$$P\mathrm{H} = \ker(T-\lambda I) = \ker(T-\lambda I)^*.$$

In particular P is self-adjoint.

Proof. Since w-hyponormal operators are isoloid by Corollary 3.5.

Then λ is an isolated point of $\sigma(T)$. Then the range of Riesz idempotent

$$P = \frac{1}{2\pi i}\int_{\partial \mathrm{D}_\lambda} (T-\lambda I)^{-1}\,\mathrm{d}\lambda$$

is aninvariant closed subspace of T and $\sigma\left(T|_{P\mathrm{H}}\right) = \{\lambda\}$. Here D_λ isa closed disc with its center λ such that $\mathrm{D}_\lambda \cap \sigma(T) = \{\lambda\}$.

If $\lambda = 0$, then $\sigma\left(T|_{P\mathrm{H}}\right) = \{0\}$ Since $T|_{P\mathrm{H}}$ is w-hyponormal by Theorem 2.2, $T|_{P\mathrm{H}} = 0$ by Lemma 3.2. Therefore, 0 is an eigenvalue of T.

If $\lambda \neq 0$, then $T|_{P\mathrm{H}}$ is an invertible w-hyponormal operator and hence $\left(T|_{P\mathrm{H}}\right)^{-1}$ is also w-hyponormal. We see that $\|T_{P\mathrm{H}}\| = |\lambda|$ and $\left\|\left(T|_{P\mathrm{H}}\right)^{-1}\right\| = \frac{1}{|\lambda|}$, Let $x \in P\mathrm{H}$ be arbitrary vector. Then

$$\|x\| \le \left\|\left(T|_{P\mathrm{H}}\right)^{-1}\right\| \|T_{P\mathrm{H}}x\| = \frac{1}{|\lambda|}\|T_{P\mathrm{H}}x\| \le \frac{1}{|\lambda|}|\lambda|\|x\| = \|x\|.$$

This implies that $\frac{1}{\lambda}T|_{P\mathrm{H}}$ is unitary with its spectrum $\sigma\left(\frac{1}{\lambda}T|_{P\mathrm{H}}\right) = \{1\}$. Hence $T_{P\mathrm{H}} = \lambda I$ and λ is an eigenvalue of *T*. Therefore, $P\mathrm{H} = \ker(T-\lambda I)$ Since $\ker(T-\lambda I) \subset \ker(T-\lambda I)^*$ by Proposition 2.16, it suffices to show that $\ker(T-\lambda I)^* \subset \ker(T-\lambda I)$. Since $\ker(T-\lambda I)$ is a reducing subspace of *T* by Proposition 2.16 and the restriction of a *w*-hyponormal to its reducing subspace is also *w*-hyponormal operator, we see that T is of the form $T = T' \oplus \lambda I$ on $\mathrm{H} = \ker(T-\lambda I) \oplus \ker(T-\lambda I)^\perp$, where T' is a w-hyponormal operator with $\ker(T'-\lambda I) = \{0\}$. Since $\lambda \in \sigma(T) = \sigma(T') \cup \{\lambda\}$ is isolated, the only two cases occur. One is $\lambda \notin \sigma(T')$ and the other is that λ is an isolated point of $\sigma(T')$. The latter case, however, does not occur otherwise we have $\lambda \in \sigma_p(T')$ and this contradicts the fact that $\ker(T'-\lambda I) = \{0\}$. $\ker(T-\lambda I) = \ker(T-\lambda I)^*$ is immediate from the injectivity of $T'-\lambda I$ as an operator on $\ker(T-\lambda I)^\perp$.

Next, we show that P is self-adjoint. Since $P\mathrm{H} = \ker(T-\lambda I) = \ker(T-\lambda I)^*$ we have

$$\left((T-zI)^*\right)^{-1} P = \overline{(z-\lambda)^{-1}}P.$$

Hence

$$P^*P = -\frac{1}{2i\pi}\int_{\partial \mathrm{D}_\lambda} \left((T-zI)^*\right)^{-1} P\,\mathrm{d}\bar{z} = -\frac{1}{2i\pi}\int_{\partial \mathrm{D}_\lambda} \overline{(z-\lambda)^{-1}}P\,\mathrm{d}\bar{z} = \overline{\left(\frac{1}{2i\pi}\int_{\partial \mathrm{D}_\lambda}\frac{1}{z-\lambda}\mathrm{d}z\right)}P = PP^*.$$

Therefore, the proof is achieved.

5. Conclusion

In the study of *w*-hyponormal operator, the Aluthge transform is a very useful tool. It is an operator transform from the class of *w*-hyponormal operator to the class of semi-hyponormal operator. By using Aluthge transform, we treat spectrum properties of *w*-hyponormal operator like some of hyponormal operator.

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