Advances in Pure Mathematics, 2011, 1, 243-244
doi:10.4236/apm.2011.14043 Published Online July 2011 (http://www.SciRP.org/journal/apm)

An Inequality for Second Order Differential Equation with Retarded Argument

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Received April 23, 2011; *revised May* 10, 2011; *accepted May* 20, 2011

Abstract

Applications of differential equations with retarded argument can be encountered in the theory of automatic control, in the theory of self-oscillatory systems, in the study of problems connected with combustion in rocket engines, in a number of problems in economics, biophysics. The problems in this areas can be solved reducing differential equations with retarded argument. In this work an important inequality for second order differential equation with retarded argument is obtained.

Keywords: Differential Equation with Retarded Argument, Inequality

1. Introduction

In this study we consider the equation

$$L(w) = w''(t) + \lambda w(t) + M(t) w(t - \Delta(t)) = 0 \quad (1)$$

on an interval I. Where λ is a real parameter; $M(t)$ and $\Delta(t)$ are continuous functions on I; $t - \Delta(t) \ge 0$ and $1 \ge \Delta(t) > 0$ for each $t \in I$.

2. An Inequality for Second Order Differential Equation with Retarded Argument

Theorem. Let us denote by every point with t_{0_j} which is satisfying the mean-value theorem for a continuous solution $w(t_j)$ of (1) on $[t_j - \Delta(t), t_j] \subseteq I$ for each $t_j \in I$ and $j \in J$, where J is an index set. Also let us assume that $\sup M(t) = M_0$ where M_0 is a real number. Then for all t_j in the equation

$$\left\| w(t_{0_j}) \right\| e^{-k|t_j - t_{0_j}|} \le \left\| w(t_j) \right\| \le \left\| w(t_{0_j}) \right\| e^{k|t_j - t_{0_j}|} \quad (2)$$

where

$$\left\| w(t_j) \right\| = \left[\left| w(t_j) \right|^2 + \left| w'(t_{0_j}) \right|^2 \right]^{1/2}, k = 1 + |\lambda| + \frac{3}{2}|M_0|$$

Proof. From the mean-value theorem we can write the followings:

$$\frac{w(t_j) - w(t_j - \Delta(t_j))}{\Delta(t_j)} = w'(t_{0_j})$$

$$w(t_j - \Delta(t_j)) = w(t_j) - w'(t_{0_j}) \Delta(t_j)$$

and

$$\left| w(t_j - \Delta(t_j)) \right| \le \left| w(t_j) \right| + \left| w'(t_{0_j}) \right| \quad (3)$$

Now we let $u(t_j) = \left\| w(t_j) \right\|^2$. Thus

$$u = w\overline{w} + w'\overline{w'}$$

where $\overline{w}(t_j) = \overline{w(t_j)}$. Then

$$u' = w'\overline{w} + w\overline{w'} + w''\overline{w'} + w'\overline{w''}$$

From the definition of a derivative it follows that $\overline{w'} = \overline{w}'$. Also $\left| w(t_j) \right| = \left| \overline{w(t_j)} \right|$. Therefore

$$\left| u'(t_j) \right| \le 2\left| w(t_j) \right| \left| w'(t_{0_j}) \right| + 2\left| w'(t_{0_j}) \right| \left| w''(t_j) \right| \quad (4)$$

Since w satisfies $L(w) = 0$ we have

$$w''(t_j) = -\lambda w(t_j) - M(t_j) w(t_j - \Delta(t_j))$$

and hence applying (3)

$$\begin{aligned} \left| w''(t_j) \right| &\le |\lambda| \left| w(t_j) \right| + |M_0| \left| w(t_j - \Delta(t_j)) \right| \\ &\le |\lambda| \left| w(t_j) \right| + |M_0| \left(\left| w(t_j) \right| + \left| w'(t_{0_j}) \right| \right) \end{aligned} \quad (5)$$

Using (5) in (4) we obtain

$$\begin{aligned}\left|u'\left(t_j\right)\right| &\le 2\left|w\left(t_j\right)\right|\left|w'\left(t_{0_j}\right)\right| + 2\left|w'\left(t_{0_j}\right)\right| \\ &\quad\cdot\left\{|\lambda|\left|w\left(t_j\right)\right| + |M_0|\left(\left|w\left(t_j\right)\right| + \left|w'\left(t_{0_j}\right)\right|\right)\right\} \\ &= 2\left(1+|\lambda|+|M_0|\right)\left|w\left(t_j\right)\right|\left|w'\left(t_{0_j}\right)\right| + 2|M_0|\left|w'\left(t_{0_j}\right)\right|^2\end{aligned}$$

Now applying the fact

$$2\left|w\left(t_j\right)\right|\left|w'\left(t_{0_j}\right)\right| \le \left|w\left(t_j\right)\right|^2 + \left|w'\left(t_{0_j}\right)\right|^2$$

we get

$$\begin{aligned}\left|u'\left(t_j\right)\right| &\le \left(1+|\lambda|+|M_0|\right)\left|w\left(t_j\right)\right|^2 \\ &\quad + \left(1+|\lambda|+3|M_0|\right)\left|w'\left(t_{0_j}\right)\right|^2 \\ &\le 2\left(1+|\lambda|+\frac{3}{2}|M_0|\right)\left(\left|w\left(t_j\right)\right|^2 + \left|w'\left(t_{0_j}\right)\right|^2\right)\end{aligned}$$

or

$$\left|u'\left(t_j\right)\right| \le 2ku\left(t_j\right)$$

This is equivalent to

$$-2ku\left(t_j\right) \le u'\left(t_j\right) \le 2ku\left(t_j\right) \qquad (6)$$

And these inequalities lead directly to (2). Indeed consider the right inequality which can be written as $u' - 2ku \le 0$. It is equivalent to

$$\mathrm{e}^{-2kt_j}\left(u' - 2ku\right) = \left(\mathrm{e}^{-2kt_j}u\right)' \le 0$$

If $t_j > t_{0_j}$ we integrate from t_0 to t_{0_j} obtaining

$$\mathrm{e}^{-2kt_j}u\left(t_j\right) - \mathrm{e}^{-2kt_{0_j}}u\left(t_{0_j}\right) \le 0$$

or

$$u\left(t_j\right) \le u\left(t_{0_j}\right)\mathrm{e}^{2k\left(t_j - t_{0_j}\right)}$$

The left inequality in (6) similarly implies

$$\left\|w\left(t_{0_j}\right)\right\|\mathrm{e}^{-k\left(t_j - t_{0_j}\right)} \le \left\|w\left(t_j\right)\right\|, \quad t_j > t_{0_j}$$

and therefore

$$\left\|w\left(t_{0_j}\right)\right\|\mathrm{e}^{-k\left|t_j - t_{0_j}\right|} \le \left\|w\left(t_j\right)\right\| \le \left\|w\left(t_{0_j}\right)\right\|\mathrm{e}^{k\left|t_j - t_{0_j}\right|}, \quad t_j > t_{0_j}$$

which is just (2) for $t_j > t_{0_j}$. The case $t_j < t_{0_j}$ may be considered analogically.

3. References

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