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An Independence Property for General Information

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Abstract

The aim of this paper was a generalization of independence property proposed by J. Kampé de Feriét and B. Forte in Information Theory without probability, called *general information*. Therefore, its application to fuzzy sets has been presented.

Keywords

Information, Functional Equations, Fuzzy Sets

1. Introduction

Since 1967-69, J. Kampé de Ferét and B. Forte have introduced, by axiomatic way, new information measures without probability [1]-[3]; later, in analogous way, with P. Benvenuti we have defined information measures without probability or fuzzy measure [4] for fuzzy sets [5] [6]. This form of information measure is again called *general information*.

In Information Theory an important role has played by an independence property with respect to a given information measures J applied to crisp sets [7]. These sets are called *J-independent* (*i.e.* independent each other with the respect to J) [8].

For this reason we will propose a generalization of J-independence property.

The paper develops in the following way: in Section 2 we recall some preliminaires; in Section 3 the generalization of J-indepedence is proposed; the result is extended to fuzzy sets in Section 4. Section 5 is devoted to the conclusion.

2. Preliminaires

Let Ω be an abstract space and $\mathcal{C}$ the σ-algebra of crisp sets $C \subset \Omega$, such that $(\Omega, \mathcal{C})$ is a measurable space. We refer to [7] for all knoledge and operations among crisp sets.

J. Kampé de Ferét and B. Forte gave the following definition [1] [2]:

Definition 2.1 *Measure of general information J for crisp sets is a mapping*

$$J(\cdot):\mathcal{C}\to[0,+\infty]$$

such that $\forall\, C, C', C_1, C_2 \in \mathcal{C}$:

(i) $C\subset C' \Rightarrow J(C)\ge J(C')$,

(ii) $J(\varnothing)=+\infty$, $J(\Omega)=0$;

(iii) $J(C_1\cap C_2)=J(C_1)+J(C_2)$, if $C_1\cap C_2\ne\varnothing$.

If the couple (C_1,C_2) satisfies the (iii), we say that C_1 and C_2 are *J*-independent, *i.e.* independent each other with respect to information *J*.

3. A Generalization of the *J*-Independence Property

In this paragraph we are going to present a generalization of the *J*-independence property.

We propose the following:

Definition 3.1 *Given a general information J, let* C_1 *and* C_2 *be two crisp sets in C such that* $C_1\cap C_2\ne\varnothing$. *We say that* C_1 *and* C_2 *are J*-independent each other *if there exists a continuous function* $\Phi:[0,+\infty]^2\to[0,+\infty]$ *such that*

$$J(C_1\cap C_2)=\Phi\big(J(C_1),J(C_2)\big) \tag{1}$$

We shall characterize the function Φ, taking into account the properties of the intersection for every $C_1, C_2, C_3, C_1'\in\mathcal{C}$:

$$\begin{cases}(p_1)\ \Phi\big(J(C_1),J(C_2)\big)=\Phi\big(J(C_2),J(C_1)\big), \text{ commutativity}\\ (p_2)\ \Phi\Big(\big(J(C_1),J(C_2)\big),J(C_3)\Big)=\Phi\Big(J(C_1),\big(J(C_2),J(C_3)\big)\Big),\\ \qquad \text{if } C_1\cap C_2\cap C_3\ne\varnothing \text{ associativity}\\ (p_3)\ \Phi\big(J(C),J(\Omega)\big)=J(C), \text{ neutral element}\\ (p_4)\ C_1\subset C_1'\Rightarrow \Phi\big(J(C_1),J(C_2)\big)\ge\Phi\big(J(C_1'),J(C_2)\big),\\ \qquad \text{if } C_1'\cap C_2\ne\varnothing \ \text{ monotonicity}\\ (p_5)\ \Phi\big(J(C_1),J(C_2)\big)\ge\bigvee\big(J(C_1),J(C_2)\big).\end{cases}$$

Putting $J(C_1)=x, J(C_2)=y, J(C_3)=z, J(C_1')=x'$, the properties [($p_1$) - ($p_5$)] have translated in the following system of functional equations and inequalities [9] [10]:

$$\begin{cases}(P_1)\ \Phi(x,y)=\Phi(y,x)\\ (P_2)\ \Phi\big(\Phi(x,y),z\big)=\Phi\big(x,\Phi(y,z)\big)\\ (P_3)\ \Phi(x,0)=x\\ (P_4)\ x\ge x'\Rightarrow\Phi(x,y)\ge\Phi(x',y)\\ (P_5)\ \Phi(x,y)\ge\bigvee(x,y).\end{cases}$$

We can give the following

Proposition 3.2 *A class of solutions of the system* [(P_1) - (P_5)] *is*

$$\Phi_h(x,y)=h^{-1}\big(h(x)+h(y)\big), \tag{2}$$

where h is any continuous, strictly increasing function $h:[0,+\infty]\to[0,+\infty]$ with $h(0)=0$ and

$h(+\infty) = +\infty$.

Proof. The class of functions (2) satisfy the equations [(P_1)-(P_3)] and the inequality (P_4) by appling the Ling Theorem about the representation of a function which is monotone, commutative, associative with neutral element [11]. The inequality (P_5) is a consequence of the monotonicity of h. □

So, from (2), we have

Proposition 3.3 *The generalization of the J-independence property for crisp sets is*

$$J(C_1 \cap C_2) = h^{-1}\left(h(J(C_1)) + h(J(C_2))\right), \forall\, C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 \neq \varnothing, \tag{3}$$

where h is any continuous, strictly increasing function $h:[0,+\infty] \to [0,+\infty]$ with $h(0) = 0$ and $h(+\infty) = +\infty$. □

Remark When h is linear, the generalization (3) coincide with the property (iii).

4. Extension to Fuzzy Setting

In this paragraph, we are considering the extension of J-independence property at fuzzy setting.

Let Ω be an abstract space and $\mathcal{F}$ the σ-algebra of fuzzy sets such that $(\Omega, \mathcal{F})$ is a measurable space [5], [6]. In [4] we have given the definition of measure of general information for fuzzy sets:

Definition 4.1 *Measure of general information in fuzzy setting is a mapping* $J'(\cdot): F \to [0,+\infty]$ *such that* $\forall\, F, F', F_1, F_2 \in \mathcal{F}$:

(i') $F \subset F' \Rightarrow J'(F) \geq J'(F')$,

(ii') $J'(\varnothing) = +\infty$, $J'(\Omega) = 0$,

(iii') $J'(F_1 \cap F_2) = J'(F_1) + J'(F_2)$, if $F_1 \cap F_2 \neq \varnothing$.

If the couple (F_1, F_2) satisfies the (iii'), we say that F_1 and F_2 are J'-independent, *i.e.* independent each other with respect to information J'.

Also in fuzzy setting, we generalize the (iii'), setting

$$J'(F_1 \cap F_2) = \Psi\left(J'(F_1), J'(F_2)\right) \text{ if } F_1 \cap F_2 \neq \varnothing. \tag{4}$$

The properties of the intersection between fuzzy sets are the similar to the [(p_1) − (p_4)] [5] [6]. Therefore, we are looking for functions (4) solutions of the system [(P_1) − (P_5)]. We have again the similar result:

Proposition 4.2 *A class of solution of the system* [(P_1) − (P_5)] *is*

$$\Psi_k(x, y) = k^{-1}\left(k(x) + k(y)\right), \tag{5}$$

where k is any continuous, strictly increasing function $k:[0,+\infty] \to [0,+\infty]$ with $k(0) = 0$ and $k(+\infty) = +\infty$.

From (5), we get

Proposition 4.3 *A generalization of the J'-independence property between two fuzzy set is*

$$J'(F_1 \cap F_2) = k^{-1}\left(k(J'(F_1)) + k(J'(F_2))\right), \forall F_1, F_2 \in F, F_1 \cap F_2 \neq \varnothing, \tag{6}$$

where k is any continuous, strictly increasing function $k:[0,+\infty] \to [0,+\infty]$ with $k(0) = 0$ and $k(+\infty) = +\infty$.

Proof. The proof is similar to that given for crisp sets. □

Remark. When k is linear, the generalization (6) coincide with the property (iii').

5. Conclusions

In this paper we have proposed a genralization of J-independence property between crisp sets:

$$J(C_1 \cap C_2) = h^{-1}\left(h(J(C_1)) + h(J(C_2))\right), \forall\, C_1, C_2 \in C, C_1 \cap C_2 \neq \varnothing,$$

where h is any continuous, strictly increasing function $h:[0,+\infty] \to [0,+\infty]$ with $h(0) = 0$ and $h(+\infty) = +\infty$.

Therefore, we have extended the result to fuzzy setting:

$$J'(F_1 \cap F_2) = k^{-1}\left(k(J'(F_1)) + k(J'(F_2))\right), \forall F_1, F_2 \in F, F_1 \cap F_2 \neq \varnothing,$$

where k is any continuous, strictly increasing function $k:[0,+\infty]\to[0,+\infty]$ with $k(0)=0$ and $k(+\infty)=+\infty$.

References

[1] Kampé de Fériet, J. and Forte, B. (1967) Information et Probabilité. *Comptes Rendus de l'Académie des Sciences Paris*, **265**, 110-114, 142-146, 350-353.

[2] Forte, B. (1969) Measures of Information: The General Axiomatic Theory. *RAIRO Informatique Théorique et Applications*, 63-90.

[3] Kampé de Feriét, J. (1970) Mesures de l'information fornie par un evénement. *Colloque International du Centre National de la Recherche Scientifique*, **186**, 191-221.

[4] Benvenuti, P., Vivona, D. and Divari, M. (1990) A General Information for Fuzzy Sets. *Uncertainty in Knowledge Bases*, *Lectures Notes in Computer Sciences*, **521**, 307-316. http://dx.doi.org/10.1007/BFb0028117

[5] Zadeh, L.A. (1965) Fuzzy Sets. *Information and Control*, **8**, 338-353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X

[6] Klir, G.J. and Folger, T.A. (1988) Fuzzy Sets, Uncertainty and Information. Prentice-Hall International Editions, Englewood Cliffs.

[7] Halmos, P.R. (1965) Measure Theory. Van Nostrand Company, Princeton.

[8] Benvenuti, P. (2004) L'opera scientifica. Roma, Ed.Univ.La Sapienza, Italia.

[9] Aczél, J. (1966) Lectures on Functional Equations and Their Applications. Academic Press, New York.

[10] Forte, B. (1970) Functional Equations in Generalized Information Theory. In: *Applications of Functional Equations and Inequalities to Information Theory*, Ed. Cremonese, Roma-Italy, 113-140.

[11] Ling, C.-H. (1995) Representation of Associative Functions. *Publicationes Mathematicae*, **12**, 189-212.