Journal of Applied Mathematics and Physics, 2015, 3, 1249-1255
Published Online October 2015 in SciRes. http://www.scirp.org/journal/jamp
http://dx.doi.org/10.4236/jamp.2015.310153

Qualitative Properties and Numerical Solution of the Kolmogorov-Fisher Type Biological Population Task with Double Nonlinear Diffusion

Dildora Kabulovna Muhamediyeva

National University of Uzbekistan, Tashkent, Uzbekistan
Email: matematichka@inbox.ru

Received 24 July 2015; accepted 23 October 2015; published 26 October 2015

Abstract

In the present work we study the global solvability of the Kolmogorov-Fisher type biological population task with double nonlinear diffusion and qualitative properties of the solution of the task based on the self-similar analysis. In additional, in this paper we consider the model of two competing population with dual nonlinear cross-diffusion.

Keywords

Double Nonlinearity, Cross-Diffusion, Biological Population, A Parabolic System of Quasilinear Equations, Convective Heat Transfer, Numerical Solution, Iterative Process, Self-Similar Solutions

1. Introduction

Let's consider in the domain $Q=\{(t,x):0<t,x\in R\}$ parabolic system of two quasilinear equations of reaction-diffusion with double nonlinear diffusion

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=\dfrac{\partial}{\partial x}\left(D_1 u_2^{m_1-1}\left|\dfrac{\partial u_1}{\partial x}\right|^{p-2}\dfrac{\partial u_1}{\partial x}\right)+k_1(t)u_1\left(1-u_1^{\beta_1}\right) \\ \dfrac{\partial u_2}{\partial t}=\dfrac{\partial}{\partial x}\left(D_2 u_1^{m_2-1}\left|\dfrac{\partial u_2}{\partial x}\right|^{p-2}\dfrac{\partial u_2}{\partial x}\right)+k_2(t)u_2\left(1-u_2^{\beta_2}\right) \end{cases} \tag{1}$$

$$u_1\big|_{t=0}=u_{10}(x),\quad u_2\big|_{t=0}=u_{20}(x),$$

How to cite this paper: Muhamediyeva, D.K. (2015) Qualitative Properties and Numerical Solution of the Kolmogorov-Fisher Type Biological Population Task with Double Nonlinear Diffusion. *Journal of Applied Mathematics and Physics*, **3**, 1249-1255. http://dx.doi.org/10.4236/jamp.2015.310153

which describes the process of biological populations of the Kolmogorov-Fisher in two-component nonlinear medium, the diffusion coefficient which is equal to $D_1 u_2^{m_1-1}\left|\frac{\partial u_1}{\partial x}\right|^{p-2}$, $D_2 u_1^{m_2-1}\left|\frac{\partial u_2}{\partial x}\right|^{p-2}$, where $m_1, m_2, p, \beta_1, \beta_2$ are positive real numbers, and $u_1 = u_1(t,x) \ge 0$, $u_2 = u_2(t,x) \ge 0$ are sought solution .

Below we investigate the qualitative properties of the considered problem by constructing self-similar system of equations for (1).

2. Self-Similar System of Equations

Self-similar system of equations we will construct by the method of nonlinear splitting [1]-[3].

Substitution in (1)

$$u_1(t,x) = \mathrm{e}^{k_1 t} v_1(t,x), \quad u_2(t,x) = \mathrm{e}^{k_2 t} v_2(t,x),$$

lead (1) to the form:

$$\begin{cases} \frac{\partial v_1}{\partial \tau_1} = \frac{\partial}{\partial x}\left(D_1 v_2^{m_1-1}\left|\frac{\partial v_1}{\partial x}\right|^{p-2}\frac{\partial v_1}{\partial x}\right) + k_1 \mathrm{e}^{\left[(\beta_1-p+2)k_1-(m_1-1)k_2\right]t} v_1^{\beta_1+1} \\ \frac{\partial v_2}{\partial \tau_2} = \frac{\partial}{\partial x}\left(D_2 v_1^{m_2-1}\left|\frac{\partial v_2}{\partial x}\right|^{p-2}\frac{\partial v_2}{\partial x}\right) + k_2 \mathrm{e}^{\left[(\beta_2-p+2)k_2-(m_2-1)k_1\right]t} v_2^{\beta_2+1} \end{cases} \tag{2}$$

$$v_1\big|_{t=0} = v_{10}(\eta), \quad v_2\big|_{t=0} = v_{20}(\eta).$$

Choosing

$$\tau = \frac{\mathrm{e}^{\left[(m_1-1)k_2+(p-2)k_1\right]t}}{(p-2)k_1+(m_1-1)k_2} = \frac{\mathrm{e}^{\left[(m_1-1)k_1+(p-2)k_2\right]t}}{(p-2)k_2+(m_2-1)k_1},$$

we get the following system of equations:

$$\begin{cases} \frac{\partial v_1}{\partial \tau} = \frac{\partial}{\partial x}\left(D_1 v_2^{m_1-1}\left|\frac{\partial v_1}{\partial x}\right|^{p-2}\frac{\partial v_1}{\partial x}\right) - a_1 \tau^{b_1} v_1^{\beta_1+1} \\ \frac{\partial v_2}{\partial \tau} = \frac{\partial}{\partial x}\left(D_2 v_1^{m_2-1}\left|\frac{\partial v_2}{\partial x}\right|^{p-2}\frac{\partial v_2}{\partial x}\right) - a_2 \tau^{b_2} v_2^{\beta_2+1} \end{cases} \tag{3}$$

where

$$a_1 = k_1 \cdot \left[(p-2)k_1 + (m_1-1)k_2\right]^{b_1}, \quad b_1 = \frac{(\beta_1 - p + 2)k_1 - (m_1-1)k_2}{(p-2)k_1 + (m_1-1)k_2}$$

$$a_2 = k_2 \cdot \left[(p-2)k_2 + (m_2-1)k_1\right]^{b_2}, \quad b_2 = \frac{(\beta_2 - p + 2)k_2 - (m_2-1)k_1}{(p-2)k_2 + (m_2-1)k_1}$$

For the purpose of obtaining self-similar system for the system of Equation (3) we find first the solution of a system of ordinary differential equations [4]-[7]

$$\begin{cases} \frac{\mathrm{d}\bar{v}_1}{\mathrm{d}\tau} = -a_1 \tau^{b_1} \bar{v}_1^{\beta_1+1}, \\ \frac{\mathrm{d}\bar{v}_2}{\mathrm{d}\tau} = -a_2 \tau^{b_2} \bar{v}_2^{\beta_2+1}, \end{cases}$$

in the form

$$\bar{v}_1(\tau) = \tau^{-\alpha_1}, \quad \bar{v}_2(\tau) = \tau^{-\alpha_2},$$

where

$$\alpha_1 = \frac{b_1+1}{\beta_1}, \quad \alpha_2 = \frac{b_2+1}{\beta_2}.$$

And then the solution of system (3) is sought in the form

$$v_1(t,x) = \bar{v}_1(t)\, w_1(\tau,x), \; v_2(t,x) = \bar{v}_2(t)\, w_2(\tau,x) \tag{4}$$

and $\tau = \tau(t)$ is selected so

$$\begin{aligned}\tau(\tau) &= \int_0^{\tau} \bar{v}_1^{(p-2)}(t)\,\bar{v}_2^{(m_1-1)}(t)\,\mathrm{d}t = \int_0^{\tau} \bar{v}_2^{(p-2)}(t)\,\bar{v}_1^{(m_2-1)}\mathrm{d}t \\ &= \begin{cases} \dfrac{1}{1-\left[\alpha_1(p-2)+\alpha_2(m_1-1)\right]}(T+\tau)^{-\left[\alpha_1(p-2)+\alpha_2(m_1-1)\right]+1}, & \text{if } 1-\left[\alpha_1(p-2)+\alpha_2(m_1-1)\right] \neq 0, \\ \ln(T+\tau), & \text{if } 1-\left[\alpha_1(p-2)+\alpha_2(m_1-1)\right] = 0, \end{cases}\end{aligned}$$

if $\alpha_1(p-2)+\alpha_2(m_1-1) = \alpha_2(p-2)+\alpha_1(m_2-1)$.

Then for $w_i(\tau,x), i=1,2$ we get the system of equations

$$\begin{cases} \dfrac{\partial w_1}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_1 w_2^{m_1-1}\left|\dfrac{\partial w_1}{\partial x}\right|^{p-2}\dfrac{\partial w_1}{\partial x}\right) + \psi_1\left(w_1^{\beta_1+1} - w_1\right) \\ \dfrac{\partial w_2}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_2 w_1^{m_2-1}\left|\dfrac{\partial w_2}{\partial x}\right|^{p-2}\dfrac{\partial w_2}{\partial x}\right) + \psi_2\left(w_2^{\beta_2+1} - w_2\right) \end{cases} \tag{5}$$

where

$$\psi_i = \frac{1}{\left(1-\left[\alpha_i(p-2)+\alpha_{3-i}(m_i-1)\right]\right)\tau}, \quad i=1,2 \tag{6}$$

Consider the self-similar solution of system (5) of the form

$$w_1(t,x) = f_1(\xi),\; w_2(t,x) = f_2(\xi),\; \xi = |x|\big/(T+\tau)^{1/p} \tag{7}$$

Then substituting (7) into (5) with respect to $f_1(\xi), f_2(\xi)$ we get the following system of nonlinear degenerate self-similar equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(f_2^{m_1-1}\left|\dfrac{\mathrm{d}f_1}{\mathrm{d}\xi}\right|^{p-2}\dfrac{\mathrm{d}f_1}{\mathrm{d}\xi}\right) + \dfrac{\xi}{p}\dfrac{\mathrm{d}f_1}{\mathrm{d}\xi} + \theta_1\left(f_1 - f_1^{\beta_1+1}\right) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(f_1^{m_2-1}\left|\dfrac{\mathrm{d}f_2}{\mathrm{d}\xi}\right|^{p-2}\dfrac{\mathrm{d}f_2}{\mathrm{d}\xi}\right) + \dfrac{\xi}{p}\dfrac{\mathrm{d}f_2}{\mathrm{d}\xi} + \theta_2\left(f_2 - f_2^{\beta_2+1}\right) = 0, \end{cases} \tag{8}$$

where $\theta_i = \dfrac{1}{\left(1-\left[\alpha_i(p-2)+\alpha_{3-i}(m_i-1)\right]\right)\tau}$, $i=1,2.$ Let's build an upper solutions for system (8).

3. Construction an Upper Solution

If

$$\beta_i = \left[(p-2)^2 - (m_1-1)(m_2-1)\right]\Big/\left[(p-1)\left(p-(m_i+1)\right)\right], \quad p > 2+\sqrt{(m_1-1)(m_2-1)}, \quad i=1,2,$$

Equation (8) has a local solution of the form

$$\bar{f}_1(\xi) = A\left(a-\xi^{\gamma}\right)_+^{n_1}, \quad \bar{f}_2(\xi) = B\left(a-\xi^{\gamma}\right)_+^{n_2},$$

where $(b)_+ = \max(0,b)$, $\gamma = p/(p-1)$,

$$n_1=\frac{(p-1)(p-(m_1+1))}{(p-2)^2-(m_1-1)(m_2-1)};\ n_2=\frac{(p-1)(p-(m_2+1))}{(p-2)^2-(m_1-1)(m_2-1)}.$$

Then in the domain Q according to the comparison principle of solutions [1] [8] we get

Theorem 1. Let $u_i(0,x)\le u_{i\pm}(0,x), x\in R$. Then the solution of the task (1) in the domain Q takes place an estimation

$$\begin{aligned}&u_1(t,x)\le u_{1+}(t,x)=e^{k_1t}\tau^{-\alpha_1}\bar{f}_1(\xi),\\&u_2(t,x)\le u_{2+}(t,x)=e^{k_2t}\tau^{-\alpha_2}\bar{f}_2(\xi),\end{aligned}\qquad \xi=|x|/\tau^{1/p}$$

where $\bar{f}_1(\xi),\bar{f}_2(\xi)\ u\ \tau(t)$—above-defined functions.

Note that the solution of system (1) when $\beta_i=\left[(p-2)^2-(m_1-1)(m_2-1)\right]/\left[(p-1)(p-(m_i+1))\right]$ has the following representation in the

$$a=\left[P_1\gamma\Big/B\left(\frac{1}{\gamma},1+n_1\right)\right]^{\frac{\gamma}{n_1}}=\left[P_2\gamma\Big/B\left(\frac{1}{\gamma},1+n_2\right)\right]^{\frac{\gamma}{n_2}}.$$

where $B(a,b)$—Beta Euler function [9].

It is proved that this view is self-similar asymptotics of solutions of systems (1).

$$\begin{cases}\tau^{-\frac{1}{\mu_1}}\int_{-\infty}^{\infty}\left(a-\xi_1^{\gamma}\right)_+^{n_1}\mathrm{d}x=P_1\\ \tau^{-\frac{1}{\mu_2}}\int_{-\infty}^{\infty}\left(a-\xi_2^{\gamma}\right)_+^{n_2}\mathrm{d}x=P_2\end{cases}$$

Thence

$$\begin{cases}\tau^{-\frac{1}{\mu_1}}\int_{-\infty}^{\infty}\left(a-\xi_1^{\gamma}\right)_+^{n_1}\mathrm{d}x=a^{\frac{n_1}{\gamma}}\frac{1}{\gamma}\int_0^1\eta^{\frac{1}{\gamma}-1}(1-\eta)^{n_1}\mathrm{d}\eta=a^{\frac{n_1}{\gamma}}\frac{1}{\gamma}B\left(\frac{1}{\gamma},1+n_1\right)=P_1\\ \tau^{-\frac{1}{\mu_2}}\int_{-\infty}^{\infty}\left(a-\xi_2^{\gamma}\right)_+^{n_2}\mathrm{d}x=a^{\frac{n_2}{\gamma}}\frac{1}{\gamma}\int_0^1\eta^{\frac{1}{\gamma}-1}(1-\eta)^{n_2}\mathrm{d}\eta=a^{\frac{n_2}{\gamma}}\frac{1}{\gamma}B\left(\frac{1}{\gamma},1+n_2\right)=P_2\end{cases}$$

$$a=\left[P_1\gamma\Big/B\left(\frac{1}{\gamma},1+n_1\right)\right]^{\frac{\gamma}{n_1}}=\left[P_2\gamma\Big/B\left(\frac{1}{\gamma},1+n_2\right)\right]^{\frac{\gamma}{n_2}}.$$

Carried out computational experiments and numerical results are obtained (see **Table 1**, **Table 2**).

4. Conclusion

Thus, it is assumed that the possibility of adequate study of nonlinear equations, biological populations with double nonlinearity based on the method of nonlinear splitting and numerical study of nonlinear processes described by equations with double nonlinearity and analysis of results on the basis of the estimates of the solutions gives a comprehensive picture of the process of multicomponent competing systems of biological populations.

5. Results

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(D_1v^{m_1-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)+k_1(t)u\left(1-u^{\beta_1}\right),\\ \frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(D_2u^{m_2-1}\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right)+k_2(t)v\left(1-v^{\beta_2}\right),\end{cases}$$

Table 1. Numerical results in the case of fast diffusion.

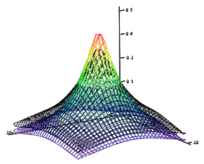
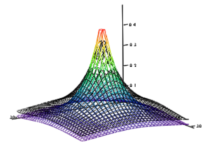
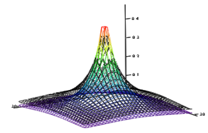
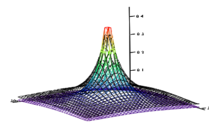
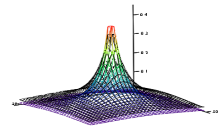
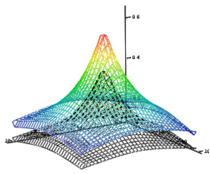
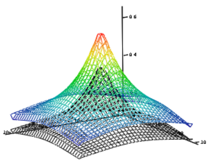
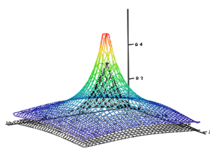
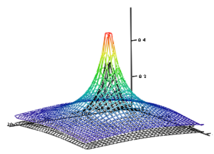
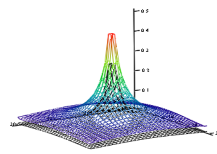
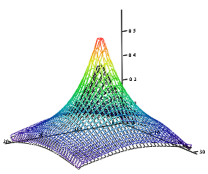
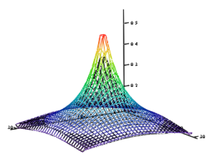
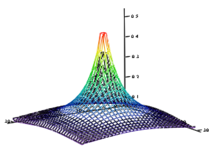
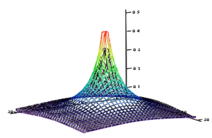
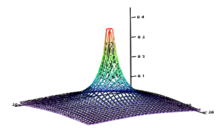
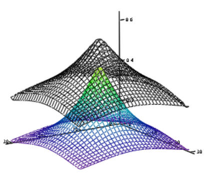
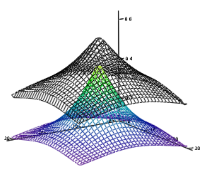
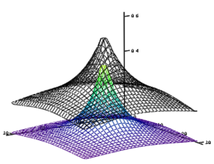
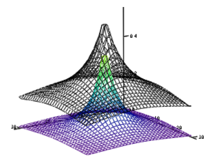
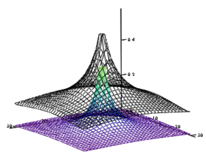
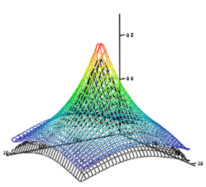
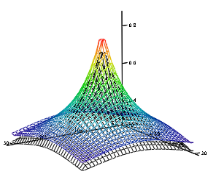
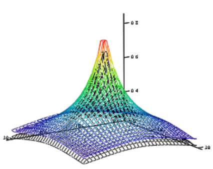
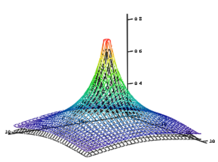
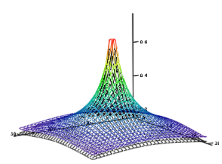
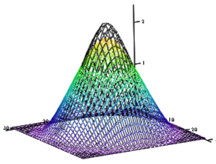
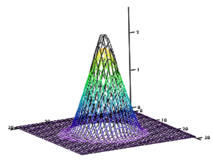
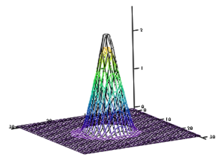
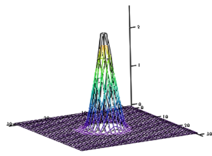
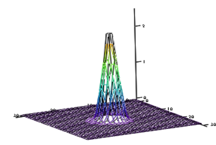
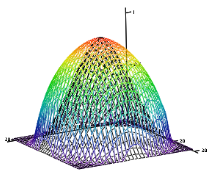
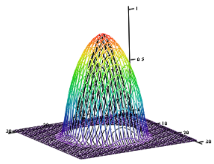
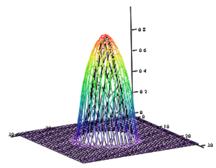
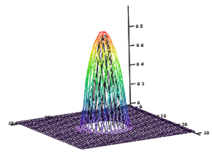
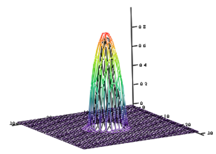
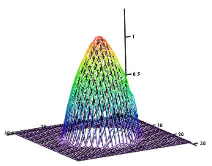
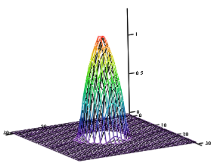
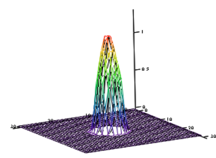
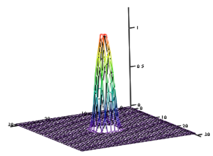
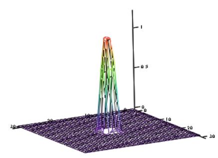
Parameter values	$x_1=1; x_2=1;$ $\lvert x\rvert=\sqrt{2}$	$x_1=2; x_2=2;$ $\lvert x\rvert=2\sqrt{2}$	$x_1=3; x_2=3;$ $\lvert x\rvert=3\sqrt{2}$	$x_1=4; x_2=4;$ $\lvert x\rvert=4\sqrt{2}$	$x_1=5; x_2=5;$ $\lvert x\rvert=5\sqrt{2}$
$m_1=2.5$, $m_2=2.4$, $p=3.1$, $eps=10^{-3}$, $n_1=0.944>0$, $n_2=0.708>0$, $q=-0.89<0$, $\beta_1=1$, $\beta_2=1$					
$m_1=3.7$, $m_2=3.8$, $p=4$, $eps=10^{-3}$ $n_1=0.59>0$, $n_2=0.674>0$, $q=-3.56<0$, $\beta_1=2$, $\beta_2=2$					
$m_1=4.1$, $m_2=4.0$, $p=4.4$, $eps=10^{-3}$, $n_1=0.672>0$, $n_2=0.576>0$, $q=-3.54<0$, $\beta_1=1$, $\beta_2=1$					
$m_1=3.7$, $m_2=3.3$, $p=4$, $eps=10^{-3}$, $n_1=0.95>0$, $n_2=0.407>0$, $q=-2.21<0$, $\beta_1=2$, $\beta_2=2$					
$m_1=7$, $m_2=7.1$, $p=4$, $eps=10^{-3}$, $n_1=0.368>0$, $n_2=0.377>0$, $q=-32.6<0$, $\beta_1=3$, $\beta_2=4$					

Table 2. Numerical results in the case of slow diffusion.

Parameter values	$x_1=1; x_2=1;$ $\lvert x\rvert=\sqrt{2}$	$x_1=2; x_2=2;$ $\lvert x\rvert=2\sqrt{2}$	$x_1=3; x_2=3;$ $\lvert x\rvert=3\sqrt{2}$	$x_1=4; x_2=4;$ $\lvert x\rvert=4\sqrt{2}$	$x_1=5; x_2=5;$ $\lvert x\rvert=5\sqrt{2}$
$m_1=1.3$, $m_2=1.2$, $p=2.5$, $eps=10^{-3}$, $n_1=1.579>0$, $n_2=2.368>0$, $q=0.19>0$, $\beta_1=1$, $\beta_2=1$					
$m_1=1.9$, $m_2=1.7$, $p=3$, $eps=10^{-3}$, $n_1=0.541>0$, $n_2=1.622>0$, $q=0.37>0$, $\beta_1=2, \beta_2=2$					
$m_1=3.7$, $m_2=3.5$, $p=5.4$, $eps=10^{-3}$, $n_1=0.64>0$, $n_2=0.823>0$, $q=4.81>0$, $\beta_1=3$, $\beta_2=4$					

1) Fast diffusion is shown in **Table 1**. As an initial approximation it is necessary to take:

$$u_0(x,t)=(T+t)^{-\alpha_1}\left(a+\xi^{\gamma}\right)^{n_1},\quad v_0(x,t)=(T+t)^{-\alpha_2}\left(a+\xi^{\gamma}\right)^{n_2},\quad \xi=\frac{|x|}{\tau^{\frac{1}{p}}},\quad \gamma=\frac{p}{p-1},$$

$$n_i=\frac{(p-1)\left[p-(m_i+1)\right]}{q},\quad i=1,2,\quad q=(p-2)^2-(m_1-1)(m_2-1)$$

Parameter values must be $n_1>0, n_2>0, q<0$. Constant a is determined from the condition $\int_{-\infty}^{\infty} u_1(x,0)\mathrm{d}x=P_1$, $\int_{-\infty}^{\infty} u_2(x,0)\mathrm{d}x=P_2$: $a=\left(P_1\gamma\Big/B\left(\frac{1}{\gamma},1+n_1\right)\right)^{\frac{\gamma}{n_1}}=\left(P_2\gamma\Big/B\left(\frac{1}{\gamma},1+n_2\right)\right)^{\frac{\gamma}{n_2}}$

2) Slow diffusion is shown in **Table 2**. As an initial approximation it is necessary to take:

$$u_0(x,t)=(T+t)^{-\alpha_1}\left(a-\xi^{\gamma}\right)_+^{n_1},\quad v_0(x,t)=(T+t)^{-\alpha_2}\left(a-\xi^{\gamma}\right)_+^{n_2},\quad \xi=\frac{|x|}{\tau^{\frac{1}{p}}},$$

$$\gamma=\frac{p}{p-1},\ \text{т},\quad i=1,2,\quad q=(p-2)^2-(m_1-1)(m_2-1)$$

Parameter values must be $n_1 > 0, n_2 > 0, q_i > 0$. Constant a is determined from the condition

$$a = \left(P_1\gamma \middle/ B\left(\frac{1}{\gamma}, 1+n_1\right) \right)^{\frac{\gamma}{n_1}} = \left(P_2\gamma \middle/ B\left(\frac{1}{\gamma}, 1+n_2\right) \right)^{\frac{\gamma}{n_2}}.$$

References

[1] Aripov, M. (1988) Method Reference Equations for the Solution of Nonlinear Boundary Value Problems. Fan, Tashkent, 137.

[2] Belotelov, N.V. and Lobanov, A.I. (1997) Population Model with Nonlinear Diffusion. *Mathematic Modeling*, **12**, 43-56.

[3] Volterra, V. (1976) The Mathematical Theory of the Struggle for Existence. Science, Moscow, 288.

[4] Gause, G.F. (1934) About the Processes of Destruction of One Species by Another in the Populations of Ciliates. *Zoological Journal*, **1**, 16-27.

[5] Aripov, M. and Muhammadiev, J. (1999) Asymptotic Behaviour of Automodel Solutions for One System of Quasilinear Equations of Parabolic Type. France. *Buletin Stiintific-Universitatea din Pitesti*, *Seria Matematica si Informatica*, 19-40.

[6] Aripov, M.M. and Muhamediyeva, D.K. (2013) To the Numerical Modeling of Self-Similar Solutions of Reaction-Diffusion System of the One Task of Biological Population of Kolmogorov-Fisher Type. *International Journal of Engineering and Technology*, **2**, 281-286.

[7] Aripov, M.M. and Muhamedieva, D.K. (2013) Approaches to the Solution of One Problem of Biological Populations. *Issues of Computational and Applied Mathematics*, **129**, 22-31.

[8] Murray, D.J. (1983) Nonlinear Diffusion Equations in Biology. Mir, Moscow, 397.

[9] Huashui, Z. (2010) The Asymptotic Behavior of Solutions for a Class of Doubly Degenerate Nonlinear Parabolic Equations. *Journal of Mathematical Analysis and Applications*, **370**, 1-10. http://dx.doi.org/10.1016/j.jmaa.2010.05.003