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On Stability of Nonlinear Differential System via Cone-Perturbing Liapunov Function Method

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Abstract

Totally equistable, totally ϕ_0-equistable, practically equistable, and practically ϕ_0-equistable of system of differential equations are studied. Cone valued perturbing Liapunov functions method and comparison methods are used. Some results of these properties are given.

Keywords

Totally Equistable, Totally ϕ_0-Equistable, Practically Equistable, Practically ϕ_0-Equistable

1. Introduction

Consider the non linear system of ordinary differential equations

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{1.1}$$

and the perturbed system

$$x' = f(t,x) + R(t,x), \quad x(t_0) = x_0. \tag{1.2}$$

Let R^n be Euclidean n-dimensional real space with any convenient norm $\|.\|$, and scalar product $(.,.) \le \|.\|\|.\|$. Let for some $\rho > 0$

$$S_\rho = \{x \in R^n, \|x\| < \rho\}$$

where $f, R \in C[J \times S_\rho, R^n]$, $J = [0,\infty)$ and $C[J \times S_\rho, R^n]$ denotes the space of continuous mappings $J \times S_\rho$ into R^n.

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Consider the scalar differential equations with an initial condition

$$u' = g_1(t,u), \quad u(t_0) = u_0, \tag{1.3}$$

$$\omega' = g_2(t,\omega), \quad \omega(t_0) = \omega_0 \tag{1.4}$$

and the perturbing equations

$$u' = g_1(t,u) + \varphi_1(t), \quad u(t_0) = u_0 \tag{1.5}$$

$$\omega' = g_2(t,\omega) + \varphi_2(t), \quad \omega(t_0) = \omega_0 \tag{1.6}$$

where $g_1, g_2 \in C[J \times R, R]$, $\varphi_1, \varphi_2 \in C[J, R]$ respectively.

Other mathematicians have been interested in properties of qualitative theory of nonlinear systems of differential equations. In last decade, in [1], some different concepts of stability of system of ordinary differential Equations (1.1) are considered namely, say totally stability, practically stability of (1.1), and (1.2); and in [2], methods of perturbing Liapunov function are used to discuss stability of (1.1). The authors in [3] discussed some stability of system of ordinary differential equations, and in [4] [5] the authors discussed totally and totally φ_0-stability of system of ordinary differential Equations (1.1) using Liapunov function method that was played essential role for determine stability of system of differential equations. In [6] the authors discussed practically stability for system of functional differential equations.

In [7], and [8], the authors discussed new concept namely, φ_0-equitable of the zero solution of system of ordinary differential equations using cone-valued Liapunov function method. In [4], the author discussed and improved some concepts stability and discussed concept mix between totally stability from one side and φ_0-stability on the other side.

In this paper, we will discuss and improve the concept of totally stability, practically stability of the system of ordinary differential Equations (1.1) with Liapunov function method, and comparison technique. Furthermore, we will discuss and improve the concept of totally φ_0-stability, and practically φ_0-stability of the system of ordinary differential Equations (1.1). These concepts are mix and lie somewhere between totally stability and practically stability from one side and φ_0-stability on the other side. Our technique depends on cone-valued Liapunov function method, and comparison technique. Also we give some results of these concepts of the zero solution of differential equations.

The following definitions [8] will be needed in the sequal.

Definition 1.1. A proper subset K of R^n is called a cone if

$$(\mathrm{i})\ \lambda K \subset K, \lambda \ge 0.\ (\mathrm{ii})\ K + K \subset K,\ (\mathrm{iii})\ \bar{K} = K,\ (\mathrm{iv})\ K^0 \ne \varnothing,\ (\mathrm{v})\ K \cap (-K) = \{0\}$$

where K and K^0 denote the closure and interior of K respectively and ∂K denotes the boundary of K.

Definition 1.2. The set $K^* = \{\phi \in R^n, (\phi, x) \ge 0, x \in K\}$ is called the adjoint cone if it satisfies the properties of the definition 3.1.

$$x \in \partial K \text{ if } (\phi, x) = 0 \text{ for some } \phi \in K_0^*, K_0 = K / \{0\}.$$

Definition 1.3. A function $g : D \to K, D \subset R^n$ is called quasimonotone relative to the cone K if

$$x, y \in D, y - x \in \partial K$$

then there exists $\phi_0 \in K_0^*$ such that

$$(\phi_0, y - x) = 0 \text{ and } (\phi_0, g(y) - g(x)) > 0.$$

Definition 1.4. A function $a(r)$ is said to belong to the class $\mathcal{K}$ if $a \in [R^+, R^+], a(0) = 0$ and $a(r)$ is strictly monotone increasing in r.

2. Totally Equistable

In this section we discuss the concept of totally equistable of the zero solution of (1.1) using perturbing Liapuniv functions method and Comparison principle method.

We define for $V \in C\left[J \times S_\rho, R^n\right]$, the function $D^+V(t,x)$ by

$$D^+V(t,x)_{1.2} = \lim_{h\to 0}\sup \frac{1}{h}\Big(V\big(t+h, x+h\big(f(t,x)+R(t,x)\big)\big)-V(t,x)\Big).$$

The following definition [1] will be needed in the sequal.

Definition 2.1. The zero solution of the system (1.1) is said to be T_1-totally equistable (stable with respect to permanent perturbations), if for every $\epsilon > 0, t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for every solution of perturbed Equation (1.2), the inequality

$$\left\|x(t,t_0,x_0)\right\| < \epsilon \quad for\ t \ge t_0$$

holds, provided that $\|x_0\| < \delta_1$ and $\|R(t,x)\| < \delta_2$.

Definition 2.2. The zero solution of the Equation (1.3) is said to be T_1-totally equistable (stable with respect to permanent perturbations), if for every $\epsilon > 0, t_0 \in J$, there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that for every solution of perturbed Equation (1.5). The inequality

$$u(t,t_0,u_0) < \epsilon, \quad t \ge t_0$$

holds, provided that $u_0 < \delta_1^*$ and $\varphi_1(t) < \delta_2^*$.

Theorem 2.1. Suppose that there exist two functions

$$g_1, g_2 \in C[J\times R, R] \text{ with } g_1(t,0) = g_2(t,0) = 0$$

and there exist two Liapunov functions

$$V_1 \in C\left[J \times S_\rho, R^n\right] \text{ and } V_{2\eta} \in C\left[J\times S_\rho \cap S_\eta^C, R^n\right] \text{ with } V_1(t,0) = V_{2\eta}(t,0) = 0$$

where $S_\eta = \left\{x \in R^n, \|x\| < \eta\right\}$ for $\eta > 0$ and S_η^C denotes the complement of S_η satisfying the following conditions:

(H_1) $V_1(t,x)$ is locally Lipschitzian in x.

$$D^+V_1(t,x) \le g_1\big(t, V_1(t,x)\big), \ \forall (t,x) \in J\times S_\rho.$$

(H_2) $V_{2\eta}(t,x)$ is locally Lipschitzian in x.

$$b\big(\|x\|\big) \le V_{2\eta}(t,x) \le a\big(\|x\|\big), \quad \forall (t,x)\in J\times S_\rho \cap S_\eta^C$$

where $a, b \in \mathcal{K}$ are increasing functions.

(H_3)

$$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \le g_2\big(t, V_1(t,x)+V_{2\eta}(t,x)\big), \quad \forall (t,x)\in J\times S_\rho\cap S_\eta^C.$$

(H_4) If the zero solution of (1.3) is equistable, and the zero solution of (1.4) is totally equistable.

Then the zero solution of (1.1) is totally equistable.

Proof. Since the zero solution of the system (1.4) is totally equistable, given $b(\epsilon) > 0$, there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that for every solution $\omega(t,t_0,\omega_0)$ of perturbed equation (1.6) the inequality

$$\omega(t,t_0,\omega_0) < \epsilon, \quad t \ge t_0 \tag{2.1}$$

holds, provided that $\omega_0 < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$.

Since the zero solution of (1.3) is equistable given $\frac{\delta_0(\epsilon)}{2}$ and $t_0 \in J$, there exists $\delta = \delta(t_0,\epsilon) > 0$ such that

$$u(t,t_0,u_0) < \frac{\delta_0(\epsilon)}{2} \tag{2.2}$$

holds, provided that $u_0 \le \delta$.

From the condition (H2) we can find $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) + \frac{\delta_0}{2} < \delta_1^*. \tag{2.3}$$

To show that the zero solution of (1.1) is T_1-totally equistable, it must show that for every $\epsilon > 0, t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for every solution $x(t, t_0, x_0)$ of perturbed Equation (1.2). The inequality

$$\|x(t, t_0, x_0)\| < \epsilon \quad \text{for } t \ge t_0$$

holds, provided that $\|x_0\| < \delta_1$ and $\|R(t, x)\| < \delta_2$.

Suppose that this is false, then there exists a solution $x(t, t_0, x_0)$ of (1.2) with $t_1 > t_0$ such that

$$\|x(t_0, t_0, x_0)\| = \delta_1, \ \|x(t_1, t_0, x_0)\| = \epsilon \tag{2.4}$$

$$\delta_1 \le \|x(t, t_0, x_0)\| \le \epsilon \quad \text{for } t \in [t_0, t_1].$$

Let $\delta_1 = \eta$ and setting $m(t, x) = V_1(t, x) + V_{2\eta}(t, x)$.

Since $V_1(t, x)$ and $V_{2\eta}(t, x)$ are Lipschitzian in x for constants M_1 and M_2 respectively. Then

$$D^+V_1(t, x)_{1.2} + D^+V_{2\eta}(t, x)_{1.2} \le D^+V_1(t, x)_{1.1} + D^+V_{2\eta}(t, x)_{1.1} + M\|R(t, x)\|$$

where $M = M_1 + M_2$ From the condition (H3) we obtain the differential inequality

$$D^+V_1(t, x) + D^+V_{2\eta}(t, x) \le g_2(t, V_1(t, x) + V_{2\eta}(t, x)) + M\|R(t, x)\|$$

for $t \in [t_0, t_1]$ Then we have

$$D^+m(t, x) \le g_2(t, m(t, x)) + M\|R(t, x)\|.$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0)$.

Applying the comparison Theorem (1.4.1) of [1], it yields

$$m(t, x) \le r_2(t, t_0, \omega_0) \qquad \text{for } t \in [t_0, t_1]$$

where $r_2(t, t_0, \omega_0)$ is the maximal solution of the perturbed Equation (1.6).

Define $\varphi_2(t) = M\|R(t, x)\|$.

To prove that

$$r_2(t, t_0, \omega_0) < b(\epsilon).$$

It must be show that

$$\omega_0 < \delta_1^* \ \text{ and } \ \varphi_2(t) < \delta_2^*.$$

Choose $u_0 = V_1(t_0, x_0)$. From the condition (H1) and applying the comparison Theorem of [1], it yields

$$V_1(t, x) \le r_1(t, t_0, u_0)$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).

From (2.2) at $t = t_0$

$$V_1(t_0, x_0) \le r_1(t_0, t_0, u_0) < \frac{\delta_0(\epsilon)}{2}. \tag{2.5}$$

From the condition (H_2) and (2.4), at $t = t_0$

$$V_{2\eta}(t_0, x_0) \le a(\|x_0\|) \le a(\delta_1). \tag{2.6}$$

From (2.3), we get

$$\omega_0 = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0) \le \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_1^*.$$

Since $\varphi_2(t) = M\|R(t,x)\| \le M\delta_2 = \delta_2^*$.
From (2.1), we get

$$m(t,x) \le r_2(t, t_0, \omega_0) < b(\epsilon). \tag{2.7}$$

Then from the condition (H_2), (2.4) and (2.7) we get $t = t_1$

$$b(\epsilon) = b(\|x(t_1)\|) \le V_{2\eta}(t_1, x(t_1)) < m(t_1, x(t_1)) \le r_2(t_1, t_0, \omega_0) < b(\epsilon).$$

This is a contradiction, then it must be

$$\|x(t, t_0, x_0)\| < \epsilon \quad \text{for } t \ge t_0$$

holds, provided that $\|x_0\| < \delta_1$ and $\|R(t,x)\| < \delta_2$.
Therefore the zero solution of (1.1) is totally equistable.

3. Totally ϕ_0-Equistable

In this section we discuss the concept of Totally ϕ_0-equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definition [4] will be needed in the sequal.

Definition 3.1. The zero solution of the system (1.1) is said to be totally ϕ_0-equistable (ϕ_0-equistable with respect to permanent perturbations), if for every $\epsilon > 0$, $t_0 \in J$ and $\phi_0 \in K_0^*$, there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < \epsilon \quad \text{for } t \ge t_0$$

holds, provided that $(\phi_0, x_0) < \delta_1$ and $\|R(t,x)\| < \delta_2$ where $x(t, t_0, x_0)$ is the maximal solution of perturbed Equation (1.2).

Let for some $\rho > 0$

$$S_\rho^* = \{x \in R^n, (\phi_0, x) < \rho, \phi_0 \in K_0^*\}.$$

Theorem 3.1. Suppose that there exist two functions

$$g_1, g_2 \in C[J \times R, R] \quad \text{with} \quad g_1(t,0) = g_2(t,0) = 0$$

and let there exist two cone valued Liapunov functions

$$V_1 \in C[J \times S_\rho^*, K] \quad \text{and} \quad V_{2\eta} \in C[J \times S_\rho^* \cap S_\eta^{*C}, K] \quad \text{with} \quad V_1(t,0) = V_{2\eta}(t,0) = 0$$

where $S_\eta^* = \{x \in K, (\phi_0, x) < \eta, \phi_0 \in K_0^*\}$ for $\eta > 0$ and S_η^{*C} denotes the complement of S_η^* satisfying the following conditions:

(h_1) $V_1(t,x)$ is locally Lipschitzian in x and

$$D^+(\phi_0, V_1(t,x)) \le g_1(t, V_1(t,x)) \quad \text{for } (t,x) \in J \times S_\rho^*.$$

(h_2) $V_{2\eta}(t,x)$ is locally Lipschitzian in x and

$$b(\phi_0, x) \le (\phi_0, V_{2\eta}(t,x)) \le a(\phi_0, x) \quad \text{for } (t, x_t) \in J \times S_\rho^* \cap S_\eta^{*C}$$

where $a, b \in \mathcal{K}$ are increasing functions.

(h_3) $D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2\eta}(t,x)) \le g_2(t, V_1(t,x) + V_{2\eta}(t,x))$ for $(t,x) \in J \times S^*_\rho \cap S^{*C}_\eta$.

(h_4) If the zero solution of (1.3) is ϕ_0-equistable, and the zero solution of (1.4) is totally ϕ_0-equistable. Then the zero solution of (1.1) is totally ϕ_0-equistable.

Proof. Since the zero solution of (1.4) is totally ϕ_0-equistable, given, given $b(\epsilon) > 0$ there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that the inequality

$$(\phi_0, r_2(t, t_0, \omega_0)) < \epsilon, \quad t \ge t_0 \tag{3.1}$$

holds, provided that $(\phi_0, \omega_0) < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$. where $r_2(t, t_0, \omega_0)$ is the maximal solution of perturbed Equation (1.6).

Since the zero solution of the system (1.3) is ϕ_0-equistable, given $\frac{\delta_0(\epsilon)}{2}$ and $t_0 \in J$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that

$$(\phi_0, r_1(t, t_0, u_0)) < \frac{\delta_0(\epsilon)}{2} \tag{3.2}$$

holds, provided that $(\phi_0, u_0) \le \delta$ where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).

From the condition (h_2) we can choose $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) + \frac{\delta_0}{2} < \delta_1^*. \tag{3.3}$$

To show that the zero solution of (1.1) is T_1-totally ϕ_0-equistable, it must be prove that for every $\epsilon > 0, t_0 \in J$ and $\phi_0 \in K_0^*$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < \epsilon \quad \text{for } t \ge t_0$$

holds, provided that $(\phi_0, x_0) < \delta_1$ and $\|R(t,x)\| < \delta_2$ where $x(t, t_0, x_0)$ is the maximal solution of perturbed Equation (1.2).

Suppose that is false, then there exists a solution $x(t, t_0, x_0)$ of (1.2) with $t_1 > t_0$ such that

$$(\phi_0, x(t_0, t_0, x_0)) = \delta_1, \ (\phi_0, x(t_1, t_0, x_0)) = \epsilon \tag{3.4}$$

$$\delta_1 \le (\phi_0, x(t, t_0, x_0)) \le \epsilon \quad \text{for } t \in [t_0, t_1].$$

Let $\delta_1 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$.

Since $V_1(t,x)$ and $V_{2\eta}(t,x)$ are Lipschitzian in x for constants M_1 and M_2 respectively.

Then

$$\begin{aligned} &D^+(\phi_0, V_1(t,x))_{1.2} + D^+(\phi_0, V_{2\eta}(t,x))_{1.2} \\ &\le D^+(\phi_0, V_1(t,x))_{1.1} + D^+(\phi_0, V_{2\eta}(t,x))_{1.1} + MR(t,x) \end{aligned}$$

where $M = M_1 + M_2$ From the condition (h_3) we obtain the differential inequality

$$D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2\eta}(t,x)) \le g_2(t, V_1(t,x) + V_{2\eta}(t,x)) + M\|R(t,x)\|$$

for $t \in [t_0, t_1]$ Then we have

$$D^+(\phi_0, m(t,x)) \le g_2(t, m(t,x)) + M\|R(t,x)\|.$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0)$. Applying the comparison Theorem of [1], yields

$$(\phi_0, m(t,x)) \le (\phi_0, r_2(t, t_0, \omega_0)) \quad \text{for } t \in [t_0, t_1].$$

Define $\varphi_2(t) = M\|R(t,x)\|$.

To prove that

$$\left(\phi_0, r_2\left(t, t_0, \omega_0\right)\right) < b(\epsilon).$$

It must be shown that

$$\left(\phi_0, \omega_0\right) < \delta_1^* \text{ and } \varphi_2(t) < \delta_2^*.$$

Choose $u_0 = V_1\left(t_0, x_0\right)$. From the condition (h$_1$) and applying the comparison Theorem [1], it yields

$$\left(\phi_0, V_1(t, x)\right) \le \left(\phi_0, r_1\left(t, t_0, u_0\right)\right).$$

From (3.2) at $t = t_0$

$$\left(\phi_0, V_1\left(t_0, x_0\right)\right) \le \left(\phi_0, r_1\left(t_0, t_0, u_0\right)\right) < \frac{\delta_0(\epsilon)}{2}. \tag{3.5}$$

From the condition (h$_2$) and (3.4), at $t = t_0$

$$\left(\phi_0, V_{2\eta}\left(t_0, x_0\right)\right) \le a\left(\phi_0, x_0\right) \le a\left(\delta_1\right). \tag{3.6}$$

From (3.3), we get

$$\left(\phi_0, \omega_0\right) = \left(\phi_0, V_1\left(t_0, x_0\right)\right) + \left(\phi_0, V_{2\eta}\left(t_0, x_0\right)\right) \le \frac{\delta_0(\epsilon)}{2} + a\left(\delta_1\right) < \delta_1^*.$$

Since $\varphi_2(t) = M\left\|R(t, x)\right\| \le M\delta_2 = \delta_2^*$.
From (3.1), we get

$$\left(\phi_0, m(t, x)\right) \le \left(\phi_0, r_2\left(t, t_0, \omega_0\right)\right) < b(\epsilon). \tag{3.7}$$

Then from the condition (h$_2$), (3.4) and (3.7) we get at $t = t_1$

$$b(\epsilon) = b\left(\phi_0, x\left(t_1\right)\right) \le \left(\phi_0, V_{2\eta}\left(t_1, x\left(t_1\right)\right)\right) < \left(\phi_0, m\left(t_1, x\left(t_1\right)\right)\right) \le \left(\phi_0, r_2\left(t_1, t_0, \omega_0\right)\right) < b(\epsilon).$$

This is a contradiction, then

$$\left(\phi_0, x\left(t, t_0, x_0\right)\right) < \epsilon \quad \text{for } t \ge t_0$$

provided that $\left(\phi_0, x_0\right) < \delta_1$ and $\left\|R(t, x)\right\| < \delta_2$ where $x\left(t, t_0, x_0\right)$ is the maximal solution of perturbed equation (1.2). Therefore the zero solution of (1.1) is totally ϕ_0-equistable.

4. Practically Equistable

In this section, we discuss the concept of practically equistable of the zero solution of (1.1) using perturbing Liapunov functions method and Comparison principle method.

The following definition [8] will be needed in the sequal.

Definition 4.1. Let $0 < \lambda < A$ be given. The system (1.1) is said to be practically equistable if for $t_0 \in J$ such that the inequality

$$\left\|x\left(t, t_0, x_0\right)\right\| < A \quad \text{for } t \ge t_0 \tag{4.1}$$

holds, provided that $\left\|x_0\right\| < \lambda$ where $x\left(t, t_0, x_0\right)$ is any solution of (1.1).

In case of uniformly practically equistable, the inequality (4.1) holds for any t_0.

We define

$$S(A) = \left\{x \in R^n : \|x\| \le A, A > 0\right\}.$$

Theorem 4.1. Suppose that there exist two functions

$$g_1, g_2 \in C[J \times R, R] \quad \text{with} \quad g_1(t,0) = g_2(t,0) = 0$$

and there exist two Liapunov functions

$$V_1 \in C\left[J \times S(A), R^n\right] \text{ and } V_{2\eta} \in C\left[J \times S(A) \cap S(B)^C, R^n\right] \text{ with } V_1(t,0) = V_{2B}(t,0) = 0$$

where $S(B) = \{x \in R^n, x < B, 0 < B < A\}$ and $S(B)^C$ denotes the complement of $S(B)$ satisfying the following conditions:

(I) $V_1(t,x)$ is locally Lipschitzian in *x*.

$$D^+V_1(t,x) \le g_1(t, V_1(t,x)), \quad \forall (t,x) \in J \times S(A).$$

(II) $V_{2B}(t,x)$ is locally Lipschitzian in *x*.

$$b(\|x\|) \le V_{2B}(t,x) \le a(\|x\|), \quad \forall (t,x) \in J \times S(A) \cap S(B)^C$$

where $a, b \in \mathcal{K}$ are increasing functions.

(III)

$$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \le g_2(t, V_1(t,x) + V_{2B}(t,x)), \quad \forall (t,x) \in J \times S(A) \cap S(B)^C.$$

(IV) If the zero solution of (1.3) is equistable, and the zero solution of (1.4) is uniformly practically equistable. Then the zero solution of (1.1) is practically equistable.

Proof. Since the zero solution of (1.4) is uniformly practically equistable, given $0 < \lambda_0 < A$ such that for every solution $\omega(t, t_0, \omega_0)$ of (1.4) the inequality

$$\omega(t, t_0, \omega_0) < b(A) \tag{4.2}$$

holds provided $\omega_0 \le \lambda_0$.

Since the zero solution of the system (1.3) is equistable, given $\frac{\lambda_0}{2}$ and $t_0 \in R_+$ there exist $\delta = \delta(t_0, \epsilon) > 0$ such that for every solution $u(t, t_0, u_0)$ of (1.3)

$$u(t, t_0, u_0) < \frac{\lambda_0}{2} \tag{4.3}$$

holds provided that $u_0 \le \delta$.

From the condition (II) we can find $\lambda > 0$ such that

$$a(\lambda) + \frac{\lambda_0}{2} \le \lambda_0. \tag{4.4}$$

To show that the zero solution of (1.1) practically equistable, it must be exist $0 < \lambda < A$ such that for any solution $x(t, t_0, x_0)$ of (1.1) the inequality

$$\|x(t, t_0, x_0)\| < A \qquad \text{for } t \ge t_0$$

holds, provided that $\|x_0\| < \lambda$.

Suppose that this is false, then there exists a solution $x(t, t_0, x_0)$ of (1.1) with $t_1 > t_0$ such that

$$\|x(t_0, t_0, x_0)\| = \lambda, \ \|x(t_1, t_0, x_0)\| = A \tag{4.5}$$

$$\lambda \le \|x(t, t_0, x_0)\| \le A \qquad \text{for } t \in [t_0, t_1].$$

Let $\lambda = B$ and setting

$$m(t,x) = V_1(t,x) + V_{2\eta}(t,x).$$

From the condition (III) we obtain the differential inequality for $t \in [t_0, t_1]$

$$D^+ m(t,x) \le g_2(t, m(t,x)).$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2B}(t_0, x_0)$.
Applying the comparison Theorem [8], yields

$$m(t,x) \le r_2(t, t_0, \omega_0) \quad \text{for } t \in [t_0, t_1]$$

where $r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4).
To prove that

$$r_2(t, t_0, \omega_0) < b(A).$$

It must be show that $\omega_0 \le \lambda_0$.
Choose $u_0 = V_1(t_0, x_0)$, from the condition (II) and applying the comparison Theorem of [1], yields

$$V_1(t,x) \le r_1(t, t_0, u_0)$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).
From (4.3) at $t = t_0$

$$V_1(t,x) \le r_1(t, t_0, u_0) < \frac{\lambda_0}{2}. \tag{4.6}$$

From the condition (II) and (4.5), at $t = t_0$

$$V_{2B}(t_0, x_0) \le a(\|x(t_0)\|) \le a(\lambda). \tag{4.7}$$

From (4.4), (4.6) and (4.7), we get

$$\omega_0 = V_1(t_0, x_0) + V_{2B}(t_0, x_0) \le \lambda_0.$$

From (4.2), we get

$$m(t,x) \le r_2(t, t_0, \omega_0) < b(A). \tag{4.8}$$

Then from the condition (II), (4.5) and (4.8), we get at $t = t_1$

$$b(A) = b(\|x(t_1)\|) \le V_{2B}(t_1, x_1) < m(t_1, x(t_1)) \le r_2(t_1, t_0, \omega_0) < b(A).$$

This is a contradiction, then

$$\|x(t, t_0, x_0)\| < A \quad \text{for } t \ge t_0$$

provided that $\|x_0\| < \lambda$.
Therefore the zero solution of (1.1) is practically equistable.

5. Practically ϕ_0-Equistable

In this section we discuss the concept of practically ϕ_0-equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and comparison principle method.

The following definitions [6] will be needed in the sequal.

Definition 5.1. Let $0 < \lambda < A$ be given. The system (1.1) is said to be practically ϕ_0-equistable, if for $t_0 \in J$ and $\phi_0 \in K_0^*$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < A \quad \text{for } t \ge t_0 \tag{5.1}$$

holds, provided that $(\phi_0, x_0) < \lambda$ where $x(t, t_0, x_0)$ is the maximal solution of (1.1).

In case of uniformly practically ϕ_0-equistable, the inequality (5.1) holds for any t_0.

We define

$$S^*(A) = \{x \in K, (\phi_0, x) < A, \phi_0 \in K_0^*\}.$$

Theorem 5.1. Suppose that there exist two functions

$$g_1, g_2 \in C[J \times R, R] \quad \text{with} \quad g_1(t,0) = g_2(t,0) = 0$$

and let there exist two cone valued Liapunov functions

$$V_1 \in C\left[J \times S^*(A), K\right] \quad \text{and} \quad V_{2B} \in C\left[J \times S^*(A) \cap S^*(B)^C, K\right] \quad \text{with} \quad V_1(t,0) = V_{2B}(t,0) = 0$$

where $S^*(B) = \{x \in K, (\phi_0, x_0) < B, 0 < B < A, \phi_0 \in K_0^*\}$ and $S^*(B)^C$ denotes the complement of $S^*(B)$ satisfying the following conditions:

(i) $V_1(t,x)$ is locally Lipschitzian in x relative to K.

$$D^+(\phi_0, V_1(t,x)) \le g_1(t, V_1(t,x)), \quad \forall (t,x) \in J \times S^*(A).$$

(ii) $V_{2B}(t,x)$ is locally Lipschitzian in x relative to K.

$$b(\phi_0, x) \le (\phi_0, V_{2B}(t,x)) \le a(\phi_0, x), \quad \forall (t,x) \in J \times S^*(A) \cap S^*(B)^C$$

where $a, b \in \mathcal{K}$ are increasing functions.

(iii) $D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2B}(t,x)) \le g_2(t, V_1(t,x) + V_{2B}(t,x)), \ \forall (t,x) \in J \times S^*(A) \cap S^*(B)^C$.

(iv) If the zero solution of (1.3) is ϕ_0-equistable, and the zero solution of (1.4) is uniformly practically ϕ_0-equistable.

Then the zero solution of (1.1) is practically ϕ_0-equistable.

Proof. Since the zero solution of the system (1.4) is uniformly practically ϕ_0-equistable, given $0 < \lambda_0 < a(B)$ for $a(B) > 0$ such that the inequality

$$(\phi_0, r_2(t, t_0, \omega_0)) < a(B) \tag{5.2}$$

holds provided $(\phi_0, \omega_0) \le \lambda_0$, where $r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4).

Since the zero solution of the system (1.3) is ϕ_0-equistable, given $\frac{\lambda_0}{2}$ and $t_0 \in R_+$ there exist $\delta = \delta(t_0, \lambda_0)$ such that the inequality

$$(\phi_0, r_1(t, t_0, u_0)) < \frac{\lambda_0}{2}. \tag{5.3}$$

From the condition (ii), assume that

$$a(B) \le b(A) \tag{5.4}$$

also we can choose $\lambda_1 > 0$ such that

$$a(\lambda) + \frac{\lambda_0}{2} \le \lambda_0. \tag{5.5}$$

To show that the zero solution of (1.1) is practically ϕ_0-equistable. It must be show that for $0 < \lambda < A, t_0 \in J$ and $\phi_0 \in K_0^*$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < A \quad \text{for } t \ge t_0$$

holds, provided that $(\phi_0, x_0) < \lambda$ where $x(t, t_0, x_0)$ is the maximal solution of (1.1).

Suppose that is false, then there exists a solution $x(t, t_0, x_0)$ of (1.1) with $t_2 > t_1 > t_0$ such that for $(\phi_0, x_0) < \lambda$ where $\lambda = \min(\lambda_0, \lambda_1)$

$$\left(\phi_0, x\left(t_1, t_0, x_0\right)\right) = \lambda_1, \quad \left(\phi_0, x\left(t_2, t_0, x_0\right)\right) = A \tag{5.6}$$

$$\lambda_1 \le \left(\phi_0, x\left(t, t_0, x_0\right)\right) \le A \quad \text{for } t \in \left[t_1, t_2\right].$$

Let $\lambda_1 = B$ and setting

$$m(t,x) = V_1(t,x) + V_{2B}(t,x).$$

From the condition (iii) we obtain the differential inequality

$$D^+\left(\phi_0, m(t,x)\right) \le \left(\phi_0, g_2\left(t, m(t,x)\right)\right) \quad \text{for } t \in \left[t_1, t_2\right].$$

Let $\omega_0 = m\left(t_1, x\left(t_1\right)\right) = V_1\left(t_1, x\left(t_1\right)\right) + V_{2B}\left(t_1, x\left(t_1\right)\right).$
Applying the comparison Theorem of [1], yields

$$\left(\phi_0, m(t,x)\right) \le \left(\phi_0, r_2\left(t, t_0, \omega_0\right)\right).$$

To prove that

$$\left(\phi_0, r_2\left(t, t_0, \omega_0\right)\right) < a(B).$$

It must be show that

$$\left(\phi_0, \omega_0\right) \le \lambda_0.$$

Choose $u_0 = V_1\left(t_0, x_0\right)$ From the condition (i) and applying the comparison Theorem of [1], yield

$$\left(\phi_0, V_1(t,x)\right) \le \left(\phi_0, r_1\left(t, t_0, u_0\right)\right).$$

From (5.3) at $t = t_1$

$$\left(\phi_0, V_1(t,x)\right) \le \left(\phi_0, r_1\left(t, t_0, u_0\right)\right) < \frac{\lambda_0}{2}. \tag{5.8}$$

From the condition (ii) and (5.6), at $t = t_1$

$$\left(\phi_0, V_{2B}\left(t_1, x\left(t_1\right)\right)\right) \le \left(\phi_0, x\left(t_1\right)\right) \le a\left(\lambda_1\right). \tag{5.9}$$

From (5.5), (5.8) and (5.9), we get

$$\left(\phi_0, \omega_0\right) = \left(\phi_0, V_1\left(t_1, x\left(t_1\right)\right)\right) + \left(\phi_0, V_{2B}\left(t_1, x\left(t_1\right)\right)\right) \le \lambda_0.$$

From (5.2), we get

$$\left(\phi_0, m(t,x)\right) \le \left(\phi_0, r_2\left(t, t_0, \omega_0\right)\right) < a(B). \tag{5.10}$$

Then from the condition (ii), (5.4), (5.6) and (5.10), we get at $t = t_2$

$$b(A) = b\left(\phi_0, x\left(t_2\right)\right) \le \left(\phi_0, m\left(t_2, x\left(t_2\right)\right)\right) < \left(\phi_0, r_2\left(t_2, t_0, \omega_0\right)\right) < a(B) \le a(A)$$

which leads to a contradiction, then it must be

$$\left(\phi_0, x\left(t, t_0, x_0\right)\right) < A \quad \text{for } t \ge t_0$$

holds, provided that $\left(\phi_0, x_0\right) < \lambda$. Therefore the zero solution of (1.1) is practically ϕ_0-equistable.

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