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Tight Monomials in Quantum Group for Type A_5 with $t \leq 6$

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Abstract

All tight monomials in quantum group for type A_5 with $t \leq 6$ are determined in this paper.

Keywords

Quantum Group, Canonical Basis, Tight Monomial

1. Introduction

The term "quantum groups" was popularized by V. G. Drinfel'd in his address to the International Congress of Mathematicians (ICM) in Berkeley (1986). However, quantum groups are actually not groups; they are nontrivial deformations of the universal enveloping algebras of semisimple Lie algebras, also called quantized enveloping algebras. These algebras were introduced independently by Drinfel'd [1] (in his definition, these algebras were infinitesimal, *i.e.*, they were Hopf algebras over the field of formal power series) and Jimbo [2] (in his definition, these algebras were Hopf algebras over the field of rational functions in one variable) in 1985 in their study of exactly solvable models in the statistical mechanics. Quantum groups play an important role in the study of Lie groups, Lie algebras, algebraic groups, Hopf algebras, etc.; they are also closely linked with conformal field theory, quiver theory and knot theory.

The positive part of a quantum group has a kind of important basis, *i.e.*, canonical basis introduced by Lusztig [3], which plays an important role in the theory of quantum groups and their representations. However, it is difficult to determine the elements in canonical basis, which is interested in seeking the simplest elements in canonical basis, *i.e.*, monomial basis elements. Some efforts have been done for monomial basis elements in quantum group of type A_n. Lusztig firstly introduced algebraic definition of canonical basis of quantum groups for the simply laced case (*i.e.*, A_n, D_n, E_n), and gave explicitly the longest monomials for type A_1, A_2, which were all of canonical basis elements (see [3]). Then, Lusztig [4] associated a quadratic form to every monomial. He showed

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that, given certain linear conditions, the monomial was tight, *i.e.*, it belonged to canonical basis (respectively, semitight, *i.e.*, it was a linear combination of elements in canonical basis with constant coefficients in $\mathbb{N}$) provided that the quadratic form satisfied a certain positivity condition (respectively, nonnegativity condition). He showed that the positivity condition (for tightness) always held in type A_3 and computed 8 longest tight monomials of type A_3. He also asked when we had (semi)tightness in type A_n. Based on Lusztig's work, Xi [5] found explicitly all 14 canonical basis elements of type A_3 (consisting of 8 longest monomials and 6 polynomials with one-dimensional support). For type A_4, Hu, Ye and Yue [6] determined all 62 longest monomials in canonical basis, Hu and Ye [7] gave all 144 polynomials with one-dimensional support in canonical basis, and Li and Hu [8] got 112 polynomials with two-dimensional support in canonical basis. For type A_n ($n \geq 5$), Marsh [9] carried out thorough investigation. He presented a semitight longest monomial for type A_5. However, he proved that a class of special longest monomials did not satisfy sufficient condition of tightness or semitightness for type A_n ($n \geq 6$) (although it might turn out that the corresponding monomials were still tight). Reineke [10] associated a new quadratic form to every monomial, and gave a sufficient and necessary condition for the monomial to be tight for the simply laced case in terms of the quadratic form. By use of this criterion, Wang [11] listed all tight monomials for type A_3, in which 8 longest tight monomials were the same as Lusztig and Xi's results.

Based on Reineke's criterion and some other results, all tight monomials for type A_5 with $t \leq 6$ are determined in this paper.

2. Preliminaries

Let $C = (c_{ij})_{i,j\in\Gamma_0}$ be a Cartan matrix of finite type, $D = \mathrm{diag}(d_i)_{i\in\Gamma_0}$ be a diagonal matrix with integer entries making the matrix DC symmetric. Let $\mathfrak{g} = \mathfrak{g}(C)$ be the complex semisimple Lie algebra associated with C, and let $\mathbf{U} = \mathbf{U}_v(\mathfrak{g})$ (here v is an indeterminate) be the corresponding quantized enveloping algebra, whose positive part $\mathbf{U}^+$ is the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}$ generated by $E_i, i \in \Gamma_0$, subject to the relations

$$\sum_{r+s=1-c_{ij}} (-1)^s E_i^{(s)} E_j E_i^{(r)} = 0, \forall i, j \in \Gamma_0,$$

where $E_i^{(s)} = E_i^s / [s]_i^!, [s]_i^! = [1]_i [2]_i \cdots [s]_i, [a]_i = \left(v^{ad_i} - v^{-ad_i}\right) / \left(v^{d_i} - v^{-d_i}\right)$. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, U^+ be the $\mathcal{A}$-subalgebra of $\mathbf{U}^+$ generated by $E_i^{(s)}, \forall i \in \Gamma_0, \forall s \in \mathbb{N}$. Corresponding to every reduced expression $\mathbf{i}$ of the longest element of the Weyl group of $\mathfrak{g}$, one constructs a PBW basis $B_{\mathbf{i}}$ of $\mathbf{U}^+$. Lusztig proved that the $\mathbb{Z}[v^{-1}]$-lattice $\mathcal{L}_{\mathbf{i}}$ spanned by $B_{\mathbf{i}}$ is independent of the choice of $\mathbf{i}$, write $\mathcal{L}$; and the image of $B_{\mathbf{i}}$ in the $\mathbb{Z}$-module $\mathcal{L}/v^{-1}\mathcal{L}$ is a basis B of $\mathcal{L}/v^{-1}\mathcal{L}$ independent of $\mathbf{i}$. Let $\bar{\mathcal{L}}$ be the image of $\mathcal{L}$ under the bar map of $\mathbf{U}^+$ defined by $E_i \mapsto E_i, i \in \Gamma_0$ and $v \mapsto v^{-1}$. Canonical basis $\mathbf{B}$ is the preimage of B under $\mathbb{Z}$-module isomorphism $\mathcal{L} \cap \bar{\mathcal{L}} \cong \mathcal{L}/v^{-1}\mathcal{L}$.

A monomial in $\mathbf{U}^+$ is an element of the form

$$E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_t}^{(a_t)} \tag{$*$}$$

where $i_1, i_2, \cdots, i_t \in \Gamma_0, a_1, a_2, \cdots, a_t \in \mathbb{N}$. When $t = v, s_{i_1} s_{i_2} \cdots s_{i_v} = w_0$ is the longest element of Weyl group, the monomial ($*$) is called the longest monomial. We say that ($*$) is tight if it belongs to $\mathbf{B}$; we say that ($*$) is semitight if it is a linear combination of elements in $\mathbf{B}$ with constant coefficients.

Let $Q = (Q_0, Q_1)$ be a finite quiver with vertex set Q_0 and arrow set Q_1. Write $\rho \in Q_1$ as $t_\rho \xrightarrow{\rho} h_\rho$, where h_ρ and t_ρ denote the head and the tail of ρ respectively. An automorphism σ of Q is a permutation on the vertices of Q and on the arrows of Q such that $\sigma(h_\rho) = h_{\sigma(\rho)}$ and $\sigma(t_\rho) = t_{\sigma(\rho)}$ for any $\rho \in Q_1$. Denote the quiver with automorphism σ as (Q, σ). Attach to the pair (Q, σ) a valued quiver $\Gamma = \Gamma(Q, \sigma) = (\Gamma_0, \Gamma_1)$ as follows. Its vertex set Γ_0 and arrow set Γ_1 are simply the sets of σ-orbits in Q_0 and Q_1, respectively. The valuation of Γ is given by $d_i = \#\{\text{vertices in the } \sigma\text{-orbit of } i\}$, $\forall i \in \Gamma_0$; $m_\rho = \#\{\text{arrows in the } \sigma\text{-orbit of } \rho\}$, $\forall \rho \in \Gamma_1$. The Euler form of Γ is defined to be the bilinear form $\langle , \rangle : \mathbb{Z}[\Gamma_0] \times \mathbb{Z}[\Gamma_0] \to \mathbb{Z}$ given by

$$\langle X, Y \rangle = \sum_{i\in\Gamma_0} d_i x_i y_i - \sum_{\rho\in\Gamma_1} m_\rho x_{t_\rho} y_{h_\rho},$$

where $X=\sum_{i\in\Gamma_0} x_i i, Y=\sum_{i\in\Gamma_0} y_i i\in\mathbb{Z}[\Gamma_0]$, so $X\cdot Y=\langle X,Y\rangle+\langle Y,X\rangle$ is the symmetric Euler form. The valued quiver Γ defines a Cartan matrix $C_\Gamma=C_{Q,\sigma}=(c_{ij})_{i,j\in\Gamma_0}$, where

$$c_{ij}=\begin{cases}2-2\sum\limits_{\substack{\rho\in\Gamma_1\\ h_\rho=t_\rho=i}}\dfrac{m_\rho}{d_i}, & i=j;\\ -\sum\limits_{\substack{\rho\in\Gamma_1\\ \{h_\rho,t_\rho\}=\{i,j\}}}\dfrac{m_\rho}{d_i}, & i\neq j.\end{cases}$$

Let t be a non-negative integer. Let $\mathbf{i}=(i_1,i_2,\cdots,i_t)\in\Gamma_0^t$ and $\mathbf{a}=(a_1,a_2,\cdots,a_t)\in\mathbb{N}^t$. We write

$$E_{\mathbf{i}}^{(\mathbf{a})}=E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\cdots E_{i_t}^{(a_t)}\in U^+.$$

Define

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A=(a_{rm})_{tt}\,\middle|\,a_{rm}\in\mathbb{N},\mathrm{ro}(A)=\mathrm{co}(A)=\mathbf{a},\,a_{rm}=0,\forall\, i_r\neq i_m\right\},$$

where

$$\mathrm{ro}(A)=\left(\textstyle\sum_{m=1}^t a_{1m},\cdots,\sum_{m=1}^t a_{tm}\right),\mathrm{co}(A)=\left(\sum_{r=1}^t a_{r1},\cdots,\sum_{r=1}^t a_{rt}\right).$$

Obviously, $D_{\mathbf{a}}=\mathrm{diag}(a_1,a_2,\cdots,a_t)\in\mathcal{M}_{\mathbf{i},\mathbf{a}}$.

The following results are very useful in the determination of tight monomials.

Theorem 2.1 [4] (Lusztig, 1993). Let $\mathbf{U}$ be the quantum group of type A_n,D_n,E_n, $\mathbf{i}\in\Gamma_0^t,\mathbf{a}\in\mathbb{N}^t$ as before. If the following quadratic form takes only values < 0 on $\mathcal{M}_{\mathbf{i},\mathbf{a}}\setminus\{D_{\mathbf{a}}\}$, then monomial $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight.

$$Q_{\mathbf{i},\mathbf{a}}(A)=\sum_{\substack{1\le m\le t\\1\le p<r\le t}}a_{pm}a_{rm}-\sum_{\substack{1\le p<r\le t\\1\le l<m\le t}}a_{pm}a_{rl}$$

Theorem 2.2 [10] (Reineke, 2001). Let $\mathbf{U}$ be the quantum group of type A_n, D_n, E_n, $\mathbf{i}\in\Gamma_0^t,\mathbf{a}\in\mathbb{N}^t$ as before, the monomial $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight if and only if the following quadratic form takes only values < 0 on $\mathcal{M}_{\mathbf{i},\mathbf{a}}\setminus\{D_{\mathbf{a}}\}$

$$Q'_{\mathbf{i},\mathbf{a}}(A)=\sum_{\substack{1\le m\le t\\1\le p<r\le t}}a_{pm}a_{rm}+\sum_{\substack{1\le p<r\le t\\1\le l<m\le t}}(i_l\cdot i_m)a_{pm}a_{rl}+\sum_{\substack{1\le r\le t\\1\le l<m\le t}}a_{rm}a_{rl}$$

If $i_1,i_2,\cdots,i_t$ are mutually different, then $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{D_{\mathbf{a}}\}$, by Theorem 2.2, we have the following Corollaries.

Corollary 2.3. When $i_1,i_2,\cdots,i_t$ are mutually different, monomial $E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\cdots E_{i_t}^{(a_t)}$ is tight.

Corollary 2.4. If $E_{i_1}^{(a_{p+1})}E_{i_2}^{(a_{p+2})}\cdots E_{i_t}^{(a_{p+t})}$ is tight, then for any mutually different $j_1,j_2,\cdots,j_p\notin\{i_1,i_2,\cdots,i_t\}$ and any mutually different $l_1,l_2,\cdots,l_q\notin\{i_1,i_2,\cdots,i_t;j_1,j_2,\cdots,j_p\}$, and $p+t+q\le l(w_0)$,

$$E_{j_1}^{(a_1)}E_{j_2}^{(a_2)}\cdots E_{j_p}^{(a_p)}E_{i_1}^{(a_{p+1})}E_{i_2}^{(a_{p+2})}\cdots E_{i_t}^{(a_{p+t})}E_{l_1}^{(a_{p+t+1})}E_{l_2}^{(a_{p+t+2})}\cdots E_{l_q}^{(a_{p+t+q})}$$

is also tight.

Theorem 2.5 [12] (Deng-Du, 2010). Let $\mathbf{i}=(i_1,i_2,\cdots,i_t)\in\Gamma_0^t$ and $\mathbf{a}=(a_1,a_2,\cdots,a_t)\in\mathbb{N}^t$. If $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight, then

(a) For $\forall 1\le r\le s\le t$, monomial $E_{i_r}^{(a_r)}E_{i_{r+1}}^{(a_{r+1})}\cdots E_{i_s}^{(a_s)}$ is also tight;

(b) For $\forall 1\le r<t$, $i_r\neq i_{r+1}$.

Theorem 2.6 [4] (Lusztig, 1993). Let Φ be the non-trivial automorphism of $\mathbf{U}^+$ induced by Dynkin diagram automorphism of $\mathfrak{g}$, and $\Psi:\mathbf{U}^+\to(\mathbf{U}^+)^{\mathrm{opp}}$ be the unique $\mathbb{Q}(v)$-algebra isomorphism such that $E_j\to E_j$. If $E_{\mathbf{i}}^{(\mathbf{a})}$ is tight, then $\Phi\left(E_{\mathbf{i}}^{(\mathbf{a})}\right)$ and $\Psi\left(E_{\mathbf{i}}^{(\mathbf{a})}\right)$ are all tight.

3. Main Results

Let $\mathbf{i}=(i_1,i_2,\cdots,i_t)\in\Gamma_0^t$, $\mathbf{a}=(a_1,a_2,\cdots,a_t)\in\mathbb{N}^t$. For convenience, we abbreviate a monomial $E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\cdots E_{i_t}^{(a_t)}$ as a word $i_1i_2\cdots i_t$ (1 as 0), an inequality $a_{j_1}+\cdots+a_{j_p}\le a_{l_1}+\cdots+a_{l_q}$ as $j_1+\cdots+j_p\le l_1+\cdots+l_q$. For example, a monomial $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}\left(a_1+a_3\le a_2\right)$ is abbreviated to $121(1+3\le 2)$, a monomial $E_1^{(a_1)}E_2^{(a_2)}E_3^{(a_3)}E_4^{(a_4)}$ to 1234, etc.

By Theorem 2.5(b), we only consider those words $i_1i_2\cdots i_t$ with $i_r\ne i_{r+1},\forall 1\le r<t$ in determining tight monomials, in this case, we call $i_1i_2\cdots i_t$ the word with *t*-value, $E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\cdots E_{i_t}^{(a_t)}$ the monomial with *t*-value. If $i_r\cdot i_{r+1}=0$ for some $1\le r<t$, we identify the word $i_1\cdots i_{r-1}i_ri_{r+1}i_{r+2}\cdots i_t$ with the word $i_1\cdots i_{r-1}i_{r+1}i_ri_{r+2}\cdots i_t$. Let us present the so called word-procedure for making the words with $(t+1)$- value from the words with *t*-value. Let $i_1i_2\cdots i_t$ be a word with *t*-value, we firstly add a number $i_{t+1}\in\{1,2,3,4,5\}$ different from i_1 (or i_t) in the front (or behind) of i_1 (or i_t), secondly delete the words with *t*-value, lastly apply the automorphism Φ and isomorphism Ψ. After the above procedure put into practice for all the words with *t*-value, we get all words with $(t+1)$-value by deleting repeated words. For example, by applying the above word-procedure to the word 13 with 2-value, we get the words with 3-value as follows: 132, 134, 135, 143, 213, 235, 325, 354, 435.

Theorem 3.1. Let M_t be the set of all tight monomials with *t*-value in quantum group for type A_5, we have the following results.

(1) $t=0$, $M_0=\{0\}$, tight monomial has only one;

(2) $t=1$, if $S_1=\{1,2,3\}$, then $M_1=\Phi(S_1)$, tight monomials have 5 families;

(3) $t=2$, if $S_2=\{12,13,14,15,23,24\}$, then $M_2=\Psi\Phi(S_2)$, tight monomials have 14 families;

(4) $t=3$, if $S_3=S_3^1\cup S_3^2$, where

$$S_3^1=\{123,124,125,132,134,135,234,243\},\quad S_3^2=\{121,212,232,323(1+3\le 2)\},$$

then $M_3=\Psi\Phi(S_3)$, tight monomials have 33 families;

(5) $t=4$, if $S_4=\bigcup_{i=1}^{3}S_4^i$, where

$$S_4^1=\{1234,1235,1243,1245,1254,1324,1325,1432\},$$

$$S_4^2=\{1213,1214,1215,2123,2124,2125,2321,2324,2325,3231,3234,3235(1+3\le 2)\},$$

$$S_4^3=\{2132,3243(1+4\le 2+3)\},$$

then $M_4=\Psi\Phi(S_4)$, tight monomials have 67 families;

(6) $t=5$, if $S_5=\bigcup_{i=1}^{8}S_5^i$, where

$$S_5^1=\{12345,12354,12435,12543,13254,14325\},$$

$$S_5^2=\{12134,12135,12143,12145,21234,21235,21243,21245,23241,\\ 23245,32341,32345(1+3\le 2)\},$$

$$S_5^3=\{12324,12325,13234,13235,42325,43235(2+4\le 3)\},$$

$$S_5^4=\{21324,21325,32431(1+4\le 2+3)\},$$

$$S_5^5=\{12321,23432,32123(1+5\le 2+4,2+4\le 3)\},$$

$$S_5^6=\{12132,23243,32312(1+3\le 2,2+5\le 3+4)\},$$

$$S_5^7=\{21232,32343(1+3\le 2,3+5\le 4)\},$$

$$S_5^8=\{31231,42342(2+5\le 3,1+4\le 3)\},$$

then $M_5 = \Psi\Phi(S_5)$, tight monomials have 125 families;

(7) If $t = 6$, $S_6 = \bigcup_{i=1}^{17} S_6^i$, where

$$S_6^1 = \{123245, 123254, 132354, 532341, 521234, 132345, 523241, 321245(2+4 \le 3)\},$$

$$S_6^2 = \{121345, 121354, 121435, 121543, 212345, 212435, 232451, 323145(1+3 \le 2)\},$$

$$S_6^3 = \{213245, 324351(1+4 \le 2+3)\},$$

$$S_6^4 = \{521324, 132435(2+5 \le 3+4)\},$$

$$S_6^5 = \{121434, 121343, 121454, 121545, 212343, 212434, 212454(1+3 \le 2, 4+6 \le 5)\},$$

$$S_6^6 = \{132343, 532343, 523212, 421232, 423212(2+4 \le 3, 4+6 \le 5)\},$$

$$S_6^7 = \{123214, 123215, 234321, 234325, 321234, 321235(1+5 \le 2+4, 2+4 \le 3)\},$$

$$S_6^8 = \{312314, 312315, 423421(1+4 \le 3, 2+5 \le 3)\},$$

$$S_6^9 = \{123243, 532312, 523243, 412132(2+4 \le 3, 3+6 \le 4+5)\},$$

$$S_6^{10} = \{121324, 121325, 232431, 232435, 323124(1+3 \le 2, 2+5 \le 3+4)\},$$

$$S_6^{11} = \{132431, 421324(1+6 \le 3, 2+5 \le 3+4)\},$$

$$S_6^{12} = \{213243(1+4 \le 2+3, 3+6 \le 4+5)\},$$

$$S_6^{13} = \{213234(1+4 \le 2+3, 3+5 \le 4)\},$$

$$S_6^{14} = \{321243, 432354(1+6 \le 2+4+5, 2+4 \le 3)\},$$

$$S_6^{15} = \{123212, 432343, 321232(2+4 \le 3, 4+6 \le 5, 1+5 \le 2+4)\},$$

$$S_6^{16} = \{121321, 232432, 323123(1+3 \le 2, 3+6 \le 5, 2+5 \le 3+4)\},$$

$$S_6^{17} = \{312312, 423423(1+4 \le 3, 2+5 \le 3, 3+6 \le 4+5)\},$$

then $M_6 = \Psi\Phi(S_6)$, tight monomials have 222 families;

4. Proof of Theorem 3.1

Consider the quiver $Q = (Q_0, Q_1)$ of type A_5, where $Q_1 = \left\{1 \xrightarrow{\rho_1} 2, 2 \xrightarrow{\rho_2} 3, 3 \xrightarrow{\rho_3} 4, 4 \xrightarrow{\rho_4} 5\right\}$, $Q_0 = \{1,2,3,4,5\}$. Let $\sigma =$ id be the identity automorphism of Q, then valued quiver of (Q,σ) is $\Gamma = (\Gamma_0, \Gamma_1) = (Q_0, Q_1) = Q$. The valuation is given by $d_1 = d_2 = d_3 = d_4 = d_5 = 1$, $m_{\rho_1} = m_{\rho_2} = m_{\rho_3} = m_{\rho_4} = 1$. Euler form $\langle , \rangle$ on $Q = \Gamma$ is

$$\langle X, Y \rangle = \sum_{i=1}^{5} d_i x_i y_i - \sum_{i=1}^{4} m_{\rho_i} x_{t_{\rho_i}} y_{h_{\rho_i}} = \sum_{i=1}^{5} x_i y_i - \sum_{i=1}^{4} x_i y_{i+1},$$

Symmetric Euler form $\cdot$ on $Q = \Gamma$ is

$$X \cdot Y = \langle X, Y \rangle + \langle Y, X \rangle = \sum_{i=1}^{5} 2 x_i y_i - \sum_{i=1}^{4} x_i y_{i+1} - \sum_{i=1}^{4} x_{i+1} y_i,$$

where $X=\sum_{i=1}^{5}x_i i, Y=\sum_{i=1}^{5}y_i i\in\mathbb{Z}[\Gamma_0]$.

By simple computation, we have

$\langle i,i\rangle=1, i\cdot i=2\ (i=1,2,3,4,5)$, $i\cdot(i+1)=-1\ (i=1,2,3,4)$, and $1\cdot 3=1\cdot 4=1\cdot 5=2\cdot 4=2\cdot 5=3\cdot 5=0$.

Let us prove Theorem 3.1.

Case 1. $t\le 2$. By Corollary 2.3, monomials with $t\le 2$ are all tight.

Case 2. t = 3. Applying the word-procedure on S_2, we get 33 words with 3-value. By considering Φ and Ψ, we get S_3. By Corollary 2.3, monomials in S_3^1 are all tight. For S_3^2, it suffices to consider $\mathbf{i}=(1,2,1)$. For any $\mathbf{a}=(a_1,a_2,a_3)\in\mathbb{N}^3$, we have $\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{A_x\,|\,0\le x\le\min\{a_1,a_3\}\}$, where

$$A_x=\begin{pmatrix} a_1-x & 0 & x\\ 0 & a_2 & 0\\ x & 0 & a_3-x\end{pmatrix}$$

and

$$\begin{aligned}q(A_x)&=\sum_{\substack{1\le m\le 3\\1\le p<r\le 3}}a_{pm}a_{rm}+\sum_{\substack{1\le p<r\le 3\\1\le l<m\le 3}}(i_l\cdot i_m)a_{pm}a_{rl}+\sum_{\substack{1\le r\le 3\\1\le l<m\le 3}}a_{rm}a_{rl}\\
&=a_{11}a_{31}+a_{13}a_{33}+a_{11}a_{13}+a_{31}a_{33}+(i_1\cdot i_2)a_{22}a_{31}+(i_2\cdot i_3)a_{13}a_{22}+(i_1\cdot i_3)a_{13}a_{31}\\
&=2x(a_1-x)+2x(a_3-x)-2a_2x+2x^2\\
&=-2x^2+2(a_1+a_3-a_2)x.\end{aligned}$$

Obviously, $q(A_x)<0$ if and only if $a_1+a_3\le a_2$. So monomial $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}\ (a_1+a_3\le a_2)$ is tight by Theorem 2.2.

Case 3. t = 4. Applying the word-procedure on S_3, we get 75 words with 4-value. By considering Φ and Ψ, we get $S_4\cup\{1212,2323\}$. When $\mathbf{i}\in\{(1,2,1,2),(2,3,2,3)\}$, for any $\mathbf{a}=(a_1,a_2,a_3,a_4)\in\mathbb{N}^4$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\{A_{x,y}\,|\,0\le x\le\min\{a_1,a_3\},0\le y\le\min\{a_2,a_4\}\},$$

where

$$A_{x,y}=\begin{pmatrix} a_1-x & 0 & x & 0\\ 0 & a_2-y & 0 & y\\ x & 0 & a_3-x & 0\\ 0 & y & 0 & a_4-y\end{pmatrix}$$

and

$$\begin{aligned}q(A_{x,y})&=\sum_{\substack{1\le m\le 4\\1\le p<r\le 4}}a_{pm}a_{rm}+\sum_{\substack{1\le p<r\le 4\\1\le l<m\le 4}}(i_l\cdot i_m)a_{pm}a_{rl}+\sum_{\substack{1\le r\le 4\\1\le l<m\le 4}}a_{rm}a_{rl}\\
&=a_{11}a_{31}+a_{22}a_{42}+a_{13}a_{33}+a_{24}a_{44}+(i_1\cdot i_2)a_{22}a_{31}+(i_1\cdot i_3)a_{13}a_{31}+(i_1\cdot i_4)a_{24}a_{31}\\
&\quad+(i_2\cdot i_3)(a_{13}a_{22}+a_{13}a_{42}+a_{33}a_{42})+(i_2\cdot i_4)a_{24}a_{42}\\
&\quad+(i_3\cdot i_4)a_{24}a_{33}+a_{13}a_{11}+a_{24}a_{22}+a_{33}a_{31}+a_{44}a_{42}\\
&=2x(a_1-x)+2y(a_2-y)+2x(a_3-x)+2y(a_4-y)-2a_2x+2x^2-2a_3y+2xy+2y^2\\
&=2(a_1+a_3-a_2)x+2(a_2+a_4-a_3)y-(x-y)^2-x^2-y^2.\end{aligned}$$

Obviously, $q(A_{x,y})<0$ if and only if $a_1+a_3\le a_2, a_2+a_4\le a_3$, this is a contradiction. Applying Φ,Ψ, one gets that the monomials corresponding to

$$\mathbf{i}\in\{(1,2,1,2),(2,1,2,1),(2,3,2,3),(3,2,3,2),(3,4,3,4),(4,3,4,3),(4,5,4,5),(5,4,5,4)\}$$

are all not tight for any $\mathbf{a}=(a_1,a_2,a_3,a_4)\in\mathbb{N}^4$.

Monomials in S_4^1 are all tight by Corollary 2.3. By S_3^2 and Corollary 2.4, monomials in S_4^2 are all tight. For S_4^3, it suffices to consider $\mathbf{i}=(2,1,3,2)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4)\in\mathbb{N}^4$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A_x \middle| 0\le x\le \min\{a_1,a_4\}\right\},$$

where

$$A_x=\begin{pmatrix} a_1-x & 0 & 0 & x\\ 0 & a_2 & 0 & 0\\ 0 & 0 & a_3 & 0\\ x & 0 & 0 & a_4-x\end{pmatrix}$$

and

$$\begin{aligned} q(A_x) &= \sum_{\substack{1\le m\le 4\\ 1\le p<r\le 4}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 4\\ 1\le l<m\le 4}} (i_l\cdot i_m)a_{pm}a_{rl} + \sum_{\substack{1\le r\le 4\\ 1\le l<m\le 4}} a_{rm}a_{rl}\\ &= a_{11}a_{41}+a_{14}a_{44}+a_{14}a_{11}+a_{44}a_{41}+(i_1\cdot i_2)a_{22}a_{41}+(i_1\cdot i_3)a_{33}a_{41}\\ &\quad +(i_1\cdot i_4)a_{14}a_{41}+(i_2\cdot i_4)a_{14}a_{22}+(i_3\cdot i_4)a_{14}a_{33}\\ &= 2x(a_1-x)+2x(a_4-x)-a_2x-a_3x+2x^2-a_2x-a_3x\\ &= 2(a_1+a_4-a_2-a_3)x-2x^2 \end{aligned}$$

$q(A_x)<0$ if and only if $a_1+a_4\le a_2+a_3$. So $E_2^{(a_1)}E_1^{(a_2)}E_3^{(a_3)}E_2^{(a_4)}\,(a_1+a_4\le a_2+a_3)$ is tight by Theorem 2.2.

Case 4. $t=5$. Applying the word-procedure on S_4, and deleting words including subwords 1212, 2121, 2323, 3232, 3434, 4343, 4545 and 5454 (considering Theorem 2.5(a)), we get 125 words with 5-value. By considering Φ and Ψ, we get S_5. By Corollary 2.3, monomials in S_5^1 are all tight. Monomials in S_5^2, S_5^3 are all tight by S_3^2 and Corollary 2.4. Monomials in S_5^4 are all tight by S_4^3 and Corollary 2.4.

For S_5^5, it suffices to consider $\mathbf{i}=(1,2,3,2,1)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathbb{N}^5$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A_{x,y} \middle| 0\le x\le \min\{a_1,a_5\}, 0\le y\le \min\{a_2,a_4\}\right\},$$

where

$$A_{x,y}=\begin{pmatrix} a_1-x & 0 & 0 & 0 & x\\ 0 & a_2-y & 0 & y & 0\\ 0 & 0 & a_3 & 0 & 0\\ 0 & y & 0 & a_4-y & 0\\ x & 0 & 0 & 0 & a_5-x\end{pmatrix}$$

and

$$\begin{aligned} q(A_{x,y}) &= \sum_{\substack{1\le m\le 5\\ 1\le p<r\le 5}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 5\\ 1\le l<m\le 5}} (i_l\cdot i_m)a_{pm}a_{rl} + \sum_{\substack{1\le r\le 5\\ 1\le l<m\le 5}} a_{rm}a_{rl}\\ &= a_{11}a_{51}+a_{22}a_{42}+a_{24}a_{44}+a_{15}a_{55}+a_{11}a_{15}+a_{24}a_{22}+a_{44}a_{42}+a_{55}a_{51}\\ &\quad +(i_1\cdot i_2)(a_{22}a_{51}+a_{42}a_{51})+(i_1\cdot i_3)a_{33}a_{51}+(i_1\cdot i_4)(a_{24}a_{51}+a_{44}a_{51})\\ &\quad +(i_1\cdot i_5)a_{15}a_{51}+(i_2\cdot i_3)a_{33}a_{42}+(i_2\cdot i_4)a_{24}a_{42}+(i_2\cdot i_5)(a_{15}a_{22}+a_{15}a_{42})\\ &\quad +(i_3\cdot i_4)a_{24}a_{33}+(i_3\cdot i_5)a_{15}a_{33}+(i_4\cdot i_5)(a_{15}a_{24}+a_{15}a_{44})\\ &= 2x(a_1-x)+2y(a_2-y)+2y(a_4-y)+2x(a_5-x)-a_2x\\ &\quad -a_4x+2x^2-a_3y+2y^2-a_2x-a_3y-a_4x\\ &= 2(a_1+a_5-a_2-a_4)x+2(a_2+a_4-a_3)y-2x^2-2y^2. \end{aligned}$$

$q\left(A_{x,y}\right)<0$ if and only if $a_1+a_5\le a_2+a_4, a_2+a_4\le a_3$. So

$$E_1^{(a_1)}E_2^{(a_2)}E_3^{(a_3)}E_2^{(a_4)}E_1^{(a_5)}\left(a_1+a_5\le a_2+a_4, a_2+a_4\le a_3\right)$$

is tight by Theorem 2.2.

For S_5^6, it suffices to consider $\mathbf{i}=(1,2,1,3,2)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathbb{N}^5$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A_{x,y}\,\middle|\,0\le x\le\min\{a_1,a_3\}, 0\le y\le\min\{a_2,a_5\}\right\},$$

where

$$A_{x,y}=\begin{pmatrix} a_1-x & 0 & x & 0 & 0\\ 0 & a_2-y & 0 & 0 & y\\ x & 0 & a_3-x & 0 & 0\\ 0 & 0 & 0 & a_4 & 0\\ 0 & y & 0 & 0 & a_5-y\end{pmatrix}$$

and

$$q\left(A_{x,y}\right)=\sum_{\substack{1\le m\le 5\\1\le p<r\le 5}}a_{pm}a_{rm}+\sum_{\substack{1\le p<r\le 5\\1\le l<m\le 5}}\left(i_l\cdot i_m\right)a_{pm}a_{rl}+\sum_{\substack{1\le r\le 5\\1\le l<m\le 5}}a_{rm}a_{rl}$$

$$\begin{aligned}&=a_{11}a_{31}+a_{22}a_{52}+a_{13}a_{33}+a_{25}a_{55}+a_{13}a_{11}+a_{25}a_{22}+a_{33}a_{31}+a_{55}a_{52}+\left(i_1\cdot i_2\right)a_{22}a_{31}\\&\quad+\left(i_1\cdot i_3\right)a_{13}a_{31}+\left(i_1\cdot i_5\right)a_{25}a_{31}+\left(i_2\cdot i_3\right)\left(a_{13}a_{22}+a_{13}a_{52}+a_{33}a_{52}\right)+\left(i_2\cdot i_4\right)a_{44}a_{52}\\&\quad+\left(i_2\cdot i_5\right)a_{25}a_{52}+\left(i_3\cdot i_5\right)a_{25}a_{33}+\left(i_4\cdot i_5\right)a_{25}a_{44}\\&=2x\left(a_1-x\right)+2y\left(a_2-y\right)+2x\left(a_3-x\right)+2y\left(a_5-y\right)-\left(a_2-y\right)x\\&\quad+2x^2-xy-a_2x-\left(a_3-x\right)y-a_4y+2y^2-\left(a_3-x\right)y-a_4y\\&=2\left(a_1+a_3-a_2\right)x+2\left(a_2+a_5-a_3-a_4\right)y-\left(x-y\right)^2-x^2-y^2.\end{aligned}$$

$q\left(A_{x,y}\right)<0$ if and only if $a_1+a_3\le a_2, a_2+a_5\le a_3+a_4$. So

$$E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_3^{(a_4)}E_2^{(a_5)}\left(a_2+a_5\le a_3+a_4, a_1+a_3\le a_2\right)$$

is tight by Theorem 2.2.

For S_5^7, it suffices to consider $\mathbf{i}=(2,1,2,3,2)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathbb{N}^5$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A=A_{x,x_1,x_2,x_3}\,\middle|\,\text{entries in matrix are all non-negative integer}\right\},$$

where

$$A=\begin{pmatrix} a_1-x-x_1 & 0 & x & 0 & x_1\\ 0 & a_2 & 0 & 0 & 0\\ x_2 & 0 & a_3-x_2-x_3 & 0 & x_3\\ 0 & 0 & 0 & a_4 & 0\\ x+x_1-x_2 & 0 & x_2+x_3-x & 0 & a_5-x_1-x_3\end{pmatrix}$$

and

$$\begin{aligned}q(A) &= \sum_{\substack{1\le m\le 5\\1\le p<r\le 5}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 5\\1\le l<m\le 5}} (i_l\cdot i_m)a_{pm}a_{rl} + \sum_{\substack{1\le r\le 5\\1\le l<m\le 5}} a_{rm}a_{rl}\\
&= a_{11}a_{31}+a_{11}a_{51}+a_{31}a_{51}+a_{13}a_{33}+a_{13}a_{53}+a_{33}a_{53}+a_{15}a_{35}+a_{15}a_{55}+a_{35}a_{55}\\
&\quad +a_{15}a_{11}+a_{13}a_{11}+a_{15}a_{13}+a_{35}a_{31}+a_{33}a_{31}+a_{35}a_{33}+a_{55}a_{51}+a_{53}a_{51}+a_{55}a_{53}\\
&\quad +(i_1\cdot i_2)(a_{22}a_{31}+a_{22}a_{51})+(i_1\cdot i_3)(a_{13}a_{31}+a_{13}a_{51}+a_{33}a_{51})+(i_1\cdot i_4)a_{44}a_{51}\\
&\quad +(i_2\cdot i_3)a_{13}a_{22}+(i_1\cdot i_5)(a_{15}a_{31}+a_{15}a_{51}+a_{35}a_{51})+(i_2\cdot i_5)a_{15}a_{22}\\
&\quad +(i_3\cdot i_4)a_{44}a_{53}+(i_3\cdot i_5)(a_{15}a_{33}+a_{15}a_{53}+a_{35}a_{53})+(i_4\cdot i_5)(a_{15}a_{44}+a_{35}a_{44})\\
&= 2(a_1+a_3-a_2)x+2(a_1+2a_3+a_5-a_2-a_4)x_1+2(a_3+a_5-a_4)x_3\\
&\quad -x^2-(x-x_2)^2-x_2^2-2x_1^2-2x_3^2-2x_1x_3-2xx_1-2x_2x_3.\end{aligned}$$

$q(A)<0$ if and only if $a_1+a_3\le a_2, a_3+a_5\le a_4$. So

$$E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_3^{(a_4)}E_2^{(a_5)}(a_1+a_3\le a_2, a_3+a_5\le a_4)$$

is tight by Theorem 2.2.

For S_5^8, it suffices to consider $\mathbf{i}=(3,1,2,3,1)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathbb{N}^5$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A_{x,y}\,\middle|\,0\le x\le \min\{a_1,a_4\}, 0\le y\le\min\{a_2,a_5\}\right\},$$

where

$$A_{x,y}=\begin{pmatrix}a_1-x&0&0&x&0\\0&a_2-y&0&0&y\\0&0&a_3&0&0\\x&0&0&a_4-x&0\\0&y&0&0&a_5-y\end{pmatrix}$$

and

$$\begin{aligned}q(A_{x,y}) &= \sum_{\substack{1\le m\le 5\\1\le p<r\le 5}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 5\\1\le l<m\le 5}} (i_l\cdot i_m)a_{pm}a_{rl} + \sum_{\substack{1\le r\le 5\\1\le l<m\le 5}} a_{rm}a_{rl}\\
&= a_{11}a_{41}+a_{22}a_{52}+a_{14}a_{44}+a_{25}a_{55}+a_{14}a_{11}+a_{25}a_{22}+a_{44}a_{41}+a_{55}a_{52}+(i_1\cdot i_2)a_{22}a_{41}\\
&\quad +(i_1\cdot i_3)a_{33}a_{41}+(i_1\cdot i_4)a_{14}a_{41}+(i_1\cdot i_5)a_{25}a_{41}+(i_2\cdot i_3)a_{33}a_{52}+(i_2\cdot i_5)a_{25}a_{52}\\
&\quad +(i_2\cdot i_4)(a_{14}a_{22}+a_{14}a_{52}+a_{44}a_{52})+(i_3\cdot i_4)a_{14}a_{33}+(i_3\cdot i_5)a_{25}a_{33}+(i_4\cdot i_5)a_{25}a_{44}\\
&= 2(a_1+a_4-a_3)x+2(a_2+a_5-a_3)y-2x^2-2y^2\end{aligned}$$

$q(A_{x,y})<0$ if and only if $a_1+a_4\le a_3, a_2+a_5\le a_3$. So

$$E_3^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_3^{(a_4)}E_1^{(a_5)}(a_1+a_4\le a_3, a_2+a_5\le a_3)$$

is tight by Theorem 2.2.

Case 5. $t=6$. Applying the word-procedure on S_5, and deleting words including subwords 1212, 2121, 2323, 3232, 3434, 4343, 4545 and 5454(considering Theorem 2.5(a)), we get 228 words with 6-value. By considering Φ and Ψ, we get $S_6\cup\{121323, 232434\}$. When $\mathbf{i}\in\{(1,2,1,3,2,3),(2,3,2,4,3,4)\}$, for any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{N}^6$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A_{x,y,z}\,\middle|\,0\le x\le\min\{a_1,a_3\}, 0\le y\le\min\{a_2,a_5\}, 0\le z\le\min\{a_4,a_6\}\right\},$$

where

$$A_{x,y,z}=\begin{pmatrix} a_1-x & 0 & x & 0 & 0 & 0 \\ 0 & a_2-y & 0 & 0 & y & 0 \\ x & 0 & a_3-x & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4-z & 0 & z \\ 0 & y & 0 & 0 & a_5-y & 0 \\ 0 & 0 & 0 & z & 0 & a_6-z \end{pmatrix}$$

and

$$\begin{aligned} q\left(A_{x,y,z}\right) &= \sum_{\substack{1\le m\le 6\\ 1\le p<r\le 6}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 6\\ 1\le l<m\le 6}} \left(i_l \cdot i_m\right) a_{pm}a_{rl} + \sum_{\substack{1\le r\le 6\\ 1\le l<m\le 6}} a_{rm}a_{rl} \\ &= a_{11}a_{31}+a_{22}a_{52}+a_{13}a_{33}+a_{44}a_{64}+a_{25}a_{55}+a_{46}a_{66}+a_{11}a_{13}+a_{22}a_{25}+a_{31}a_{33} \\ &\quad +a_{44}a_{46}+a_{52}a_{55}+a_{64}a_{66}+\left(i_1\cdot i_2\right)a_{22}a_{31}+\left(i_1\cdot i_3\right)a_{13}a_{31}+\left(i_1\cdot i_5\right)a_{25}a_{31} \\ &\quad +\left(i_2\cdot i_3\right)\left(a_{13}a_{22}+a_{13}a_{52}+a_{33}a_{52}\right)+\left(i_2\cdot i_4\right)a_{44}a_{52}+\left(i_2\cdot i_5\right)a_{25}a_{52}+\left(i_2\cdot i_6\right)a_{46}a_{52} \\ &\quad +\left(i_3\cdot i_5\right)a_{25}a_{33}+\left(i_4\cdot i_5\right)\left(a_{25}a_{44}+a_{25}a_{64}+a_{55}a_{64}\right)+\left(i_4\cdot i_6\right)a_{46}a_{64}+\left(i_5\cdot i_6\right)a_{46}a_{55} \\ &= 2x\left(a_1-x\right)+2y\left(a_2-y\right)+2x\left(a_3-x\right)+2z\left(a_4-z\right)+2y\left(a_5-y\right) \\ &\quad +2z\left(a_6-z\right)-\left(a_2-y\right)x+2x^2-xy-a_2x-a_3y+xy-\left(a_4-z\right)y \\ &\quad +2y^2-yz-\left(a_3-x\right)y-a_4y-a_5z+yz+2z^2-\left(a_5-y\right)z \\ &= 2\left(a_1+a_3-a_2\right)x+2\left(a_2+a_5-a_3-a_4\right)y+2\left(a_4+a_6-a_5\right)z \\ &\quad -\left(x-y\right)^2-\left(y-z\right)^2-x^2-z^2. \end{aligned}$$

$q\left(A_{x,y,z}\right)<0$ if and only if $a_1+a_3\le a_2, a_2+a_5\le a_3+a_4, a_4+a_6\le a_5$. This is a contradiction. Applying Φ,Ψ, one gets that the monomials corresponding to

$$\mathbf{i}\in\left\{(1,2,1,3,2,3),(3,2,3,1,2,1),(3,4,3,5,4,5),(5,4,5,3,4,3),(4,3,4,2,3,2),(2,3,2,4,3,4)\right\}$$

are all not tight for any $\mathbf{a}=\left(a_1,a_2,a_3,a_4,a_5,a_6\right)\in\mathbb{N}^6$.

By Corollary 2.4, we have $S_3^2\Rightarrow S_6^1,S_6^2,S_6^5$, $S_4^3\Rightarrow S_6^3,S_6^4$, $S_5^7\Rightarrow S_6^6$, $S_5^5\Rightarrow S_6^7$, $S_5^8\Rightarrow S_6^8$, $S_5^6\Rightarrow S_6^9,S_6^{10},S_6^{13}$, $S_3^2,S_4^3\Rightarrow S_6^{11}$, and $S_4^3\Rightarrow S_6^{12}$.

For S_6^{14}, it suffices to consider $\mathbf{i}=(3,2,1,2,4,3)$. For any $\mathbf{a}=\left(a_1,a_2,a_3,a_4,a_5,a_6\right)\in\mathbb{N}^6$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A_{x,y}\,\middle|\,0\le x\le\min\left\{a_1,a_6\right\},0\le y\le\min\left\{a_2,a_4\right\}\right\}$$

where

$$A_{x,y}=\begin{pmatrix} a_1-x & 0 & 0 & 0 & 0 & x \\ 0 & a_2-y & 0 & y & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & y & 0 & a_4-y & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ x & 0 & 0 & 0 & 0 & a_6-x \end{pmatrix}$$

and

$$\begin{aligned} q\left(A_{x,y}\right) &= \sum_{\substack{1\le m\le 6\\ 1\le p<r\le 6}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 6\\ 1\le l<m\le 6}} \left(i_l\cdot i_m\right)a_{pm}a_{rl} + \sum_{\substack{1\le r\le 6\\ 1\le l<m\le 6}} a_{rm}a_{rl} \\ &= a_{11}a_{61} + a_{22}a_{42} + a_{24}a_{44} + a_{16}a_{66} + a_{11}a_{16} + a_{22}a_{24} + a_{42}a_{44} + a_{61}a_{66} \\ &\quad + \left(i_1\cdot i_2\right)\left(a_{22}a_{61} + a_{42}a_{61}\right) + \left(i_1\cdot i_3\right)a_{33}a_{61} + \left(i_1\cdot i_4\right)\left(a_{24}a_{61} + a_{44}a_{61}\right) \\ &\quad + \left(i_1\cdot i_5\right)a_{55}a_{61} + \left(i_1\cdot i_6\right)a_{16}a_{61} + \left(i_2\cdot i_3\right)a_{33}a_{42} + \left(i_2\cdot i_4\right)a_{24}a_{42} \\ &\quad + \left(i_2\cdot i_6\right)\left(a_{16}a_{22} + a_{16}a_{42}\right) + \left(i_3\cdot i_4\right)a_{24}a_{33} + \left(i_3\cdot i_6\right)a_{16}a_{33} \\ &\quad + \left(i_4\cdot i_6\right)\left(a_{16}a_{24} + a_{16}a_{44}\right) + \left(i_5\cdot i_6\right)a_{16}a_{55} \\ &= 2x\left(a_1 - x\right) + 2y\left(a_2 - y\right) + 2y\left(a_4 - y\right) + 2x\left(a_6 - x\right) - a_2x - a_4x \\ &\quad - a_5x + 2x^2 - a_3y + 2y^2 - a_2x - a_3y - a_4x - a_5x \\ &= 2\left(a_1 + a_6 - a_2 - a_4 - a_5\right)x + 2\left(a_2 + a_4 - a_3\right)y - 2x^2 - 2y^2. \end{aligned}$$

$q\left(A_{x,y}\right) < 0$ if and only if $a_2 + a_4 \le a_3, a_1 + a_6 \le a_2 + a_4 + a_5$. So

$$E_3^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}E_4^{(a_5)}E_3^{(a_6)}\left(a_2 + a_4 \le a_3, a_1 + a_6 \le a_2 + a_4 + a_5\right)$$

is tight by Theorem 2.2.

For S_6^{15}, it suffices to consider $\mathbf{i} = (1,2,3,2,1,2)$. For any $\mathbf{a} = (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{N}^6$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}} = \left\{A = A_{x,y,y_1,y_2,y_3} \,\middle|\, \text{entries in matrix are all non-negative integer}\right\}$$

where

$$A = \begin{pmatrix} a_1 - x & 0 & 0 & 0 & x & 0 \\ 0 & a_2 - y - y_1 & 0 & y & 0 & y_1 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & y_2 & 0 & a_4 - y_2 - y_3 & 0 & y_3 \\ x & 0 & 0 & 0 & a_5 - x & 0 \\ 0 & y + y_1 - y_2 & 0 & y_2 + y_3 - y & 0 & a_6 - y_1 - y_3 \end{pmatrix}$$

and

$$\begin{aligned} q(A) &= \sum_{\substack{1\le m\le 6\\ 1\le p<r\le 6}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 6\\ 1\le l<m\le 6}} \left(i_l\cdot i_m\right)a_{pm}a_{rl} + \sum_{\substack{1\le r\le 6\\ 1\le l<m\le 6}} a_{rm}a_{rl} \\ &= a_{11}a_{51} + a_{22}a_{42} + a_{22}a_{62} + a_{42}a_{62} + a_{24}a_{44} + a_{24}a_{64} + a_{44}a_{64} + a_{15}a_{55} + a_{26}a_{46} + a_{26}a_{66} + a_{46}a_{66} \\ &\quad + \left(i_1\cdot i_2\right)\left(a_{22}a_{51} + a_{42}a_{51}\right) + \left(i_1\cdot i_3\right)a_{33}a_{51} + \left(i_1\cdot i_4\right)\left(a_{24}a_{51} + a_{44}a_{51}\right) + \left(i_1\cdot i_5\right)a_{15}a_{51} \\ &\quad + \left(i_1\cdot i_6\right)\left(a_{26}a_{51} + a_{46}a_{51}\right) + \left(i_2\cdot i_3\right)\left(a_{33}a_{42} + a_{33}a_{62}\right) + \left(i_2\cdot i_4\right)\left(a_{24}a_{42} + a_{24}a_{62} + a_{44}a_{62}\right) \\ &\quad + \left(i_2\cdot i_5\right)\left(a_{15}a_{22} + a_{15}a_{42} + a_{15}a_{62} + a_{55}a_{62}\right) + \left(i_2\cdot i_6\right)\left(a_{26}a_{42} + a_{26}a_{62} + a_{46}a_{62}\right) \\ &\quad + \left(i_3\cdot i_4\right)a_{24}a_{33} + \left(i_3\cdot i_5\right)a_{15}a_{33} + \left(i_3\cdot i_6\right)a_{26}a_{33} + \left(i_4\cdot i_5\right)\left(a_{15}a_{24} + a_{15}a_{44} + a_{15}a_{64} + a_{55}a_{64}\right) \\ &\quad + \left(i_4\cdot i_6\right)\left(a_{26}a_{44} + a_{26}a_{64} + a_{46}a_{64}\right) + \left(i_5\cdot i_6\right)\left(a_{26}a_{55} + a_{46}a_{55}\right) + a_{11}a_{15} + a_{22}a_{24} + a_{22}a_{26} \\ &\quad + a_{24}a_{26} + a_{42}a_{44} + a_{42}a_{46} + a_{44}a_{46} + a_{51}a_{55} + a_{62}a_{64} + a_{62}a_{66} + a_{64}a_{66} \\ &= 2\left(a_1 + a_5 - a_2 - a_4\right)x + 2\left(a_2 + a_4 - a_3\right)y + 2\left(a_2 + 2a_4 + a_6 - a_3 - a_5\right)y_1 + 2\left(a_4 + a_6 - a_5\right)y_3 \\ &\quad - \left(y - y_2\right)^2 - \left(x - y_3\right)^2 - \left(x - y_1\right)^2 - y^2 - y_2^2 - y_3^2 - 2y_1y_3 - 2y_2y_3 - 2y_1y. \end{aligned}$$

$q(A) < 0$ if and only if $a_2 + a_4 \le a_3, a_4 + a_6 \le a_5, a_1 + a_5 \le a_2 + a_4$. So

$$E_1^{(a_1)}E_2^{(a_2)}E_3^{(a_3)}E_2^{(a_4)}E_1^{(a_5)}E_2^{(a_6)}\left(a_2 + a_4 \le a_3, a_1 + a_5 \le a_2 + a_4, a_4 + a_6 \le a_5\right)$$

is tight by Theorem 2.2.

For S_6^{16}, it suffices to consider $\mathbf{i}=(1,2,1,3,2,1)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{N}^6$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A=A_{x,x_1,x_2,x_3,y}\,\middle|\,\text{entries in matrix are all non-negative integer}\right\}$$

where

$$A=\begin{pmatrix} a_1-x-x_1 & 0 & x & 0 & 0 & x_1 \\ 0 & a_2-y & 0 & 0 & y & 0 \\ x_2 & 0 & a_3-x_2-x_3 & 0 & 0 & x_3 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & y & 0 & 0 & a_5-y & 0 \\ x+x_1-x_2 & 0 & x_2+x_3-x & 0 & 0 & a_6-x_1-x_3 \end{pmatrix}$$

and

$$\begin{aligned} q(A)=&\sum_{\substack{1\le m\le 6\\1\le p<r\le 6}}a_{pm}a_{rm}+\sum_{\substack{1\le p<r\le 6\\1\le l<m\le 6}}(i_l\cdot i_m)a_{pm}a_{rl}+\sum_{\substack{1\le r\le 6\\1\le l<m\le 6}}a_{rm}a_{rl}\\ =&\,a_{11}a_{31}+a_{11}a_{61}+a_{31}a_{61}+a_{22}a_{52}+a_{13}a_{33}+a_{13}a_{63}+a_{33}a_{63}+a_{25}a_{55}+a_{16}a_{36}+a_{16}a_{66}\\ &+a_{36}a_{66}+(i_1\cdot i_2)(a_{22}a_{31}+a_{22}a_{61}+a_{52}a_{61})+(i_1\cdot i_3)(a_{13}a_{31}+a_{13}a_{61}+a_{33}a_{61})\\ &+(i_1\cdot i_4)a_{44}a_{61}+(i_1\cdot i_5)(a_{25}a_{31}+a_{25}a_{61}+a_{55}a_{61})+(i_1\cdot i_6)(a_{16}a_{31}+a_{16}a_{61}+a_{36}a_{61})\\ &+(i_2\cdot i_3)(a_{13}a_{22}+a_{13}a_{52}+a_{33}a_{52})+(i_2\cdot i_4)a_{44}a_{52}+(i_2\cdot i_5)a_{25}a_{52}\\ &+(i_2\cdot i_6)(a_{16}a_{22}+a_{16}a_{52}+a_{36}a_{52})+(i_3\cdot i_4)a_{44}a_{63}+(i_3\cdot i_5)(a_{25}a_{33}+a_{25}a_{63}+a_{55}a_{63})\\ &+(i_3\cdot i_6)(a_{16}a_{33}+a_{16}a_{63}+a_{36}a_{63})+(i_4\cdot i_5)a_{25}a_{44}+(i_4\cdot i_6)(a_{16}a_{44}+a_{36}a_{44})\\ &+(i_5\cdot i_6)(a_{16}a_{25}+a_{16}a_{55}+a_{36}a_{55})+a_{11}a_{13}+a_{11}a_{16}+a_{13}a_{16}+a_{22}a_{25}+a_{31}a_{33}\\ &+a_{31}a_{36}+a_{33}a_{36}+a_{52}a_{55}+a_{61}a_{63}+a_{61}a_{66}+a_{63}a_{66}\\ =&\,2(a_1+a_3-a_2)x+2(a_2+a_5-a_3-a_4)y+2(a_1+2a_3+a_6-a_2-a_5)x_1+2(a_3+a_6-a_5)x_3\\ &-(x-x_2)^2-(x_2-y)^2-(x_3-y)^2-x^2-x_3^2-2x_1^2-2x_2x_3-2x_1x-2x_1x_3. \end{aligned}$$

$q(A)<0$ if and only if $a_1+a_3\le a_2, a_3+a_6\le a_5, a_2+a_5\le a_3+a_4$. So

$$E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_3^{(a_4)}E_2^{(a_5)}E_1^{(a_6)}\left(a_1+a_3\le a_2, a_2+a_5\le a_3+a_4, a_3+a_6\le a_5\right)$$

is tight by Theorem 2.2.

For S_6^{17}, it suffices to consider $\mathbf{i}=(3,1,2,3,1,2)$. For any $\mathbf{a}=(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{N}^6$, we have

$$\mathcal{M}_{\mathbf{i},\mathbf{a}}=\left\{A=A_{x,y,z}\,\middle|\,0\le x\le\min\{a_1,a_4\},0\le y\le\min\{a_2,a_5\},0\le z\le\min\{a_3,a_6\}\right\}$$

where

$$A=\begin{pmatrix} a_1-x & 0 & 0 & x & 0 & 0 \\ 0 & a_2-y & 0 & 0 & y & 0 \\ 0 & 0 & a_3-z & 0 & 0 & z \\ x & 0 & 0 & a_4-x & 0 & 0 \\ 0 & y & 0 & 0 & a_5-y & 0 \\ 0 & 0 & z & 0 & 0 & a_6-z \end{pmatrix}$$

and

$$
\begin{aligned}
q(A) &= \sum_{\substack{1\le m\le 6\\ 1\le p<r\le 6}} a_{pm}a_{rm} + \sum_{\substack{1\le p<r\le 6\\ 1\le l<m\le 6}} (i_l\cdot i_m) a_{pm}a_{rl} + \sum_{\substack{1\le r\le 6\\ 1\le l<m\le 6}} a_{rm}a_{rl} \\
&= a_{11}a_{41} + a_{22}a_{52} + a_{33}a_{63} + a_{14}a_{44} + a_{25}a_{55} + a_{36}a_{66} + a_{11}a_{14} + a_{22}a_{25} + a_{33}a_{36} + a_{41}a_{44} + a_{52}a_{55} + a_{63}a_{66} \\
&\quad + (i_1\cdot i_2)a_{22}a_{41} + (i_1\cdot i_3)a_{33}a_{41} + (i_1\cdot i_4)a_{14}a_{41} + (i_1\cdot i_5)a_{25}a_{41} + (i_1\cdot i_6)a_{36}a_{41} + (i_2\cdot i_3)a_{33}a_{52} \\
&\quad + (i_2\cdot i_4)(a_{14}a_{22} + a_{14}a_{52} + a_{44}a_{52}) + (i_2\cdot i_5)a_{25}a_{52} + (i_2\cdot i_6)a_{36}a_{52} + (i_3\cdot i_4)(a_{14}a_{33} + a_{14}a_{63} + a_{44}a_{63}) \\
&\quad + (i_3\cdot i_5)(a_{25}a_{33} + a_{25}a_{63} + a_{55}a_{63}) + (i_3\cdot i_6)a_{36}a_{63} + (i_4\cdot i_5)a_{25}a_{44} + (i_4\cdot i_6)a_{36}a_{44} + (i_5\cdot i_6)a_{36}a_{55} \\
&= 2(a_1 + a_4 - a_3)x + 2(a_2 + a_5 - a_3)y + 2(a_3 + a_6 - a_4 - a_5)z - (x-z)^2 - (y-z)^2 - x^2 - y^2.
\end{aligned}
$$

$q(A) < 0$ if and only if $a_2 + a_5 \le a_3, a_3 + a_6 \le a_4 + a_5, a_1 + a_4 \le a_3$. So

$$E_3^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_3^{(a_4)}E_1^{(a_5)}E_2^{(a_6)}\left(a_1 + a_4 \le a_3, a_2 + a_5 \le a_3, a_3 + a_6 \le a_4 + a_5\right)$$

is tight by Theorem 2.2.

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