

The Adomian Decomposition Method and the Differential Transform Method for Numerical Solution of Multi-Pantograph Delay Differential Equations

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Abstract

In this paper, the Adomian Decomposition Method (ADM) and the Differential Transform Method (DTM) are applied to solve the multi-pantograph delay equations. The sufficient conditions are given to assure the convergence of these methods. Several examples are presented to demonstrate the efficiency and reliability of the ADM and the DTM; numerical results are discussed, compared with exact solution. The results of the ADM and the DTM show its better performance than others. These methods give the desired accurate results only in a few terms and in a series form of the solution. The approach is simple and effective. These methods are used to solve many linear and nonlinear problems and reduce the size of computational work.

Keywords

Multi-Pantograph Delay Differential Equations, Adomian Decomposition Method (ADM), Differential Transform Method (DTM), Convergence of Adomian Decomposition Method

1. Introduction

Pantograph is a device located on the electriclocomotive. The first time, electric locomotive was made in America in 1851. It was commissioned in 1895. Mathematical model of pantograph was first developed by Taylor and Ockendon (1971) [1]. Pantograph equations belong to a special class of functional-differential equations with proportional delays and arise in many applications such as, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, electro dynamics, quantum mechanics and cell growth, number theory, mixing problems, population models, etc.

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A numerical method based on the Adomian Decomposition Method (ADM) which has been used from the 1970s to the 1990s by George Adomian [13] [14]. The differential transform method (DTM) has been successfully developed by Zhou (1986) in electric circuit analysis. DTM has been used to solve linear and nonlinear differential equations [15].

ADM and DTM have been shown to solve effectively, easily and accurately a large class of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equations with approximate solutions which converge rapidly to accurate solutions [3] [4] [13] [14]. The basic motivation of this work is to apply the ADM and DTM to the DDE. It is well known now in the literature that this algorithm provides the solution in a rapidly convergent series [3] [4]. ADM and DTM are very effective and convenient for solving multi-pantograph equations [3] [4].

This study is presented as follows: In second section, we start by presenting ADM and DTM to solve multi-pantograph delay differential equations. In third section, we continue to the presentation of the convergence of ADM with Theorem 3.1 and Definition 3.2. In fourth section, these methods are shown and compared by four examples by taking various values for t and error evaluation is made. Also, we have plotted the graphs for numerical solutions of ADM and DTM and exact solution.

We examined that multi-pantograph delay differential equations are solved by several methods. Thus, we wanted to show up that may be more efficient, simpler and reliable the solution treatment of the ADM for multi-pantograph delay differential equations. The results show that the ADM is more powerful method than other methods for multi-pantograph delay differential equations.

2. Analysis of Adomian Decomposition Method and the Differential Transform

In this paper, we consider the following multi-pantograph equations [16],

$$y'(t) = a(t)y(t) + \sum_{i=1}^{k} b_i(t)y(q_it) + f(t), \ 0 \le t < T$$
(1)

$$\mathbf{y}(t) = \mathbf{y}_0,\tag{2}$$

 $a(t), b_i(t), f(t)$ are analytical functions, $0 < q_i < 1, i = 1, \dots, k$. Using the ADM, the differential operator *L* is given by

$$L(.) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(.) \tag{3}$$

The inverse operator L^{-1} , this is *n*-fold integral operator defined by

$$L^{-1}(.) = \int_0^t (.)_{n-\text{times}} \,\mathrm{d}t \tag{4}$$

operating with L^{-1} on Equation (1), it then follows

$$Ly + Ry + Ny = f \tag{5}$$

where the method defines Ny the nonlinear term by the Adomian polynomials [3] [13] [14]

$$Ny = \sum_{n=0}^{\infty} A_n,$$

 A_{n} are Adomian polynomials that can be generated for all forms of nonlinearity as

$$A_n = \frac{1}{n!} \frac{d^n}{d\alpha^n} \left[N\left(\sum_{i=1}^n \alpha^i y_i\right) \right]_{\alpha=0},$$

where $i = 0, 1, 2, 3, \cdots$

$$A_{0} = f(y_{0})$$

$$A_{1} = y_{1} \left(\frac{d}{dy_{0}}\right) f(y_{0})$$

$$A_{2} = y_{2} \left(\frac{d}{dy_{0}}\right) f(y_{0}) + \left(\frac{y_{1}^{2}}{2!}\right) \left(\frac{d^{2}}{dy^{2}}\right) f(y_{0})$$

$$A_{3} = y_{3} \left(\frac{d}{dy_{0}}\right) f(y_{0}) + y_{1} y_{2} \left(\frac{d^{2}}{dy_{0}^{2}}\right) f(y_{0}) + \left(\frac{y_{1}^{3}}{3!}\right) \left(\frac{d^{3}}{dy_{0}^{3}}\right) f(y_{0}).$$

Operating with L^{-1} on Equation (5), it then follows

$$L^{-1}Ly + L^{-1}Ry + L^{-1}Ny = L^{-1}f$$
(6)

$$y = y(0) + L^{-1}f - L^{-1}Ry - L^{-1}Ny$$
(7)

$$\sum_{n=0}^{\infty} y_n = y_0 + L^{-1} f - L^{-1} R \sum_{n=0}^{\infty} y_n - L^{-1} \sum_{n=0}^{\infty} A_n$$
(8)

to determine the components

 $y_n(t), n \ge 0.$

First, we identify the zero component $y_0(t)$ by all terms that arise from the boundary conditions at t = 0 and from integrating the source term if it exists. Second, the remaining components of y(t) can be determined in a way such that each component is determined by using the preceding components [3] [13] [14]

$$y_0 = y(0) + L^{-1}f,$$
(9)

and Equation (8) gives for $n = 1, 2, 3, \cdots$ in other words, the method introduces the recursive relation

$$y_{1} = -L^{-1}Ry_{0} - L^{-1}A_{0}$$

$$y_{2} = -L^{-1}Ry_{1} - L^{-1}A_{1}$$

$$y_{3} = -L^{-1}Ry_{2} - L^{-1}A_{2}$$

$$\vdots$$

$$y_{n+1} = -L^{-1}Ry_{n} - L^{-1}A_{n}$$
(10)

The Adomian decomposition method assumes that the unknown function y(t) can be expressed by an infinite series of the form [3] [13] [14]

$$y(t) = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + \cdots$$

so that the components $y_n(t)$ will be determined recursively [3] [13] [14]

Differential Transform Method

Differential transform of function y(x) is defined as follows [17],

$$Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0}$$
(11)

In Equation (11), y(x) is original function and Y(k) is the transformed function, which is called the T-function. Differential inverse transform of Y(k) is defined as

$$Y(k) = \sum_{k=0}^{\infty} x^k Y(k)$$
(12)

From Equations (12) and (11), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0}$$
(13)

Equation (13) implies that concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically.

In actual applications, the function y(x) is expressed by the a finite series and Equation (12) can be written as

$$y(x) = \sum_{k=0}^{m} x^k Y(k) \tag{14}$$

Equation (13) implies that is

$$y(x) = \sum_{k=m+1}^{\infty} x^{k} Y(k)$$

is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

The following theorems that can be deduced from Equations (11) and (12) are given below, see [17] [18]. **Theorem 1** If y(t) = g(t) + h(t), then Y(k) = G(k) + H(k). **Theorem 2** If y(t) = cg(t), then Y(k) = cG(k).

Theorem 3 If $y(t) = \frac{d^k g(t)}{d^k}$, then $Y(k) = \frac{(k+n)!}{k!} G(k+n)$. Theorem 4 If y(t) = g(t+a), then $Y(k) = \sum_{m=k}^{N} {h \choose k} a^{h-k} G(m)$, for $N \to \infty$. Theorem 5 If $y(t) = g\left(\frac{t}{a}\right) a \ge 1$, then

$$Y(k) = \sum_{h=k}^{N} (-1)^{h-k} \frac{(a-1)^{h-k}}{a^{h}} t_0^{h-k} {h \choose k} a^{h-k} G(h), \text{ for } N \to \infty.$$

3. The Convergence of ADM

The Adomian Decomposition Method is equivalent to the sequence defined as follows [19] [20]

$$S_n = y_1 + y_2 + y_3 + \dots + y_n,$$

by using the iterative scheme

$$S_{0} = 0, S_{n+1} = N(y_{0} + S_{n})$$
(15)

and related to the functional equation

$$S = N(y_0 + S).$$

The numerical solution of Equation (15) was used fixed-point theorem by Cherruault [19] [20].

Theorem 3.1 Let *N* be an operator from Hilbert space *H* in to *H* and *y* be the exact solution of functional equation. $\sum_{i=0}^{\infty} y_i$, which is obtained by ADM iterative scheme, converges to *y* when $\exists 0 \le \alpha < 1$, $\|y_{k+1}\| \le \alpha \|y_k\|$,

 $\forall k \in N \bigcup \{0\}.$

Proof See [19] [20]. **Definition 3.2** For every $i \in N \bigcup \{0\}$ we define

$$\alpha_{i} = \begin{cases} \frac{\|y_{i+1}\|}{\|y_{i}\|}, \|y_{i}\| \neq 0, \\ 0, & \|y_{i}\| = 0, \end{cases}$$

Corollary 3.3 In Theorem 4.1, $\sum_{i=0}^{\infty} y_i$ converges to exact solution *y*, when

$$0 \le \alpha_i < 1, i = 1, 2, 3, \cdots$$

4. Numerical Examples

In this section, four experiments of multi-pantograph delay differential equations are given to illustrate the efficiency of the ADM and the DTM. The examples are computed using Maple 15. Results obtained by the methods are compared with the exact solution of each example and found to be good agreement with each other. The absolute errors in tables are given at selected points.

Example 4.1 Consider the following linear multi-pantograph delay equation of the first-order [16]

$$y'(t) = -\frac{5}{6}y(t) + 4y\left(\frac{t}{2}\right) + 9y\left(\frac{t}{3}\right) + t^2 - 1, \ 0 < t \le 1,$$

$$y(0) = 1,$$

which has the exact solution,

$$y(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$$

Using the ADM, we get according to Equations (3)-(10), we obtain recursive formula for $k = 0, 1, 2, 3, \cdots$,

$$y_{k+1} = L^{-1} \left(-\frac{5}{6} y_k(t) + 4 y_k \left(\frac{t}{2} \right) + 9 y_k \left(\frac{t}{3} \right) \right)$$
$$y_0 = 1 + \frac{1}{3} t^3 - t$$
$$y_1 = \frac{73}{6} t - \frac{25}{12} t^2$$
$$y_2 = \frac{1825}{72} t^2 - \frac{175}{216} t^3$$
$$y_3 = \frac{12775}{1296} t^3$$

Thus, we obtain

$$y_{\text{ADM}}(t) = \sum_{n=0}^{\infty} y_n = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3 + 0 + 0 + \cdots$$

The solution by DTM method:

By using Theorems of DTM, we have following recurrence relation:

$$Y(k+1) = \frac{Y(k)}{k+1} \left[-\frac{5}{6} + \frac{4}{2^k} + \frac{9}{3^k} \right] + \delta(k-2) - \delta(k), \ k \ge 1, \ Y(0) = 1$$

Utilizing the recurrence relation, we find

$$Y(1) = \frac{67}{6}, Y(2) = \frac{1675}{72}, Y(3) = \frac{12157}{1296}, Y(4) = 0, Y(5) = 0, \cdots$$

Finally, the differential inverse transform of Y(k) gives

$$y(t) = \sum_{k=0}^{\infty} Y(k) t^{k}$$

we obtain the following series solution

$$y_{\text{DTM}}(t) = \sum_{k=0}^{\infty} Y(k) t^{k} = 1 + \frac{67}{6}t + \frac{1675}{72}t^{2} + \frac{12157}{1296}t^{3} + 0 + 0 + \cdots$$

the closed form of above solution is

$$y(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$$

which is exactly the same as the exact solution.

The obtained results (ADM and DTM) are exactly the same with the one found by exact solution. It is clear from Table 1 and Figure 1 that the three results not only give rapidly convergent series but also accurately compute the solutions.

Using our methods, we choose 6 points on [0, 1] respectively. The numerical results are given in the following **Table 1**.

Example 4.2 Solve the following nonlinear pantograph delay equation of first-order [3]

$$y'(t) = 1 - 2y^2\left(\frac{t}{2}\right), \ 0 \le t \le 1$$

 $y(0) = 0,$

which has the exact solution, $y(t) = \sin t$.

The solution by ADM method:

By applying the ADM, according to Equations (3)-(10), we obtain

$$Ly = 1 - 2y^{2}\left(\frac{t}{2}\right)$$
$$L^{-1}Ly = L^{-1}\left(-2y^{2}\left(\frac{t}{2}\right)\right) + L^{-1}(1)$$
$$y(t) = L^{-1}\left(-2y^{2}\left(\frac{t}{2}\right)\right) + t$$



Figure 1. The obtained solution of the multi-pantograph delay equation [ADM solution (red), DTM solution (black), EXACT solution (blue)].

Table 1. Comparison of exact solution, the (ADM) and the (DTM) of y(t).				
	Solution			
t –	y _{ADM} /y _{DTM}	<i>Y</i> _{exact}	Error	
0	1	1	0	
0.2	4.238932099	4.238932099	0	
0.4	9.789234568	9.789234568	0	
0.6	18.10116667	18.10116667	0	
0.8	29.62498766	29.62498766	0	
1	44.8109568	44.8109568	0	

We have solved this problem using the proposed method. Recursive formula and the sequence of approximate solution are obtained as follows:

$$y_{k+1}(t) = -L^{-1}\left(2y_k^2\left(\frac{t}{2}\right)\right)$$
$$k = 0, y_1(t) = -\frac{t^3}{6}$$
$$= 1, y_2(t) = \frac{t^5}{120}, \quad k = 2, y_3(t) = -\frac{t^7}{5040}$$

thus, we obtain:

$$y_{\text{ADM}}(t) = \sum_{n=0}^{\infty} y_n = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

Using to convergence of ADM's method,

k

$$\alpha_{i} = \left\| \frac{y_{i+1}}{y_{i}} \right\|, 0 \le \alpha_{i} < 1, i = 0, 1, 2, 3, \cdots$$

$$\alpha_{0} = 0.500000000 < 1$$

$$\alpha_{1} = 0.02380952381 < 1$$

$$\alpha_{2} = 0.01302729713 < 1$$

$$\vdots$$

Here, the values of α_i are less than one and hence ADM is convergent. The solution by DTM method:

By using Theorems of DTM, we have following recurrence relation:

$$(k+1)Y(k+1) = \delta(k) - 2\sum_{r=0}^{k} Y(r)Y(k-r), k \ge 1, Y(0) = 1$$

Utilizing the recurrence relation, we find

$$Y(1) = 1, Y(2) = 0, Y(3) = -\frac{1}{3!}, Y(4) = 0, Y(5) = \frac{1}{5!}, \cdots$$

Finally, the differential inverse transform of Y(k) gives

$$y(t) = \sum_{k=0}^{\infty} Y(k) t^{k}$$

we obtain the following series solution

$$y_{\text{DTM}}(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

The obtained results (ADM and DTM) are exactly the same with each other. Increasing the approximation order up to the absolute differences between the numerical solutions are calculated for and comparisons have been made with known results as reported in Table 2 and Figure 2.

Example 4.3 Consider the following linear multi-pantograph delay equation of the first-order [21].

$$y'(t) = -3y(t) + 0.4y(0.4t) + y(0.1t), 0 < t \le 1,$$

y(0) = 2.127,

The solution by ADM method:

By applying the ADM, according to Equations (3)-(10), we obtain

$$Ly = -3y(t) + 0.4y(0.4t) + y(0.1t)$$
$$L^{-1}Ly = L^{-1}(-3y(t) + 0.4y(0.4t) + u(0.1t))$$
$$y(t) = 2.127 - L^{-1}(3y(t) - y(0.4t) - y(0.1t))$$

We have solved this problem using the proposed method. Recursive formula and the sequence of approximate solution are obtained as follows:

$$y_{k+1} = L^{-1} \left(-3y_k(t) + 0.4y_k(0.4t) + y_k(0.1t) \right)$$



Figure 2. The obtained solution of the multi-pantograph delay equation [ADM solution (red), DTM solution (blue), EXACT solution (black)].

<i>t</i>	Solution			
l	${\cal Y}_{ m ADM}$	${\mathcal{Y}}_{ m DTM}$	$y_{\rm exact}$	
0	0	0	0	
0.2	0.198669331	0.198669	0.198669333	
0.4	0.389418342	0.389418	0.389418342	
0.6	0.564642446	0.564642	0.564642446	
0.8	0.717355723	0.717356	0.717355723	
1	0.841468254	0.841471	0.841468254	

Table 2. Comparison of the ADM, the DTM and the EXACT solution.

$$y_0 = 2.127,$$

 $k = 0, y_1 = -3.403200000t$
 $k = 1, y_2 = 4.662384000t^2$
 $k = 2, y_3 = -4.547378527t^3$
 $k = 3, y_4 = 3.380293828t^4$

Thus, we obtain:

$$y_{ADM}(t) = 2.127 - 3.40320000t + 4.662384000t^{2} - 4.547378527t^{3} + 3.380293828t^{4} - \cdots$$

The solution by DTM method:

By using Theorems of DTM, we have following recurrence relation:

$$Y(k+1) = \frac{Y(k)}{k+1} \left[-3 + \left(\frac{2}{5}\right)^{k+1} + \frac{1}{10^k} \right], k \ge 1, Y(0) = 2.127$$

Using the recurrence relation, we find

$$Y(1) = -3.403200000, Y(2) = 4.662384000, Y(3) = -4.547378528$$
$$Y(4) = 3.380293829, Y(5) = -2.021185850, \cdots$$

Finally, the differential inverse transform of Y(k) gives

$$y(t) = \sum_{k=0}^{\infty} Y(k) t^{k}$$

we obtain the following series solution

 $y(t) = 2.127 - 3.403200000t + 4.662384000t^{2} - 4.547378528t^{3} + 3.380293829t^{4} - 2.021185850t^{5} + \cdots$

The obtained results (ADM and DTM) are exactly the same with the one found by exact solution. It is clear from Table 3 and Figure 3 that the two results not only give rapidly convergent series but also accurately compute the solutions.

Example 4.4 Consider the following linear multi-pantograph delay equation of the first-order [21].

$$y'(t) = -y(t) - 0.8y(0.5t) + 0.1y(0.25t), 0 < t \le 1,$$

 $y(0) = 1,$

The solution by ADM method:

By applying the ADM, according to Equations (3)-(10), we obtain

$$Ly = -y(t) - 0.8y(0.5t) + 0.1y(0.25t)$$
$$Ly = L^{-1}(-y(t) + 0.8y(0.5t) + 0.1u(0.25t))$$
$$y(t) = 1 - L^{-1}(y(t) + 0.8y(0.5t) - 0.1y(0.25t))$$

We have solved this problem using the proposed method. Recursive formula and the sequence of approximate solution are obtained as follows:

$$y_{k+1} = L^{-1} \left(-y_k \left(t \right) + 0.8 y_k \left(0.5t \right) + 0.1 y_k \left(0.25t \right) \right)$$
$$y(0) = 1, y_0 = 1,$$
$$k = 0, y_1 = -1.70000000t$$
$$k = 1, y_2 = 1.68750000t^2$$



Figure 3. The obtained solution of the multi-pantograph delay equation [ADM solution (red), DTM solution (blue)].

	Solution		
1 —	y _{adm}	y _{DTM}	
0	2.127	2.127	
0.2	1.601884802	1.601884802	
0.4	1.307204736	1.307204736	
0.6	1.219390558	1.219390558	
0.8	1.444676306	1.444676306	
1	2.219099301	2.219099301	

l'able 3.	Comparison of the ADM	solution y_{ADM}	with the solution b	by DTM is illustrated.
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 $k = 2, y_3 = -0.4650651040t^3$ $k = 3, y_4 = 0.1277112376t^4$

Thus, we obtain:

$$y_{\text{ADM}}(t) = \sum_{n=0}^{\infty} y_n = 1 - 1.70000000t + 1.68750000t^2 - 0.4650651040t^3 + \cdots$$

Using to convergence of ADM's method,

$$\begin{split} \alpha_i &= \left\| \frac{y_{i+1}}{y_i} \right\|, \ 0 \leq \alpha_i < 1, \ i = 0, 1, 2, 3, \cdots. \\ \alpha_0 &= 0.9926470588 < 1 \\ \alpha_1 &= 0.2755941357 < 1 \\ \alpha_2 &= 0.2746093751 < 1 \\ &: \end{split}$$

according to the obtained results ADM is convergent to the exact solution.

The solution by DTM method:

By using Theorems of DTM, we have following recurrence relation,

$$Y(k+1) = \frac{Y(k)}{k+1} \left[-1 + \frac{0.8}{2^k} + \frac{0.1}{4^k} \right], k \ge 1, Y(0) = 1$$

Utilizing the recurrence relation, we find

$$Y(1) = -1.7, Y(2) = 1.168750000t, Y(3) = -0.4650651042, Y(4) = 0.127712376, \cdots$$

Finally, the differential inverse transform of Y(k) gives

$$y(t) = \sum_{k=0}^{\infty} Y(k) t^{k}$$

we obtain the following series solution

$$y(t) = 1 - 1.7t + 1.168750000t^2 - 0.4650651042t^3 + 0.127712376t^4 - \cdots$$

The obtained results (ADM and DTM) are exactly the same with the one found by exact solution. It is clear from Table 4 and Figure 4 that the two results not only give rapidly convergent series but also accurately compute the solutions.

5. Conclusion

It has been the aim of this paper to show that it appears natural to approximate the solution of multi-pantograph delay differential equation by ADM and DTM. We obtain the high approximate solutions or the exact solutions within a few iterations. It is concluded from figures and tables that the successive approximations methods are an accurate and efficient method to solve multi-pantograph delay differential equations. Some numerical examples have been provided to illustrate that the present method is effective in accuracy and convergence speed. In a word, the ADM and DTM show that the techniques are reliable, powerful and promising methods for linear





The comparison of the ribbit solution y _{ADM} with the solution by D rivers indistributed.				
t	Sol	ution		
	${\cal Y}_{ m ADM}$	У _{ртм}		
0.0	1	1		
0.2	0.703233817	0.703233817		
0.4	0.480505241	0.480505241		
0.6	0.316847313	0.316847313		
0.8	0.202197189	0.202197189		
1.0	0.131396133	0.131396133		

Fable 4. Compariso	on of the ADM solution	y_{ADM} with th	e solution by	DTM is illustrated.
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and nonlinear problems.

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