Open Journal of Discrete Mathematics, 2015, 5, 54-64
Published Online July 2015 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2015.53005

Scientific Research Publishing

On the Signed Domination Number of the Cartesian Product of Two Directed Cycles

Ramy Shaheen

Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria
Email: shaheenramy2010@hotmail.com

Received 21 March 2015; accepted 21 July 2015; published 24 July 2015

Abstract

Let D be a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$. A function $f: V(D) \to \{-1,1\}$ is called a signed dominating function (SDF) if $f\left(N_D^-[v]\right) \geq 1$ for each vertex $v \in V$. The weight $\omega(f)$ of f is defined by $\sum_{v\in V} f(v)$. The signed domination number of a digraph D is $\gamma_s(D) = \min\{\omega(f) \mid f \text{ is an SDF of } D\}$. Let $C_m \times C_n$ denotes the cartesian product of directed cycles of length m and n. In this paper, we determine the exact values of $\gamma_s(C_m \times C_n)$ for $m = 8, 9, 10$ and arbitrary n. Also, we give the exact value of $\gamma_s(C_m \times C_n)$ when $m, n \equiv 0 \pmod 3$ and bounds for otherwise.

Keywords

Directed Graph, Directed Cycle, Cartesian Product, Signed Dominating Function, Signed Domination Number

1. Introduction

Throughout this paper, a digraph $D(V,A)$ always means a finite directed graph without loops and multiple arcs, where $V = V(D)$ is the vertex set and $A = A(D)$ is the arc set. If uv is an arc of D, then say that v is an out-neighbor of u and u is an in-neighbor of v. For a vertex $v \in V(D)$, let $N_D^+(v)$ and $N_D^-(v)$ denote the set of out-neighbors and in-neighbors of v, respectively. We write $d_D^+(v) = \left|N_D^+(v)\right|$ and $d_D^-(v) = \left|N_D^-(v)\right|$ for the out-degree and in-degree of v in D, respectively (shortly $d^+(v)$, $d^-(v)$). A digraph D is r-regular if $d_D^+(v) = d_D^-(v) = r$ for any vertex $v \in D$. Let $N_D^+[v] = N_D^+(v) \cup \{v\}$ and $N_D^-[v] = N_D^-(v) \cup \{v\}$. The maximum out-degree and in-degree of D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively (shortly Δ^+, Δ^-). The minimum out-degree and in-degree of D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively (shortly δ^+, δ^-).

How to cite this paper: Shaheen, R. (2015) On the Signed Domination Number of the Cartesian Product of Two Directed Cycles. *Open Journal of Discrete Mathematics*, **5**, 54-64. http://dx.doi.org/10.4236/ojdm.2015.53005

A *signed dominating function* of D is defined in [1] as function $f: V \to \{-1,1\}$ such that $f\left(N_D^-[v]\right) \ge 1$ for every vertex $v \in V$. The *signed domination number* of a directed graph D is $\gamma_s(D) = \min\{\omega(f) | f \text{ is an SDF of } D\}$. Also, a signed k-*dominating* function (SKDF) of D is a function $f: V \to \{-1,1\}$ such that $f\left(N_D^-[v]\right) \ge k$ for every vertex $v \in V$. The k-*signed domination number* of a digraph D is $\gamma_{ks}(D) = \min\{\omega(f) | f \text{ is an SKDF of } D\}$. Consult [2] for the notation and terminology which are not defined here.

The Cartesian product $D_1 \times D_2$ of two digraphs D_1 and D_2 is the digraph with vertex set $V(D_1 \times D_2) = V(D_1) \times V(D_2)$ and $((u_1,u_2),(v_1,v_2)) \in A(D_1 \times D_2)$ if and only if either $u_1 = v_1$ and $(u_2,v_2) \in A(D_2)$ or $u_2 = v_2$ and $(u_1,v_1) \in A(D_1)$.

In the past few years, several types of domination problems in graphs had been studied [3]-[7], most of those belonging to the vertex domination. In 1995, Dunbar *et al.* [3], had introduced the concept of signed domination number of an undirected graph. Haas and Wexler in [1], established a sharp lower bound on the signed domination number of a general graph with a given minimum and maximum degree and also of some simple grid graph. Zelinka [8] initiated the study of the signed domination numbers of digraphs. He studied the signed domination number of digraphs for which the in-degrees did not exceed 1, as well as for acyclic tournaments and the circulant tournaments. Karami *et al.* [9] established lower and upper bounds for the signed domination number of digraphs. Atapour *et al.* [10] presented some sharp lower bounds on the signed k-domination number of digraphs. Shaheen and Salim in [11], were studied the signed domination number of two directed cycles $C_m \times C_n$ when $m = 3, 4, 5, 6, 7$ and arbitrary n. In this paper, we study the Cartesian product of two directed cycles C_m and C_n for $mn \ge 8n$. We mainly determine the exact values of $\gamma_s(C_8 \times C_n)$, $\gamma_s(C_9 \times C_n)$, $\gamma_s(C_{10} \times C_n)$ and for some values of m and n. Some previous results:

Theorem 1.1 (Zelinka [8]). Let D be a directed cycle or path with n vertices. Then $\gamma_s(D) = n$.

Lemma 1.2 (Zelinka [8]). Let D be a directed graph with n vertices. Then $\gamma_s(D) \equiv n \pmod 2$.

Corollary 1.3 (Karami *et al.* [9]). Let D be a directed of order n in which $d^+(v) = d^-(v) = k$ for each $v \in V$, where k is a nonnegative integer. Then $\gamma_s(D) \ge \dfrac{n}{1+k}$.

In [11], the following results are proved.

Theorem 1.4 [11]:

$\gamma_s(C_3 \times C_n) = n : n \equiv 0 \pmod 3$, otherwise $\gamma_s(C_3 \times C_n) = n+2$. $\gamma_s(C_4 \times C_n) = 2n$.
$\gamma_s(C_5 \times C_n) = 2n : n \equiv 0 \pmod{10}$, $\gamma_s(C_5 \times C_n) = 2n+1 : n \equiv 3,5,7 \pmod{10}$,
$\gamma_s(C_5 \times C_n) = 2n+2 : n \equiv 2,4,6,8 \pmod{10}$, $\gamma_s(C_5 \times C_n) = 2n+3 : n \equiv 1,9 \pmod{10}$.
$\gamma_s(C_6 \times C_n) = 2n : n \equiv 0 \pmod 3$, otherwise $\gamma_s(C_6 \times C_n) = 2n+4$. $\gamma_s(C_7 \times C_n) = 3n$.

2. Main Results

In this section we calculate the signed domination number of the Cartesian product of two directed cycles C_m and C_n for $m = 8, 9, 10$ and $m \equiv 0 \pmod 3$ and arbitrary n.

The vertices of a directed cycle C_n are always denoted by the integers $\{1,2,\cdots,n\}$ considered modulo n. The ith row of $V(C_m \times C_n)$ is $R_i = \{(i,j) : j = 1,2,\cdots,n\}$ and the jth column $K_j = \{(i,j) : i = 1,2,\cdots,m\}$. For any vertex $(i,j) \in V(C_m \times C_n)$, always we have the indices i and j are reduced modulo m and n, respectively.

Let us introduce a definition. Suppose that f is a signed dominating function for $C_m \times C_n$, and assume that $1 \le j,h \le n$. We say that the hth column of $f(C_m \times C_n)$ is a *t-shift* of the jth column if $f(i,j) = f(i+t,h)$ for each vertex $(i,j) \in K_j$, where the indices i, t, $i+t$ are reduced modulo m and j, h are reduced modulo n.

Remark 2.1: Let f is a $\gamma_s(C_m \times C_n)$-function. Then $f[(r,s)] \ge 1$ for each $1 \le r \le m$ and each $1 \le s \le n$. Since $C_m \times C_n$ is 2-regular, it follows from $f((i,j)) = -1$ that $f((i \pm 1, j)) = f((i, j \pm 1)) = 1$ because $f[(i,j)] \ge 1$, $f((i+1, j-1)) = 1$ because $f[(i+1,j)] \ge 1$ and $f((i-1, j+1)) = 1$ because $f[(i, j+1)] \ge 1$. On the other hand, if $f((i \pm 1, j)) = f((i, j \pm 1)) = 1$, $f((i+1, j-1)) = 1$ and $f((i-1, j+1)) = 1$, then we must have $f((i,j)) = -1$ since f is a minimum signed dominating function.

Remark 2.2. Since the case $f((i,j)) = f((i+1,j)) = -1$ is not possible, we get $s_j \ge 0$. Furthermore, s_j is odd if m is odd and even when m is even.

Let f be a signed dominating function for $C_m \times C_n$, then we denote $f(K_j) = \sum_{i=1}^{m} f((i,j))$ of the weight of a column K_j and put $s_j = f(K_j)$. The sequence $(s_1, s_2, \cdots, s_n)$ is called a signed dominating sequence corresponding to f. We define

$$X_i = \left|\{j : s_j = i\}\right|, i = 0,1,\cdots,m.$$

Then we have

$$X_0 + X_1 + \cdots + X_m = n.$$

$$\omega(f) = X_1 + 2X_2 + \cdots + mX_m.$$

For the remainder of this section, let f be a signed domination function of $C_m \times C_n$ with signed dominating sequence $(s_1, \cdots, s_n)$. We need the following Lemma:

Lemma 2.3. If $s_j = k$ then $s_{j-1}, s_{j+1} \geq m - 2k$. Furthermore, $s_{j-1} + s_j \geq m - k$ and $s_j + s_{j+1} \geq m - k$.

Proof. Let $s_j = k$, then there are $(m-k)/2$ of vertices in K_j which get value −1. By Remark 2.1, K_{j+1} include at least $2(m-k)/2$ of vertices which get the value 1 and at most $m-(m-k) = k$ of vertices which has value −1. Hence, $s_{j+1} \geq m - 2k$. Furthermore, $s_j + s_{j+1} \geq m - k$. By the same argument, we get $s_{j-1} \geq m - 2k$ and $s_{j-1} + s_j \geq m - k$. □

Theorem 2.4.

$$\gamma_s(C_8 \times C_n) = \begin{cases} 3n & : n \equiv 0 \pmod{16}, \\ 3n+1 : & n \equiv 3,13 \pmod{16}, \\ 3n+2 : & n \equiv 6,10 \pmod{16}, \\ 3n+3 : & n \equiv 5,7,9,11 \pmod{16}, \end{cases}$$

$$3n+2 \leq \gamma_s(C_8 \times C_n) \leq 3n+4 : n \equiv 2,4,8,12,14 \pmod{16},$$

$$3n+3 \leq \gamma_s(C_8 \times C_n) \leq 3n+5 : n \equiv 1,15 \pmod{16},$$

Proof. We define a signed dominating function f as follows:

$f((i,2j-1)) = f((i+2,2j-1)) = f((i+5,2j-1)) = -1$ for $1 \leq j \leq \lceil n/2 \rceil$ and $i \equiv (7j-6) \pmod 8$,

$f((i,2j)) = f((i+3,2j)) = -1$ for $1 \leq j \leq \lfloor n/2 \rfloor$ and $i \equiv (7j-3) \pmod 8$, and

$f((i,j)) = 1$ otherwise. Also we define $f_n((i,n)) = 1$ for $i = 1,\cdots,8$.

By the definition of f, we have $s_j = 2$ for j is odd and $s_j = 4$ for j is even. Notice, f is a SDF for $C_8 \times C_n$ when $n \equiv 0 \pmod{16}$. Therefore, there is a problem with the vertices of K_1 when $n \equiv 1,\cdots,15 \pmod{16}$.

Now, let us define the following functions:

$$f_1((i,j)) = \begin{cases} f((i,j)) & \text{if } j \neq n, \\ +1 & \text{if } i = 1,2,3,4,5,6,7,8,\ j = n, \end{cases} \quad f_2((i,j)) = \begin{cases} f((i,j)) & \text{if } j \neq n, \\ -1 & \text{if } i = 5,8,\ j = n, \\ +1 & \text{if } i = 1,2,3,4,6,7,\ j = n, \end{cases}$$

$$f_3((i,j)) = \begin{cases} f((i,j)) & \text{if } j \neq n, \\ -1 & \text{if } i = 5,\ j = n, \\ +1 & \text{if } i = 1,2,3,4,6,7,8,\ j = n, \end{cases} \quad f_4((i,j)) = \begin{cases} f((i,j)) & \text{if } j \neq n, \\ -1 & \text{if } i = 8,\ j = n, \\ +1 & \text{if } i = 1,2,3,4,5,6,7,\ j = n, \end{cases}$$

We note:

f_1 is a SDF of $C_8 \times C_n$ when $n \equiv 1,2,4,8,12,14,15 \pmod{16}$.

f_2 is a SDF of $C_8 \times C_n$ when $n \equiv 3,13 \pmod{16}$.

f_3 is a SDF of $C_8 \times C_n$ when $n \equiv 6,9,11 \pmod{16}$.

f_4 is a SDF of $C_8 \times C_n$ when $n \equiv 5,7,10 \pmod{16}$.

Hence,

$$
\begin{aligned}
&\gamma_s(C_8\times C_n)\le 3n : n\equiv 0(\text{mod}16)\\
&\gamma_s(C_8\times C_n)\le 3n+1 : n\equiv 3,13(\text{mod}16)\\
&\gamma_s(C_8\times C_n)\le 3n+2 : n\equiv 6,10(\text{mod}16)\\
&\gamma_s(C_8\times C_n)\le 3n+3 : n\equiv 5,7,9,11(\text{mod}16)\\
&\gamma_s(C_8\times C_n)\le 3n+4 : n\equiv 2,4,8,12,14(\text{mod}16)\\
&\gamma_s(C_8\times C_n)\le 3n+5 : n\equiv 1,15(\text{mod}16)
\end{aligned}
\tag{1}
$$

For example, f_1 is a SDF of $C_8 \times C_{12}$, where $\gamma_s(C_8\times C_{12})\le 40=3(12)+4$, see **Figure 1**.

{Here, we must note that, for simplicity of drawing the Cartesian products of two directed cycles $C_m \times C_n$, we do not draw the arcs from vertices in last column to vertices in first column and the arcs from vertices in last row to vertices in first row. Also for each figure of $C_m \times C_n$, we replace it by a corresponding matrix by signs − and + which descriptions −1 and +1 on figure of $f(C_m\times C_n)$, respectively}.

By Remark 2.2, for any minimum signed dominating function f of $C_8 \times C_n$ with signed dominating sequence $(s_1,\cdots,s_n)$, we have s_j = 0, 2, 4, 6 or 8 for $j=1,\cdots,n$. By Lemma 2.3, if $s_j = 0$ then $s_{j-1},s_{j+1}\ge 8$, and if $s_j = 2$ then $s_{j-1},s_{j+1}\ge 4$. This implies that

$$\omega(f)=\sum_{j=1}^{n}s_j\ge 3n \quad \text{for } n\equiv 0(\text{mod}\,2). \tag{2}$$

$$\omega(f)=\sum_{j=1}^{n}s_j\ge 3n+1 \quad \text{for } n\equiv 1(\text{mod}\,2). \tag{3}$$

Hence, by (1), (2) and (3) we get

$$\gamma_s(C_8\times C_n)=3n \quad \text{for } n\equiv 0(\text{mod}16).$$

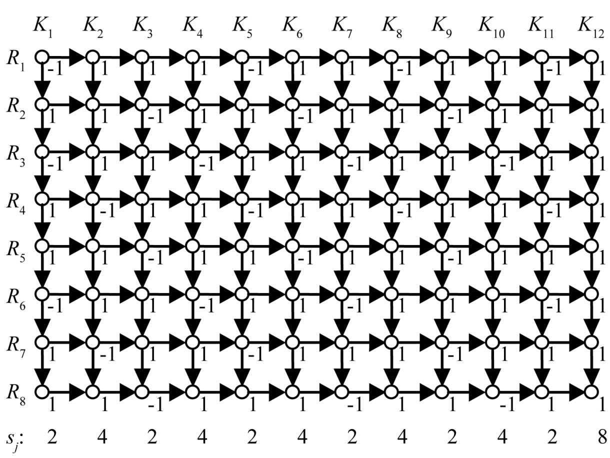
$$\gamma_s(C_8\times C_n)=3n+1 \quad \text{for } n\equiv 3,13(\text{mod}16).$$

Assume that $n\not\equiv 0,3,13(\text{mod}16)$.

Let f' ba a signed dominating function with signed dominating sequence $(s_1',s_2',\cdots,s_n')$.

If m, $n \le 7$, then by Theorem 1.4 is the required (because $C_m\times C_n\cong C_n\times C_m$). Let m, $n \ge 8$. We prove the following claim:

Claim 2.1. For $k \ge 2$, we have $\sum_{d=j+1}^{j+k}s_d'\ge 3k$ if k is even and $\sum_{d=j+1}^{j+k}s_d'\ge 3k-1$ when k is odd.



(a)

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}
R_1	−	+	+	+	−	+	+	−	+	+	−	+
R_2	+	+	−	+	+	−	+	+	−	+	+	+
R_3	−	+	+	−	+	+	−	+	+	−	+	+
R_4	+	−	+	+	−	+	+	−	+	+	−	+
R_5	+	+	−	+	+	−	+	+	−	+	+	+
R_6	−	+	+	−	+	+	−	+	+	+	−	+
R_7	+	−	+	+	−	+	+	+	−	+	+	+
R_8	+	+	−	+	+	+	−	+	+	−	+	+
s_j:	2	4	2	4	2	4	2	4	2	4	2	8

(b)

Figure 1. (a) A signed dominating function of $C_8 \times C_{12}$; (b) A corresponding matrix of a signed dominating function of $C_8 \times C_{12}$.

Proof of Claim 2.1. We have the subsequence $(s'_{j+1},\cdots,s'_k)$ is including at least two terms. Then, immediately from Remark 2.2 and Lemma 2.3, gets the required. The proof of Claim 2.1 is complete. □

Now, if $s'_j=0$ for some *j*, then $s'_{j-1}=s'_{j+1}=8$. Without loss of generality, we can assume that $s'_2=0$. Then Claim 2.1, imply that

$$\omega(f')=\sum_{j=1}^{n}s'_j=\sum_{j=1}^{3}s'_j+\sum_{j=4}^{n}s'_j\geq 16+3(n-3)-1=3n+7. \tag{4}$$

Assume that $s'_j\geq 2$ for all $j=1,\cdots,n$. We have three cases:

Case 1. If $s'_j=8$ for some *j*. Let $s'_1=8$. Then from Claim 2.1, we get

$$\omega(f')=\sum_{j=1}^{n}s'_j=s'_1+\sum_{j=2}^{n}s'_j\geq 8+3(n-1)-1=3n+4,\ \text{ when } n\equiv 0(\text{mod } 2). \tag{5}$$

$$\omega(f')=\sum_{j=1}^{n}s'_j=s'_1+\sum_{j=2}^{n}s'_j\geq 8+3(n-1)=3n+5,\ \text{ when } n\equiv 1(\text{mod } 2). \tag{6}$$

Case 2. Let $2\leq s'_j\leq 6$. If $(s'_1,\cdots,s'_n)$ include at least two terms which are equals 6, then

$$\omega(f')=\sum_{j=1}^{n}s'_j\geq 3n+4. \tag{7}$$

For $n\equiv 1(\text{mod } 2)$, then 8*n* is even. By Lemma 1.2, $\gamma_s(C_8\times C_n)=\omega(f')$ must be even number. Hence, from (7) is

$$\omega(f')=\sum_{j=1}^{n}s'_j\geq 3n+5. \tag{8}$$

Assume that $2\leq s'_j\leq 4$ for all $j=1,\cdots,n$ except once which equals 6. Thus,

$$\omega(f')=\sum_{j=1}^{n}s'_j\geq 3n+2\ \text{ for } n\equiv 0(\text{mod } 2). \tag{9}$$

$$\omega(f')=\sum_{j=1}^{n}s'_j\geq 3n+3\ \text{ for } n\equiv 1(\text{mod } 2). \tag{10}$$

For the case 3, we need the following claim:

Claim 2.2. Let *f*′ be a minimum signed dominating function of $C_8\times C_n$ with signed dominating sequence $(s'_1,s'_2,\cdots,s'_n)$. Then for $(s'_j,s'_{j+1},s'_{j+1},s'_{j+2})=(2,4,2,4)$, and up to isomorphism, there is only one possible configuration for *f*″, it is shown in **Figure 2**. The prove is immediately by drawing. □

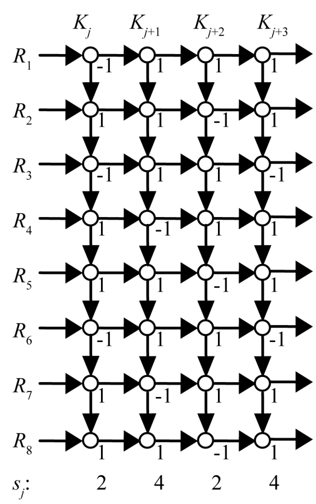


Figure 2. The form $(s'_j,s'_{j+1},s'_{j+1},s'_{j+2})=(2,4,2,4)$.

Case 3. Let $2 \le s'_j \le 4$ for all $j = 1, \cdots, n$. We define

$$X_i = \left| j : s'_j = i \right|, i = 2, 4.$$

Then we have

$$X_2 + X_4 = n.$$

$$\omega(f') = 2X_2 + 4X_4.$$

Since the case $(s'_j, s'_{j+1}) = (2,2)$ is not possible, we have $X_4 \ge X_2$.

If $X_4 \ge \lceil n/2 \rceil + 2$. Then $\omega(f') \ge 2(n - \lceil n/2 \rceil - 2) + 4(\lceil n/2 \rceil + 2) = 2n + 2\lceil n/2 \rceil + 4$. Thus

$$\omega(f') = \sum_{j=1}^{n} s'_j \ge 3n + 4 \quad \text{for } n \equiv 0 (\text{mod } 2). \tag{11}$$

$$\omega(f') = \sum_{j=1}^{n} s'_j \ge 3n + 5 \quad \text{for } n \equiv 1 (\text{mod } 2). \tag{12}$$

If $X_4 = \lceil n/2 \rceil + 1$. Then $\omega(f') \ge 2(n - \lceil n/2 \rceil - 1) + 4(\lceil n/2 \rceil + 1) = 2n + 2\lceil n/2 \rceil + 2$. Hence

$$\omega(f') = \sum_{j=1}^{n} s'_j \ge 3n + 2 \quad \text{for } n \equiv 0 (\text{mod } 2). \tag{13}$$

$$\omega(f') = \sum_{j=1}^{n} s'_j \ge 3n + 3 \quad \text{for } n \equiv 1 (\text{mod } 2). \tag{14}$$

Let $X_4 = \lceil n/2 \rceil$ and $X_2 = \lfloor n/2 \rfloor$.

Then we have one possible is as the form $(s'_1, s'_2, \cdots, s'_n) = (2,4,2,4,\cdots,2,4,\cdots)$. This implies that $\omega(f') = 3n$ for $n \equiv 0(\text{mod } 2)$ and $\omega(f') = 3n + 1$ for $n \equiv 1(\text{mod } 2)$. By Claim 2.2, we have f' is as the function f, which defined in forefront of Theorem 2.4. However, f is not be a signed dominating function for $C_8 \times C_n$ when $n \not\equiv 0,3,13(\text{mod}16)$. Thus

$$\gamma_s(C_8 \times C_n) > 3n \quad \text{for } n \equiv 0(\text{mod } 2).$$

$$\gamma_s(C_8 \times C_n) > 3n + 1 \quad \text{for } n \equiv 1(\text{mod } 2).$$

By Lemma 1.2, and above arguments, we conclude that

$$\gamma_s(C_8 \times C_n) \ge 3n + 2 \quad \text{for } n \equiv 0(\text{mod } 2). \tag{15}$$

$$\gamma_s(C_8 \times C_n) \ge 3n + 3 \quad \text{for } n \equiv 1(\text{mod } 2). \tag{16}$$

Hence, from (1), (15) and (16), deduce that

$$\gamma_s(C_8 \times C_n) \ge 3n + 2 \quad \text{for } n \equiv 6,10(\text{mod}16).$$

$$\gamma_s(C_8 \times C_n) \ge 3n + 3 \quad \text{for } n \equiv 5,7,9,11(\text{mod}16).$$

$$3n + 2 \le \gamma_s(C_8 \times C_n) \le 3n + 4 \quad \text{for } n \equiv 2,4,8,12,14(\text{mod}16).$$

$$3n + 3 \le \gamma_s(C_8 \times C_n) \le 3n + 5 \quad \text{for } n \equiv 1,15(\text{mod}16).$$

Finally, we result that:

$$\gamma_s(C_8 \times C_n) = 3n \quad \text{for } n \equiv 0(\text{mod}16).$$

$$\gamma_s(C_8 \times C_n) = 3n + 1 \quad \text{for } n \equiv 3,13(\text{mod}16).$$

$$\gamma_s(C_8 \times C_n) = 3n + 2 \quad \text{for } n \equiv 6,10(\text{mod}16).$$

$$\gamma_s(C_8 \times C_n) = 3n + 3 \quad \text{for } n \equiv 5,7,9,11(\text{mod}16).$$

$$3n+2 \le \gamma_s\left(C_8 \times C_n\right) \le 3n+4 \text{ for } n \equiv 2,4,8,12,14 \pmod{16}.$$

$$3n+3 \le \gamma_s\left(C_8 \times C_n\right) \le 3n+5 \text{ for } n \equiv 1,15 \pmod{16}. \quad \square$$

Theorem 2.5.

$$\gamma_s\left(C_9 \times C_n\right) = \begin{cases} 3n & : n \equiv 0 \pmod 3, \\ 3n+6 : & n \equiv 1,2 \pmod 3. \end{cases}$$

Proof. We define a signed dominating function f as follows: $f\left((i,j)\right) = f\left((i+3,j)\right) = f\left((i+6,j)\right) = -1$ for $1 \le j \le n$ and $i \equiv j \pmod 9$, and $f\left((i,j)\right) = 1$ otherwise. Also, let us define the following function:

$$f_1\left((i,j)\right) = \begin{cases} f\left((i,j)\right) & \text{if } j \ne n, \\ +1 & \text{if } i = 1,2,3,4,5,6,7,8,9,\ j = n. \end{cases}$$

By define f, we have $s_j = 3$ for $1 \le j \le n$. Notice, f is a SDF for $C_9 \times C_n$ for $n \equiv 0 \pmod 3$. And f_1 is a SDF of $C_9 \times C_n$ for $n \equiv 1,2 \pmod 3$. For an illustration $\gamma_s\left(C_9 \times C_6\right)$, see Figure 3. Hence,

$$\gamma_s\left(C_9 \times C_n\right) \le 3n \text{ for } n \equiv 0 \pmod 3. \tag{17}$$

$$\gamma_s\left(C_9 \times C_n\right) \le 3n+6 \text{ for } n \equiv 1,2 \pmod 3. \tag{18}$$

From Corollary 1.3 is $\gamma_s\left(C_9 \times C_n\right) \ge 3n$. Then by (17), $\gamma_s\left(C_9 \times C_n\right) = 3n$ for $n \equiv 0 \pmod 3$.

For $n \equiv 1,2 \pmod 3$.

If $4 \le n \le 8$, then by Theorems 1.4 and 2.4, gets the required. Assume that $n \ge 9$.

By Remark 2.2, we have s_j = 1, 3, 5, 7 or 9. By Lemma 2.3, if $s_j = 1$ then $s_{j-1}, s_{j+1} \ge 7$, $s_j = 3$ then $s_{j-1}, s_{j+1} \ge 3$ and $s_j = 5$ then $s_{j-1}, s_{j+1} \ge 3$ (because if $s_{j-1}, s_{j+1} < 3$, then we need $s_j \ge 7$). By Lemma 2.3, the cases $\left(s_{j+1}, s_{j+2}\right) = (1,3),(3,1)$ are not possible. Hence, $\sum_{d=j+1}^{j+k} s_d \ge 3k$, for $k \ge 2$. This implies that,

$$\sum_{d=1}^{n-1} s_d \ge 3(n-1). \tag{19}$$

We define

$$X_i = \left|\left\{ j : s_j = i \right\}\right|, i = 1,3,5,7,9.$$

Then we have

	K_1	K_2	K_3	K_4	K_5	K_6
R_1	−	+	+	−	+	+
R_2	+	−	+	+	−	+
R_3	+	+	−	+	+	−
R_4	−	+	+	−	+	+
R_5	+	−	+	+	−	+
R_6	+	+	−	+	+	−
R_7	−	+	+	−	+	+
R_8	+	−	+	+	−	+
R_9	+	+	−	+	+	−
s_j:	3	3	3	3	3	3

Figure 3. A corresponding matrix of a signed dominating function of $C_9 \times C_6$.

$$X_1 + X_3 + X_5 + X_7 + X_9 = n.$$

$$\omega(f) = X_1 + 3X_3 + 5X_5 + 7X_7 + 9X_9.$$

If we have one case from the cases $X_9 \geq 1$, $X_7 \geq 2$, $X_5 + X_7 \geq 2$ or $X_5 \geq 3$. Then by (19) is $\omega(f) \geq 3n+6$.

Assume the contrary, *i.e.*, ($X_9 = 0$, $X_7 < 2$, $X_5 + X_7 < 2$ and $X_5 < 3$).

Hence, $\omega(f) = X_1 + 3X_3 + 5X_5 + 7X_7$. We consider the cases $X_7 < 2$ and $X_5 < 3$, which are including the remained cases, *i.e.*, $X_7 = 1$ and $X_5 = 2$. First, we give the following Claim:

Claim 2.3. There is only one possible for $(s_j, s_{j+1}) = (3,3)$ is $f((i,j)) = f((i+3,j)) = f((i+6,j)) = f((i+1,j+1)) = f((i+4,j+1)) = f((i+7,j+1)) = -1$ and $f((i,j)) = f((i,j+1)) = 1$, otherwise for $1 \leq i \leq 9$.

The proof comes immediately by the drawing. □

Case 1. $X_7 = 1$ and $X_5 = X_9 = 0$. Without loss of generality, we can assume $s_n = 7$. Then we have the form $(3,3,\cdots,3,7)$. By Claim 2.3, for $j < n-1$, each column K_{j+1} is 1-shift of K_j, K_{j+2} is 2-shift of K_j and K_{j+3} is 3-shift = (0-shift) of K_j. Without loss of generality, we can assume $f((1,1)) = f((4,1)) = f((7,1)) = -1$ and $f((i,1)) = 1$ otherwise. We consider two subcases:

Subcase 1.1. For $n \equiv 1 (\text{mod } 3)$. Then K_{n-1} is $(n-2)$-shift = (2-shift) of K_1. This implies that $f((3,n-1)) = f((6,n-1)) = f((9,n-1)) = -1$. Hence, we need $f((i,n)) = 1$ for all $i = 1,\cdots,9$. This is a contradiction with $\omega(f(K_n)) = 7$. Thus, $\omega(f) \geq 3X_3 + 9X_9 = 3(n-1) + 9 = 3n+6$.

Subcase 1.2. For $n \equiv 2 (\text{mod } 3)$. Then K_{n-1} is $(n-2)$-shift = (0-shift) of K_1. This implies that $f((1,n-1)) = f((4,n-1)) = f((7,n-1)) = -1$. So, we need $f((i,n)) = 1$ for all $i = 1,\cdots,9$. Again, we get a contradiction with $\omega(f(K_n)) = 7$. Thus, $\omega(f) \geq 3X_3 + 9X_9 = 3(n-1) + 9 = 3n+6$.

Case 2. $X_5 = 2$ and $X_7 = X_9 = 0$. Here we have $s_k = s_{k+d} = 5$ and $s_j = 3$ otherwise. By the same argument similar to the Case 1, we have K_j is $(j-1)$-shift of K_1. Thus, if $j \equiv 1 (\text{mod } 3)$, then $f((1,j)) = f((4,j)) = f((7,j)) = -1$ and for $j \equiv 2 (\text{mod } 3)$ is $f((2,j)) = f((5,j)) = f((8,j)) = -1$. Also, for position the vertices of K_1, we always have $f((1,n)) = f((2,n)) = f((4,n)) = f((5,n)) = f((7,n)) = f((8,n)) = 1$. We consider four Subcases:

Subcase 2.1. $d = 1$, without loss of generality, we can assume $s_{n-1} = s_n = 5$.

For $n \equiv 1 (\text{mod } 3)$, $f((2,n-2)) = f((5,n-2)) = f((8,n-2)) = -1$. Then $f((1,n-1)) = f((2,n-1)) = f((4,n-1)) = f((5,n-1)) = f((7,n-1)) = f((8,n-1)) = 1$. The three remaining vertices from each K_{n-1} and K_n, most including two values −1, and this is impossible. The same arguments is for $n \equiv 2 (\text{mod } 3)$.

Subcase 2.2. $d = 2$, let $s_{n-2} = s_n = 5$. Then we have the form $(s_1, s_2, \cdots, s_n) = (3,3,\cdots,3,5,3,5)$.

If $n \equiv 1 (\text{mod } 3)$, then $n-3 \equiv 1 (\text{mod} 3)$. This implies that K_{n-3} is 0-shift of K_1. Therefore, $f((1,n-3)) = f((4,n-3)) = f((7,n-3)) = -1$. Hence, the three columns K_{n-2}, K_{n-1}, K_n must be including seven values of −1, two in K_{n-2}, three in K_{n-1} and two in K_n and this impossible. The same argument is for $n \equiv 2 (\text{mod } 3)$.

Subcase 2.3. $d = 3$, let $s_{n-3} = s_n = 5$. We have the form $(s_1, s_2, \cdots, s_n) = (3,3,\cdots,3,5,3,3,5)$. Then for $n \equiv 1 (\text{mod } 3)$, K_{n-4} is 2-shift of K_1. Therefore $f((3,n-4)) = f((6,n-4)) = f((9,n-4)) = -1$. Also, $s_{n-2} = s_{n-1} = 3$. Therefore, two vertices of $\{(1,n-3),(4,n-3),(7,n-3)\}$ must has value −1. By symmetry, let $f((1,n-3)) = f((4,n-3)) = -1$. Then by Claim 2.3, there is one case for $(s_{n-2}, s_{n-1}) = (3,3)$. Hence, $f((2,n-2)) = f((5,n-2)) = f((8,n-2)) = f((3,n-1)) = f((6,n-1)) = f((9,n-1)) = -1$. Therefore, we need two vertices from K_n with value −1. This is a contradiction, (because the vertices of the first column must be a signed dominates by the vertices of the last column). The same argument is for $n \equiv 2 (\text{mod } 3)$.

Subcase 2.4. $d \geq 4$, let $s_{n-d} = s_n = 5$ (by symmetry is $n-d \geq 4$).

We have the form $(s_1, s_2, \cdots, s_n) = (3,3,\cdots,3,5,3,\cdots,3,5)$. By Claim 2.3, if $(s_j, s_{j+1}, \cdots) = (3,3,\cdots)$ then for each two vertices $f((i,j)) = f((q,j)) = -1$ we must have $|i-q| = 3$ and so for $K_{j+1}, \cdots, K_{n-d-1}$. Since $s_j = 3 (j \leq n-d-1)$ and $s_{n-d} = 5$, then K_{n-d} including two vertices with value −1 by 1-shift of two vertices in K_{n-d-1}. Also, K_{n-d+1} including two vertices with value −1 by 1-shift of vertices in K_{n-d} and the third vertex must be distance 3 from any one has value −1 (Since $s_{n-d+1} = s_{n-d+1} = \cdots = 3$, Claim 2.3). Thus, the order of the values −1 or 1 of the vertices $K_{n-d+1}, \cdots, K_{n-1}$ does not change. Hence the vertices of K_{n-1} has the same order of K_{n-1} when we have the signed dominating sequence $(3,3,\cdots,3,3)$ and this impossible is signed dominating sequence of $C_9 \times C_n$ for $n \equiv 1,2 (\text{mod } 3)$. In Subcases 2.1, 2.2, 2.3 and 2.4 there are many details, we will be omitted it.

Finally, we deduce that does not exist a signed dominating function f of $C_9 \times C_n$ for $n \equiv 1,2 \pmod 3$ with $\omega(f) \le 3n+4$. Hence,

$$\gamma_s(C_9 \times C_n) \ge 3n+6 : n \equiv 1,2 \pmod 3. \tag{20}$$

From (18) and (20) is $\gamma_s(C_9 \times C_n) = 3n+6 : n \equiv 1,2 \pmod 3$. □

Theorem 2.6. $\gamma_s(C_{10} \times C_n) = 4n$.

Proof. We define a signed dominating function f as follows:

$f((i,j)) = f((i+3,j)) = f((i+6,j)) = -1$ for $1 \le j \le n$ and $i \equiv j \pmod{10}$, and $f((i,j)) = 1$ otherwise. Also, we define

$$f_{n-7}((3,n-7)) = f_{n-7}((7,n-7)) = f_{n-7}((10,n-7)) = -1,$$

$$f_{n-6}((1,n-6)) = f_{n-6}((5,n-6)) = f_{n-6}((8,n-6)) = -1,$$

$$f_{n-5}((3,n-5)) = f_{n-5}((6,n-5)) = f_{n-5}((9,n-5)) = -1,$$

$$f_{n-4}((1,n-4)) = f_{n-4}((4,n-4)) = f_{n-4}((7,n-4)) = -1,$$

$$f_{n-3}((2,n-3)) = f_{n-3}((5,n-3)) = f_{n-3}((9,n-3)) = -1,$$

$$f_{n-2}((3,n-2)) = f_{n-2}((7,n-2)) = f_{n-2}((10,n-2)) = -1,$$

$$f_{n-1}((1,n-1)) = f_{n-1}((5,n-1)) = f_{n-1}((8,n-1)) = -1,$$

$$f_n((3,n)) = f_n((6,n)) = f_n((9,n)) = -1,$$

and $f_j((i,j)) = 1$ otherwise for $j = n-5, n-4, n-3, n-2, n-1, n$.

By define f and $f_{n-7}, f_{n-6}, f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n$ we have $s_j = 4$ for all $1 \le j \le n$. Notice that: f is a SDF for $C_{10} \times C_n$ when $n \equiv 0,3, \pmod{10}$.

$$\{f \setminus \{f(K_{n-5}) \cup f(K_{n-4}) \cup f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\}\} \\ \cup \{f_{n-5} \cup f_{n-4} \cup f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 1 \pmod{10}$.

$$\{f \setminus \{f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\}\} \cup \{f_{n-2} \cup f_{n-1} \cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 2 \pmod{10}$.

$$\{f \setminus \{f(K_{n-6}) \cup f(K_{n-5}) \cup f(K_{n-4}) \cup f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\}\} \\ \cup \{f_{n-6} \cup f_{n-5} \cup f_{n-4} \cup f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 4 \pmod{10}$.

$$\{f \setminus \{f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\}\} \cup \{f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 5 \pmod{10}$.

$$\{f \setminus \{f(K_n)\}\} \cup \{f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 6 \pmod{10}$.

$$\{f \setminus \{f(K_{n-7}) \cup f(K_{n-6}) \cup f(K_{n-5}) \cup f(K_{n-4}) \cup f(K_{n-3}) \cup f(K_{n-2}) \cup f(K_{n-1}) \cup f(K_n)\}\} \\ \cup \{f_{n-7} \cup f_{n-6} \cup f_{n-5} \cup f_{n-4} \cup f_{n-3} \cup f_{n-2} \cup f_{n-1} \cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 7 \pmod{10}$.

$$\{f\setminus\{f(K_{n-4})\cup f(K_{n-3})\cup f(K_{n-2})\cup f(K_{n-1})\cup f(K_n)\}\}\cup\{f_{n-4}\cup f_{n-3}\cup f_{n-2}\cup f_{n-1}\cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 8(\text{mod } 10)$.

$$\{f\setminus\{f(K_{n-1})\cup f(K_n)\}\}\cup\{f_{n-1}\cup f_n\}$$

is a SDF for $C_{10} \times C_n$ when $n \equiv 9(\text{mod } 10)$.

For an illustration $\gamma_s(C_{10}\times C_{11})$ see **Figure 4**, (here for $n \equiv 1(\text{mod } 10)$, we are changing the functions of the columns: $K_{n-5}, K_{n-4}, K_{n-3}, K_{n-2}, K_{n-1}, K_n$). In all the cases we have $\gamma_s(C_{10}\times C_n)\le 4n$.

By Remark 2.2, we have s_j = 0, 2, 4, 6, 8 or 10. Also by Lemma 2.3, if s_j = 0, then $s_{j-1}, s_{j+1} \ge 10$ and when s_j = 2, is $s_{j-1}, s_{j+1} \ge 6$ and s_j = 4 is $s_{j-1}, s_{j+1} \ge 4$ (because if $s_{j-1} = 2$ or $s_{j+1} = 2$, then $s_j \ge 6$). This implies that

$$\gamma_s(C_{10}\times C_n)=\sum_{j=1}^{n} s_j \ge 4n.$$

So, we get $\gamma_s(C_{10}\times C_n) = 4n$. □

Corollary 2.7. For $m \equiv 0(\text{mod } 3)$, we have

$$\gamma_s(C_m\times C_n)=\frac{mn}{3} \quad \text{if } n\equiv 0(\text{mod}\,3).$$

$$\frac{mn}{3}\le\gamma_s(C_m\times C_n)=\frac{mn}{3}+\frac{2m}{3} \quad \text{if } n\equiv 1,2(\text{mod}\,3).$$

Proof. By Corollary 1.3 we have

$$\gamma_s(C_m\times C_n)\ge\frac{mn}{3}. \tag{21}$$

Let us a signed dominating function f as follows: $f((3i-2,3j-2))=-1$ for $1\le i\le m/3$, $1\le j\le n/3$, $f((3i-1,3j-1))=-1$ for $1\le i\le m/3$, $1\le j\le n/3$, and $f((3i,3j))=-1$ for $1\le i\le m/3$, $1\le j\le n/3$.

By define f, we have $s_j = m/3$ for $1\le j\le n$. Notice, f is a SDF for $C_m \times C_n$ for $m,n\equiv 0(\text{mod } 3)$. Hence, $\gamma_s(C_m\times C_n)\le mn/3$. Then from (21), is $\gamma_s(C_m\times C_n)=mn/3$ for $m,n\equiv 0(\text{mod } 3)$.

For $n \equiv 1, 2(\text{mod } 3)$.

Let $f_n((i,n))=1$ for $1\le i\le m$. Notice, $\{f\setminus\{f(K_n)\}\cup\{f_n\}\}$ is a SDF for $C_m \times C_n$ for $n\equiv 1,2(\text{mod } 3)$.

Thus, $\gamma_s(C_m\times C_n)\le m(n-1)/3+m=mn/3+2m/3$. Hence, by (21) is $mn/3\le\gamma_s(C_m\times C_n)\le mn/3+2m/3$ if $n\equiv 1,2(\text{mod } 3)$. □

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}
R_1	−	+	+	+	−	+	−	+	+	−	+
R_2	+	−	+	+	+	+	+	−	+	+	+
R_3	+	+	−	+	+	−	+	+	−	+	−
R_4	−	+	+	−	+	+	−	+	+	+	+
R_5	+	−	+	+	−	+	+	−	+	−	+
R_6	+	+	−	+	+	−	+	+	+	+	−
R_7	−	+	+	−	+	+	−	+	−	+	+
R_8	+	−	+	+	−	+	+	+	+	−	+
R_9	+	+	−	+	+	−	+	−	+	+	−
R_{10}	+	+	+	−	+	+	+	+	−	+	+
s_j:	4	4	4	4	4	4	4	4	4	4	4
						f_{n-5}	f_{n-4}	f_{n-3}	f_{n-2}	f_{n-1}	f_n

Figure 4. A corresponding matrix of a signed dominating function of $C_{10} \times C_{11}$.

3. Conclusions

This paper determined that exact value of the signed domination number of $C_m \times C_n$ for $m = 8, 9, 10$ and arbitrary n. By using same technique methods, our hope eventually lead to determination $\gamma_s(C_m \times C_n)$ for general m and n.

Based on the above (Lemma 2.3 and Theorems 1.4, 2.4, 2.5 and 2.6), also by the technique which used in this paper, we again rewritten the following conjecture (This conjecture was mention in [11]):

Conjecture 3.1.

$$\gamma_s(C_m \times C_n) = \left\lceil \frac{m}{3} \right\rceil n \quad \text{when } n \equiv 0 (\bmod 2m) \text{ or } n \equiv 1 (\bmod 3).$$

References

[1] Haas, R. and Wexler, T.B. (1999) Bounds on the Signed Domination Number of a Graph. *Discrete Mathematics*, **195**, 295-298. http://dx.doi.org/10.1016/S0012-365X(98)00189-7

[2] West, D.B. (2000) Introduction to Graph Theory. Prentice Hall, Inc., Upper Saddle River.

[3] Dunbar, J.E., Hedetniemi, S.T., Henning, M.A. and Slater, P.J. (1995) Signed Domination in Graphs, Graph Theory, Combinatorics and Application. John Wiley & Sons, Inc., Hoboken, 311-322.

[4] Cockayne, E.J. and Mynhart, C.M. (1996) On a Generalization of Signed Domination Functions of Graphs. *Ars Combinatoria*, **43**, 235-245.

[5] Hattingh, J.H. and Ungerer, E. (1998) The Signed and Minus *k*-Subdomination Numbers of Comets. *Discrete Mathematics*, **183**, 141-152. http://dx.doi.org/10.1016/S0012-365X(97)00051-4

[6] Xu, B. (2001) On Signed Edge Domination Numbers of Graphs. *Discrete Mathematics*, **239**, 179-189. http://dx.doi.org/10.1016/S0012-365X(01)00044-9

[7] Broere, I., Hattingh, J.H., Henning, M.A. and McRae, A.A. (1995) Majority Domination in Graphs. *Discrete Mathematics*, **138**, 125-135. http://dx.doi.org/10.1016/0012-365X(94)00194-N

[8] Zelinka, B. (2005) Signed Domination Numbers of Directed Graphs. *Czechoslovak Mathematical Journal*, **55**, 479-482. http://dx.doi.org/10.1007/s10587-005-0038-5

[9] Karami, H., Sheikholeslami, S.M. and Khodkar, A. (2009) Lower Bounds on the Signed Domination Numbers of Directed Graphs. *Discrete Mathematics*, 3**09**, 2567-2570. http://dx.doi.org/10.1016/j.disc.2008.04.001

[10] Atapour, M., Sheikholeslami, S., Hajjypory, R. and Volkmann, L. (2010) The Signed *k*-Domination Number of Directed Graphs. *Central European Journal of Mathematics*, **8**, 1048-1057. http://dx.doi.org/10.2478/s11533-010-0077-5

[11] Shaheen, R. and Salim, H. (2015) The Signed Domination Number of Cartesian Products of Directed Cycles. Submitted to Utilitas Mathematica.