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# Uniform Convergence and Dynamical Behavior of a Discrete Dynamical System

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## Abstract

**In this paper we study the dynamical behavior of a system $f$ approximated uniformly by a sequence $f_n$ of chaotic maps. We give examples to show that properties like sensitivity and denseness of periodic points need not be preserved under uniform convergence. We derive conditions under which some of the dynamical properties of the maps $f_n$ are preserved in $f$.**

## Keywords

**Uniform Convergence, Weakly Mixing, Topological Mixing**

## 1. Introduction

Let $(X,d)$ be a compact metric space and let $f: X \to X$ be a continuous map. Then the rule $x_{n+1} = f(x_n)$ defines a discrete dynamical system. For an given initial condition $x \in X$, the set $\{f^n(x): n \geq 0\}$ defines the orbit of the system.

A point $x \in X$ is called *periodic* if $f^k(x) = x$ for some positive integer $k$, where $f^k = f \circ f \circ f \circ \ldots \circ f$ ($k$ times). The least such $k$ is called the *period* of the point $x$. A map $f$ is called *transitive* if for any pair of non-empty open sets $U,V$ in $X$, there exist a positive integer $k$ such that $f^k(U) \bigcap V \neq \phi$. A map $f$ is called *weakly mixing* if for any pairs of non-empty open sets $U_1,U_2$ and $V_1,V_2$ in $X$, there exists $k \in \mathbf{N}$ such that $f^k(U_i) \bigcap V_i \neq \phi$ for $i = 1,2$. It is known that for any continuous self map $f$, if $f$ is weakly mixing and $U_1,U_2,\ldots,U_n,V_1,V_2,\ldots,V_n$ are non-empty open sets, then there exists a $k \geq 1$ such that $f^k(U_i) \bigcap V_i \neq \phi$ for $i = 1,2,\ldots,n$. A map $f$ is called mixing or *topologically mixing* if for each pair of non-empty open sets $U,V$ in $X$, there exists a positive integer $k$ such that $f^n(U) \bigcap V \neq \phi$ for all $n \geq k$. A map $f$ is called sensitive if there exists a $\delta > 0$ such that for each $x \in X$, $\varepsilon > 0$ there exists $y \in X$ and a positive integer $n$ such that $d(x,y) < \varepsilon$ and $d(f^n(x), f^n(y)) > \delta$. A map $f$ is called strongly sensitive if there exists a $\delta > 0$ such that for each $x \in X$ and each neighborhood $U$ of $x$, there exists a positive integer $n_0$ such that $diam(f^n(U)) > \delta$ for all $n \geq n_0$. For Details refer [1]-[3].

For non-empty open subsets $U,V$ of $X$, Define,

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$$S(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$$

$$N(f,U,V) = \{n \in \mathbf{N} : f^n(U) \bigcap V \neq \phi\}$$

$$N(f,x,V) = \{n \in \mathbf{N} : f^n(x) \in V\}$$

$$N(f,U,\delta) = \{n \in \mathbf{N} : diam(f^n(U)) > \delta\}$$

For any two continuous self maps $f,g$ on $X$, define

$$d_H(f,g) = \sup_{x \in X} d(f(x), g(x)) .$$

It can be seen that $d_H$ defines a metric on the space of all continuous functions.

Let $(f_k)$ be sequence of continuous maps on $X$. The sequence is said to converge uniformly to a function $f$ if for each $\varepsilon > 0$, $\exists r \in \mathbf{N}$ such that $d_H(f_k, f) < \varepsilon, \forall k \geq r$.

In real systems, it is often observed that due to natural constraints, any modeling of a system yields a discrete or continuous system which approximates the behavior of the original system. Thus, it is interesting to see how the dynamics of approximations effect the dynamics of the original system. In this direction, we prove that many of the dynamical properties of the system cannot be concluded even with the strongest form of approximation. We prove that if a sequence of sensitive maps converges uniformly to a function $f$, the map $f$ need not be sensitive. Similar conclusions can be made for dynamical properties like transitivity and dense set of periodic points. We derive conditions under which uniform limit of weakly mixing/topologically mixing maps is weakly mixing/topologically mixing. In recent times, some of these questions have grabbed attention. In [4], authors claimed that uniform limit of a sequence of transitive maps is transitive. However, the claim proved to be false, was corrected in [5] [6] where authors proved that uniform convergence of transitive maps need not be transitive.

In this paper we try to answer some of the above raised questions. We prove that many of the above mentioned properties are not preserved even under the strongest notion of convergence. In particular we prove that if the maps $f_k$ have positive topological entropy, the limit map need not have a positive topological entropy. We derive conditions under which properties like transitivity, dense periodic points and sensitive dependence on initial conditions are preserved.

## 2. Main Results

We first give some examples to show that properties under consideration need not be preserved under uniform convergence.

**Example 1:** Let $(\lambda_n)$ be a sequence of irrational numbers converging to 1. Let $f_n : S^1 \to S^1$ defined as,

$$f_n(e^{i\theta}) = e^{i(\theta + 2\pi\lambda_n)}$$

It may be noted that as $\lambda_n$ are irrationals, each $f_n$ is transitive but the limit is the identity map which is not transitive.

If we take the sequence $\lambda_n$ of rational numbers converging to an irrational number $\lambda$, then the sequence $(f_n)$ is a sequence of maps with dense periodic; points but the limit function does not contain any periodic point.

**Example 2:** Let $I$ be the unit interval and $S^1$ be the unit circle. Then the product $I \times S^1$ is a cylinder and the metric $\rho((x,e^{i\alpha}),(y,e^{i\beta})) = \max\{|x-y|,|e^{i\alpha} - e^{i\beta}|\}$ gives the product topology on it.

For each $n$, define $f_n : I \times S^1 \to I \times S^1$ as,

$$f_n(x,e^{i\theta}) = (x, e^{i(\theta + \frac{x}{n})})$$

Then for the map $f_n$, as any two points at different heights rotate with different velocities and move apart in finitely many iterates, for each $x \in I \times S^1$ and any neighborhood $U$ of $x$, there are points in $U$ (at different height) which move apart in finitely many iterates and hence each $f_n$ is strongly sensitive. However, the limit of the sequence $(f_n)$ is the identity map which is not sensitive and hence sensitivity/strong sensitivity is not preserved under uniform convergence.

From the examples above, it is proved that the dynamical behavior of a sequence of dynamical systems need not be preserved even under the strongest form of convergence. We now give some necessary and sufficient conditions under which some of the dynamical properties of a sequence of dynamical systems $f_n$ are preserved in the limit map $f$.

**Result 1:** Let $(X,d)$ be a compact metric space and let $(f_n)$ be a sequence of self maps on $X$ converging uniformly to $f$. Then, $f$ is transitive if and only if for any pair of non-empty open sets in $X$, $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,V)\neq\phi$.

*Proof.* Let $f$ be transitive and let $U,V$ be two non-empty open sets in $X$. As $f$ is transitive, there exists $p\in\mathbf{N}$ such that $f^p(U)\bigcap V\neq\phi$. Therefore, there exists $x\in U$ such that $f^p(x)\in V$. As $f_n^p\to f^p$, there exists $k\in\mathbf{N}$ such that $f_n^p(x)\in V\quad\forall n\geq k$. Thus, $p\in N(f_n,U,V),\forall n\geq k$. Consequently $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,V)\neq\phi$ and proof of the forward part is complete.

Conversely, let $U,V$ be two non-empty open sets in $X$. Let $u\in U$ and $v\in V$. Choose $\varepsilon>0$ such that $u\in S(u,\varepsilon)\subset U$ and $v\in S(v,\varepsilon)\subset V$. Let $U_1=S(u,\frac{\varepsilon}{4})$ and $V_1=S(v,\frac{\varepsilon}{4})$. By given condition, there exists $k\in\mathbf{N}$ such that $\bigcap_{n=k}^{\infty}N(f_n,U_1,V_1)\neq\phi$. Let $p\in\bigcap_{n=k}^{\infty}N(f_n,U_1,V_1)$. As $f_n^p\to f^p$ uniformly, there exists $n_0\in\mathbf{N}$ such that $d(f_n^p(x),f^p(x))<\frac{\varepsilon}{4}\quad\forall n\geq n_0,\forall x\in X$. Choose $s\geq\max\{k,n_0\}$. Then, $f_s^p(U_1)\bigcap V_1$ implies that there exist $x\in U_1$ such that $f_s^p(x)\in V_1$. Further, as $s\geq n_0$, $d(f_s^p(x),f^p(x))<\frac{\varepsilon}{4}$. Thus, $f^p(x)\in S(v,\varepsilon)\subset V$. Thus $f^p(U)\bigcap V\neq\phi$. As the proof can be replicated for any pair of non-empty open sets $U,V$, $f$ is transitive.

**Result 2:** Let $(X,d)$ be a compact metric space and let $(f_n)$ be a sequence of self maps on $X$ converging uniformly to $f$. Then, $f$ is weakly mixing if and only if for any pair of non-empty open sets $U_1,V_1$, $U_2,V_2$ in $X$, $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}[N(f_n,U_1,V_1)\cap N(f_n,U_2,V_2)]\neq\phi$.

*Proof.* Let $f$ be weakly mixing and let $U_1,V_1,U_2,V_2$ be two non-empty open sets in $X$. As $f$ is weakly mixing, there exists $p\in\mathbf{N}$ such that $f^p(U_i)\bigcap V_i\neq\phi. i=1,2$. Therefore, there exists $x_i\in U_i$ such that $f^p(x_i)\in V_i$. As $f_n^p\to f^p$ uniformly, there exists $k\in\mathbf{N}$ such that $f_n^p(x_i)\in V_i\quad\forall n\geq k$. Thus, $p\in[N(f_n,U_1,V_1)\cap N(f_n,U_2,V_2)],\forall n\geq k$. Consequently $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}[N(f_n,U_1,V_1)\cap N(f_n,U_2,V_2)]\neq\phi$.

Conversely, let $U_1,V_1,U_2,V_2$ be non-empty open sets in $X$. Let $u_i\in U_i$ and $v_i\in V_i$ for $i=1,2$. Choose $\varepsilon>0$ such that $u_i\in S(u_i,\varepsilon)\subset U_i$ and $v_i\in S(v_i,\varepsilon)\subset V_i$. Let $U_i^*=S(u_i,\frac{\varepsilon}{4})$ and $V_i^*=S(v_i,\frac{\varepsilon}{4})$. By given condition, there exists $k\in\mathbf{N}$ such that $\bigcap_{n=k}^{\infty}[N(f_n,U_1^*,V_1^*)\cap N(f_n,U_2^*,V_2^*)]\neq\phi$. Let $p\in\bigcap_{n=k}^{\infty}[N(f_n,U_1^*,V_1^*)\cap N(f_n,U_2^*,V_2^*)]$. As $f_n^p\to f^p$ uniformly, there exists $n_0\in\mathbf{N}$ such that $d(f_n^p(x),f^p(x))<\frac{\varepsilon}{4}\quad\forall n\geq n_0,\forall x\in X$. Choose $s\geq\max\{k,n_0\}$. Then, $f_s^p(U_i^*)\bigcap V_i^*$ implies that there exist $x_i\in U_i^*$ such that $f_s^p(x_i)\in V_1^*$. Further, as $s\geq n_0$, $d(f_s^p(x_i),f^p(x_i))<\frac{\varepsilon}{4}$. Thus, $f^p(x_i)\in S(v_i,\varepsilon)\subset V_i$. Thus $f^p(U_i)\bigcap V_i\neq\phi$. As the proof can be replicated for any collection of non-empty open sets in X, $f$ is weakly mixing.

**Corollary 1:** Let $(X,d)$ be a compact metric space and let $(f_n)$ be a sequence of self maps on $X$ converging uniformly to $f$. Then, $f$ is topologically mixing if and only if for any pair of non-empty open sets $U,V$ in $X$, $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,V)$ is cofinite.

*Proof.* If $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,V)$ is cofinite for a pair of open sets $U,V$ in $X$, then there exists a $n_0,r\in\mathbf{N}$ such that $U$ and $V$ interact under the map $f_n$ at time instant $r$, for all $n\geq n_0$. Using previous result, $f^r(U)\cap V\neq\phi$. However as $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,V)$ is cofinite, such interaction happens for all $n\geq r$ and hence $f^n(U)\cap V\neq\phi$ for all $n\geq r$. Thus $f$ is topologically mixing.

**Result 3:** Let $(X,d)$ be a compact metric space and let $(f_n)$ be a sequence of self maps on $X$ converging uniformly to $f$. Then, $f$ is sensitive if and only if there exists a $\delta>0$ such that for any non-empty open set $U$ in $X$, $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,\delta)\neq\phi$.

*Proof.* Let $f$ be sensitive with sensitivity constant $\delta$ and let $U$ be a non-empty open set in $X$. As $f$ is sensitive, there exists $u_1,u_2\in U$ and $p\in\mathbf{N}$ such that $d(f^p(u_1),f^p(u_2))>\delta$. As $f_n^p\to f^p$ uniformly, there exists $n_0\in\mathbf{N}$ such that $d(f_n^p(x),f^p(x))<\frac{\delta}{8},\forall n\geq n_0$. Consequently $d(f_n^p(u_1),f_n^p(u_2))>\frac{\delta}{2}$ for all $n\geq n_0$. Thus $p\in N(f_n,U,\frac{\delta}{2})$ for all $n\geq n_0$. Consequently, $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,\frac{\delta}{2})\neq\phi$ and proof of the forward part is complete.

Conversely, let $U$ be an arbitrary non-empty open set and let $p\in\bigcap_{n=k}^{\infty}N(f_n,U,\delta)\neq\phi$. As $f_n^p\to f^p$ uniformly, there exists $n_0\in\mathbf{N}$ such that $d(f_n^p(x),f^p(x))<\frac{\delta}{8}$, $\forall n\geq n_0,\forall x\in X$. Choose $s\geq\max\{k,n_0\}$. Then, there exists $u_1,u_2\in U$ such that $d(f_s^p(u_1),f_s^p(u_2))>\delta$. Also, as $s\geq n_0$, $d(f_s^p(x),f^p(x))<\frac{\delta}{8}\forall x\in X$. Consequently, $d(f^p(u_1),f^p(u_2))>\frac{\delta}{2}$. As $U$ was arbitrary, $f$ is sensitive.

**Corollary 2:** Let $(X,d)$ be a compact metric space and let $(f_n)$ be a sequence of self maps on $X$ converging uniformly to $f$. Then, $f$ is strongly sensitive if and only if there exists a $\delta>0$ such that for any non-empty open set $U$ in $X$, $\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,U,\delta)$ is cofinite.

*Proof.* Similar

**Result 4:** Let $(X,d)$ be a compact metric space and let $(f_n)$ be a sequence of self maps on $X$ converging uniformly to $f$. Then, $f$ has dense set of periodic points if and only if for any non-empty open set $U$ in $X$, $\bigcup_{x\in U}\bigcap_{N_x}\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,x,N_x)\neq\phi$, where $N_x$ varies over all neighborhoods of $x$.

*Proof.* Let $f$ has dense set of periodic points and let $U$ be a non empty open set in $X$. Let $x\in U$ be periodic with period $p$. As $f_n^p\to f^p$ uniformly, $f_n^p(x)$ converges to $x$. Thus, for each neighborhood $N_x$ of $x$, there exists $k\in\mathbf{N}$ such that $f_n^p(x)\in N_x$ for all $n\geq k$. Thus, $p\in N(f_n,x,N_x)$ for all $n\geq k$ which implies $p\in\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,x,N_x)$. As the steps can be repeated for any neighborhood $N_x$ of $x$, $p\in\bigcap_{N_x}\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,x,N_x)$. Consequently, $\bigcup_{x\in U}\bigcap_{N_x}\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}N(f_n,x,N_x)\neq\phi$.

Conversely, let $U$ be an arbitrary non-empty open set in $X$ and let $p \in \bigcup_{x \in U} \bigcap_{N_x} \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} N(f_n, x, N_x) \neq \phi$. Thus, there exists $x \in U, p \in \mathbf{N}$ such that for each neighborhood $N_x$ of $x$, $p \in \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} N(f_n, x, N_x)$, *i.e.* every neighborhood $N_x$ of $x$ contains the tail of the sequence $(f_n^p(x))$. Thus $f_n^p(x)$ converges to $x$. But as $f_n^p \to f^p$ uniformly, $f^p(x) = x$ and $x$ is a periodic point in $U$. As $U$ was arbitrary, $f$ has a dense set of periodic points.

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