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On No-Node Solutions of the Lazer-McKenna Suspension Bridge Models

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Abstract

In this paper, we are concerned with the existence and multiplicity of no-node solutions of the Lazer-McKenna suspension bridge models by using the fixed point theorem in a cone.

Keywords

Differential Equations, Periodic Solution, Cone, Fixed Point Theorem

1. Introduction

In [1], the Lazer-McKenna suspension bridge models are proposed as following

$$\begin{cases} m_1 v_{tt} - T v_{xx} + \delta_2 v_t - k(u-v)^+ = \varepsilon f_1(t,x), \\ m_2 u_{tt} + EIu_{xxxx} + \delta_1 u_t + k(u-v)^+ = W_1(x), \\ u(0,T) = u(l.t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ v(0,t) = v(L,T) \\ u(\bullet,x) = v(\bullet,x) \quad are \quad 2\pi - periodic \quad \text{in} \quad \text{t}. \end{cases}$$

If we look for no-node solutions of the form $u(x,t) = \lambda y(t)\sin(\pi x/L)$, $v(x,t) = \lambda z(t)\sin(\pi x/L)$ and impose a forcing term of the form $f_i(x,t,v,u) = \sin(\pi x/L)h_i(u,v)$, then via some computation, we can obtain the following system:

$$\begin{cases} y'' + \delta_1 y' + a_{11}(t)y + a_{12}(t)z = \lambda h_1(y,z), \\ z'' + \delta_2 z' + a_{21}(t)y + a_{22}(t)z = \lambda h_2(y,z), \\ y(t) = y(t+T), y'(t) = y'(t+T), z(t) = z(t+T), z'(t) = z'(t+T). \end{cases} \tag{1}$$

In this paper, by combining the analysis of the sign of Green's functions for the linear damped equation, together with a famous fixed point theorem, we will obtain some existence results for (1) if the nonlinearities satisfy the following semipositone condition

(**H**) The function $h_i(y,z)$ is bounded below, and maybe change sign, namely, there exists a sufficiently large constant M > 0 such that $h_i(y,z) + M > 0$.

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Such case is called as semipositone problems, see [2]. And one of the common techniques is the Krasnoselskii fixed point theorem on compression and expansion of cones.

Lemma 1.1 [3]. Let E be a Banach space, and K be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, Let $A: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(i) $\|Ax\| \le \|x\|, \forall x \in K \cap \partial\Omega_1$; $\|Ax\| \ge \|x\|, \forall x \in K \cap \partial\Omega_2$; or

(ii) $\|Ax\| \le \|x\|, \forall x \in K \cap \partial\Omega_2$; $\|Ax\| \ge \|x\|, \forall x \in K \cap \partial\Omega_1$;

Then, A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$

2. Preliminaries

If the linear damped equation

$$x'' + h(t)x' + a(t)x = 0, \tag{2}$$

is nonresonant, namely, its unique T-periodic solution is the trivial one, then as a consequence of Fredholm's alternative in [4], the nonhomogeneous equation $x'' + h(t)x' + a(t)x = e(t)$, admits a unique T-periodic solution which can be written as $x(t) = \int_0^T G(t,s)e(s)ds$, where G(t; s) is the Green's function of (2). For convenience, we will assume that the following standing hypothesis is satisfied throughout this paper:

(H1) $\delta_i(t), a_{ij}(t)$ are T-periodic functions such that the Green's function $G_i(t,s)$, associated with the linear damped equation

$$x'' + \delta_i(t)x' + a_{ii}(t)x = 0,$$

is positive for all $(t,s) \in [0,T] \times [0,T]$, and $0 < m_i = \min_{t,s\in[0,T]} G_i(t,s), M_i = \max_{t,s\in[0,T]} G_i(t,s).$

(H2) $a_{12}(t), a_{121}(t)$ are negative T-periodic functions, and satisfy:

$$a_{11}(t) + a_{12}(t) > 0, a_{21}(t) + a_{22}(t) > 0,\ M_1 \|a_{12}(t)\|_{L^1} \le \frac{1}{4}, M_2 \|a_{21}(t)\|_{L^1} \le \frac{1}{4}.$$

Let E denote the Banach space $C[0,T] \times C[0,T]$ with the norm $\|(y,z)\| = \max_{t\in[0,T]} |y(t)| + \max_{t\in[0,T]} |z(t)|$. for $(y,z) \in E$. Define K to a cone in E by $K = \{(y,z) \in E : y \ge 0, z \ge 0, y + z \ge \theta \|(y,z)\|\}$, where $\theta = \min_{i=1,2} \frac{m_i}{M_i}$.

Also, for $r > 0$ a positive number, let $K_r = \{(y,z) \in K : \|(y,z)\| < r\}$, $\partial K_r = \{(y,z) \in K : \|(y,z)\| = r\}$.

If **(H)**, **(H1)** and **(H2)** hold, let $\tilde{y} = y + \xi, \tilde{z} = z + \xi$, (1) is transformed into

$$\begin{cases} \tilde{y}'' + \delta_1 \tilde{y}' + a_{11}(t)\tilde{y} + a_{12}(t)\tilde{z} = a_{11}(t)\xi + a_{12}(t)\xi + h_1(\tilde{y} - \xi, \tilde{z} - \xi), \\ \tilde{z}'' + \delta_2 \tilde{z}' + a_{21}(t)\tilde{y} + a_{22}(t)\tilde{z} = a_{21}(t)\xi + a_{22}(t)\xi + h_2(\tilde{y} - \xi, \tilde{z} - \xi), \end{cases} \tag{3}$$

where ξ is chosen such that

$$a_{11}(t)\xi + a_{12}(t)\xi + \lambda h_1(\tilde{y} - \xi, \tilde{z} - \xi) > 1,$$
$$a_{21}(t)\xi + a_{22}(t)\xi + \lambda h_2(\tilde{y} - \xi, \tilde{z} - \xi) > 1.$$

Let $B: K \to E$ be a map, which defined by $B(\tilde{y},\tilde{z})(t) = (B_1(\tilde{y},\tilde{z})(t), B_2(\tilde{y},\tilde{z})(t))$, where

$$B_1(\tilde{y},\tilde{z})(t) = \int_0^T G_1(t,s)[-a_{12}(s)z(s) + F_1(\tilde{y}(s),\tilde{z}(s))]ds,$$

$$B_2(\tilde{y},\tilde{z})(t) = \int_0^T G_2(t,s)[-a_{21}(s)y(s) + F_2(\tilde{y}(s),\tilde{z}(s))]ds,$$

$$F_1(\tilde{y}(s),\tilde{z}(s)) = a_{11}(t)\xi + a_{12}(t)\xi + \lambda h_1(\tilde{y} - \xi, \tilde{z} - \xi),$$
$$F_2(\tilde{y}(s),\tilde{z}(s)) = a_{21}(t)\xi + a_{22}(t)\xi + \lambda h_2(\tilde{y} - \xi, \tilde{z} - \xi).$$

t is straightforward to verify that the solution of (1) is equivalent to the fixed point Equation $B(\tilde{y},\tilde{z})(t) = (\tilde{y}(t),\tilde{z}(t))$.

Lemma 2.1 Assume that **(H)**, **(H1)** and **(H2)** hold. Then $B: K \to K$ is compact and continuous.

For convenience, define $h_{i,\infty} = \lim_{y+z\to+\infty} \frac{h_i(y,z)}{y+z}$, for any $y,z>0$.

Lemma 2.2 [2] Assume that **(H)**, **(H1)** and **(H2)** hold. If $h_{i,\infty}=0$, then, for i = 1, 2, the functions F_i are continuous on $R^+\times R^+$, $F_i(\tilde y(s),\tilde z(s))>1$ for $(\tilde y(s),\tilde z(s))\in R^+\times R^+$, and $\lim_{\tilde y+\tilde z\to+\infty}\frac{F_i(\tilde y,\tilde z)}{\tilde y+\tilde z}=0.$

Lemma 2.3 [2] Assume that **(H)**, **(H1)** and **(H2)** hold. If $h_{i,\infty}=+\infty$, then, for i = 1, 2, the functions F_i are continuous on $R^+\times R^+$, $F_i(\tilde y(s),\tilde z(s))>1$ for $(\tilde y(s),\tilde z(s))\in R^+\times R^+$, and $\lim_{\tilde y+\tilde z\to+\infty}\frac{F_i(\tilde y,\tilde z)}{\tilde y+\tilde z}=+\infty.$

3. Main Results

Theorem 3.1 Assume that **(H)**, **(H1)** and **(H2)** hold.

(I) Then there exists a $\lambda^*>0$ such that (1) has a positive periodic solution for $0<\lambda<\lambda^*$;

(II) If $h_{i,\infty}=0$, then for an $\lambda>0$, (1) has a positive periodic solution;

(III) If $h_{i,\infty}=+\infty$, then (1) has two positive periodic solutions for all sufficiently small λ.

Proof. (I) On one hand, take R > 0 such that

$$\xi\cdot\max\{M_1,M_2\}\cdot\max\{\|a_{11}\|_{L^1}+\|a_{12}\|_{L^1},\|a_{21}\|_{L^1}+\|a_{22}\|_{L^1}\}<\frac{R}{8}.$$

Set $\Psi_i(R)=\max\{h_i(\tilde y-\xi,\tilde z-\xi):\theta R\le\|(\tilde y,\tilde z)\|\le R\}$. Then, for each $(\tilde y,\tilde z)\in\partial K_R$, we have

$$\begin{aligned}\max_{t\in[0,T]}B_1(\tilde y,\tilde z)(t)&=\max_{t\in[0,T]}\int_0^T G_1(t,s)[-a_{12}(s)z(s)+F_1(\tilde y(s),\tilde z(s))]ds,\\&\le M_1\|a_{12}\|_{L^1}\cdot\|(\tilde y,\tilde z)\|+\lambda M_1\Psi_1(R)+M_1\|a_{11}\|_{L^1}\xi+M_1\|a_{12}\|_{L^1}\xi\\&\le\frac{R}{4}+\frac{R}{8}+\lambda M_1\Psi_1(R).\end{aligned}$$

Then from the above inequalities, it follows that there exists a $\lambda_1^*>0$ such that

$$\lambda M_1\Psi_1(R)\le\frac{R}{8},\ for\ \ 0<\lambda<\lambda_1^*.$$

Furthermore, for any $(\tilde y,\tilde z)\in\partial K_R$, we obtain $\max_{t\in[0,T]}B_1(\tilde y,\tilde z)(t)\le\frac{R}{2}=\frac{\|(\tilde y,\tilde z)\|}{2}$.

In the similar way, there exists a $\lambda_2^*>0$, such that $\lambda M_2\Psi_2(R)\le\frac{R}{8},\ for\ \ 0<\lambda<\lambda_2^*.$ and we also have

$$\max_{t\in[0,T]}B_2(\tilde y,\tilde z)(t)\le\frac{R}{2}=\frac{\|(\tilde y,\tilde z)\|}{2},\ for\ (\tilde y,\tilde z)\in\partial K_R.$$

So let us choose $\lambda=\min\{\lambda_1^*,\lambda_2^*\}$, and we can obtain

$$\|B(\tilde y,\tilde z)\|<\frac{R}{2}+\frac{R}{2}=\|(\tilde y,\tilde z)\|,\ for\ any\ (\tilde y,\tilde z)\in\partial K_R,0<\lambda<\lambda^*.$$

On the other hand, from the condition $F_i(\tilde y(s),\tilde z(s))>1$ for all $(\tilde y(s),\tilde z(s))\in R^+\times R^+$, it follows that there is a sufficient small r > 0 such that $F_i(\tilde y,\tilde z)\ge\eta(\tilde y+\tilde z)$ for $(\tilde y(s),\tilde z(s))\in R^+\times R^+$, and $\tilde y(s)+\tilde z(s)\le r$, where η is chosen such that $\eta\cdot\min\{m_1,m_2\}>\frac{1}{2}$.

Then, for any $(\tilde y,\tilde z)\in\partial K_r$, we obtain

$$B_1(\tilde y,\tilde z)(t)=\int_0^T G_1(t,s)[-a_{12}(s)z(s)+F_1(\tilde y(s),\tilde z(s))]ds\ge\theta\eta m_1\|(\tilde y,\tilde z)\|>\frac{\|(\tilde y,\tilde z)\|}{2}$$

$$B_2(\tilde{y},\tilde{z})(t)=\int_0^T G_2(t,s)[-a_{21}(s)y(s)+F_2(\tilde{y}(s),\tilde{z}(s))]ds \ge \theta\eta m_2\|(\tilde{y},\tilde{z})\| > \frac{\|(\tilde{y},\tilde{z})\|}{2}.$$

So we have $\|B(\tilde{y},\tilde{z})\| > \|(\tilde{y},\tilde{z})\|$, *for any* $(\tilde{y},\tilde{z}) \in \partial K_r$.

Therefore, from **Lemma 1.1**, it follows that the operator B has at least one fixed point $(\tilde{y},\tilde{z})$ in $\bar{K}_R \setminus K_r$, for $0<\lambda<\lambda^*$.

(II) Since $h_{i,\infty}=0$, then from **Lemma 2.1**, it follows that $\lim\limits_{\tilde{y}+\tilde{z}\to+\infty}\frac{F_i(\tilde{y},\tilde{z})}{\tilde{y}+\tilde{z}}=0.$ Define a function $\tilde{F}_{i\lambda}:R^+\to R^+$ as $\tilde{F}_{i\lambda}(s)=\max\limits_{\tilde{y}+\tilde{z}<s}F_i(\tilde{y},\tilde{z})$. By **Lemma 2.5** in [2], it is easy to see that $\lim\limits_{s\to+\infty}\frac{\tilde{F}_{i\lambda}(s)}{s}=0.$ Thus by the definition, there is an $\bar{R}>2r$ such that $\tilde{F}_{i\lambda}(\bar{R})\le\varepsilon\bar{R}$, where ε satisfying $\varepsilon T\max\{M_1,M_2\}\le\frac{1}{4}$.

Then, for each $(\tilde{y},\tilde{z})\in\partial K_{\bar{R}}$, we have

$$\max_{t\in[0,T]}B_1(\tilde{y},\tilde{z})(t)=\max_{t\in[0,T]}\int_0^T G_1(t,s)[-a_{12}(s)z(s)+F_1(\tilde{y}(s),\tilde{z}(s))]ds,$$

$$\le M_1\|a_{12}\|_{L^1}\cdot\|(\tilde{y},\tilde{z})\|+\varepsilon M_1T\|(\tilde{y},\tilde{z})\| < \frac{\|(\tilde{y},\tilde{z})\|}{2}.$$

In the similar way, for any $(\tilde{y},\tilde{z})\in\partial K_R$, we also have $\max\limits_{t\in[0,T]}B_2(\tilde{y},\tilde{z})(t)<\frac{\|(\tilde{y},\tilde{z})\|}{2}$. Furthermore, from The above inequalities, we get $\|B(\tilde{y},\tilde{z})\|<\|(\tilde{y},\tilde{z})\|$, *for any* $(\tilde{y},\tilde{z})\in\partial K_{\bar{R}}$.

Therefore, from **Lemma 1.1**, it follows that B has one fixed point $(\tilde{y},\tilde{z})$ in $\bar{K}_{\bar{R}}\setminus K_r$ for any $\lambda>0$.

(III) Since $h_{i,\infty}=+\infty$, then from **Lemma 2.2**, it follows that $\lim\limits_{\tilde{y}+\tilde{z}\to+\infty}\frac{F_i(\tilde{y},\tilde{z})}{\tilde{y}+\tilde{z}}=+\infty.$ By the definition, there exists $R'>0$, such that $F_i(\tilde{y},\tilde{z})\ge\vartheta(\tilde{y}+\tilde{z})$, where ϑ is chosen such that $\vartheta\theta T\min\{m_1,m_2\}>\frac{1}{2}$.

Choosing $\hat{R}=\max\{R+1,\frac{R'}{\theta}\}$, and for any $(\tilde{y},\tilde{z})\in\partial K_{\hat{R}}$, we have $\tilde{y}+\tilde{z}\ge\theta\|(\tilde{y},\tilde{z})\|>R'$ and

$$B_1(\tilde{y},\tilde{z})(t)=\int_0^T G_1(t,s)[-a_{12}(s)z(s)+F_1(\tilde{y}(s),\tilde{z}(s))]ds\ge\theta\vartheta m_1T\|(\tilde{y},\tilde{z})\|>\frac{\|(\tilde{y},\tilde{z})\|}{2},$$

$$B_2(\tilde{y},\tilde{z})(t)=\int_0^T G_2(t,s)[-a_{21}(s)z(s)+F_2(\tilde{y}(s),\tilde{z}(s))]ds\ge\theta\vartheta m_2T\|(\tilde{y},\tilde{z})\|>\frac{\|(\tilde{y},\tilde{z})\|}{2}.$$

Thus from the above inequalities, we can get $\|B(\tilde{y},\tilde{z})\|>\|(\tilde{y},\tilde{z})\|$, *for any* $(\tilde{y},\tilde{z})\in\partial K_{\hat{R}}$.

Therefore, from **Lemma 1.1**, it follows that the operator B has at least two fixed points $(\tilde{y}_1,\tilde{z}_1)$ in $\bar{K}_R\setminus K_r$ and $(\tilde{y}_2,\tilde{z}_2)$ in $\bar{K}_{\hat{R}}\setminus K_R$. Namely, system (1) has two solutions for sufficiently small $\lambda>0$.

References

[1] Lazer, A.C. and McKenna, P.J. (1990) Large-Amplitude Periodic Oscillations in Suspension Bridges: Some New Connections with Nonlinear Analysis. *Siam Review*, **32**, 537-578. http://dx.doi.org/10.1137/1032120

[2] Wang, H. (2009) Periodic Solutions to Non-Autonomous Second-Order Systems. *Nonlinear Analysis*: *Theory*, *Methods & Applications*, **71**, 1271-1275. http://dx.doi.org/10.1016/j.na.2008.11.079

[3] Dajun, G. and Lakshmikantham, V. (1988) Nonlinear Problems in Abstract Cones.

[4] Chu, J., Fan, N. and Torres, P.J. (2012) Periodic Solutions for Second Order Singular Damped Differential Equations. *Journal of Mathematical Analysis and Applications*, **388**, 665-675. http://dx.doi.org/10.1016/j.jmaa.2011.09.061