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Doubly Periodic Riemann Boundary Value Problem of Non-Normal Type for Analytic Functions on Two Parallel Curves

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Abstract

In this paper, we present and study a kind of Riemann boundary value problem of non-normal type for analytic functions on two parallel curves. Making use of the method of complex functions, we give the method for solving this kind of doubly periodic Riemann boundary value problem of non-normal type and obtain the explicit expressions of solutions and the solvable conditions for it.

Keywords

Doubly Periodic, Holder Continuous Functions, Riemann Boundary Problem, Non-Normal Type

1. Introduction

Classical Riemann boundary value problems (RBVPs), doubly periodic or quasi-periodic RBVPs and Dirichlet Problems for analytic functions or for polyanalytic functions, on closed curves or on open arcs, have been widely investigated in papers [1]-[8]. The main approach is to use the decomposition of polyanalytic functions and their generalization to transform the given boundary value problems to their corresponding boundary value problems for analytic functions, and the fundamental and important tool for which is the Plemelj formula. Professor L. Xing proposed the Periodic Riemann Boundary Value Inverse Problems in paper [9], and then various inverse RBVPs for generalized analytic functions or bianalytic functions have been investigated in papers [10]-[13].

In present paper, we present a kind of doubly periodic RBVP of non-normal type for analytic functions on two parallel curves. On the basis of the results for normal type in paper [14], we give the method for solving this kind of doubly periodic RBVP of non-normal type and obtain the explicit expressions of solutions and the solvable conditions for it.

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2. Doubly Periodic RBVP of Non-Normal Type on Two Parallel Curves

Suppose that ω_1, ω_2 are complex constants with $\operatorname{Im}(\omega_1/\omega_2) \neq 0$, and **P** denotes the fundamental period parallelogram with vertices $\pm\omega_1 \pm \omega_2$. The function

$$\zeta(z) = 1/z + \sum_{m,n}{}' \left[1/(z - \Omega_{mn}) + 1/\Omega_{mn} + z/\Omega_{mn}^2 \right]$$

is called the Weierstrass ζ-function, where $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$, and $\sum_{m,n}{}'$ denotes the sum for all $m,\ n = 0, \pm1, \pm2, \cdots$, except for $m = n = 0$.

Let $L_0 = \sum_{j=1}^{2} L_{0j}$ be the set of two parallel curves, lying entirely in the fundamental period parallelogram **P**, not passing the origin O, with endpoints being periodic congruent and having the same tangent lines at the periodic congruent points. Let D_1, D_2, D_3 denote the domains entirely in the fundamental period parallelogram **P**, cut by L_{01} and L_{02}, respectively. Without loss of generality, we suppose that $O \in D_2$ see **Figure 1**. Let L_{01}^*, L_{02}^* be the curves periodically extended for L_{01} and L_{02} with period $2\omega_1$, respectively. And L_{nj}^* $(j = 1, 2; n = 0, \pm1, \cdots)$ be the curves periodically extended for L_{0j}^* with $2n\omega_1$.

We aim to is to find sectionally holomorphic, doubly periodic functions $F(z)$ and $\Omega(z)$, satisfying the following boundary conditions

$$\begin{cases} F^+(\tau) = \dfrac{\Pi_{11}(\tau)}{\Pi_{12}(\tau)} D_1(\tau)\Omega^-(\tau) + g_1(\tau), & \tau \in L_{01}, \\ \Omega^+(\tau) = \dfrac{\Pi_{21}(\tau)}{\Pi_{22}(\tau)} D_2(\tau) F^-(\tau) + g_2(\tau), & \tau \in L_{02}, \end{cases} \tag{1}$$

where $D_j(\tau)$, $g_j(\tau) \in H$ with $D_j(\tau) \neq 0\,(j = 1, 2)$, and $D_j(\tau)$, $g_j(\tau)$ are doubly periodic with $2\omega_1$, $2\omega_2$. $F^{\pm}(\tau)$ are the boundary values of the function $F(z)$, which is analytic in D_1 and D_3, belonging to the class $h(a_j)$ on L_{0j}, satisfying the boundary conditions (1), and $\Omega^{\pm}(\tau)$ are the boundary values of the function $\Omega(z)$, which is analytic in D_2, belonging to the class $h(a_j)$ on L_{0j}, satisfying the boundary conditions (1). While

$$\Pi_{11}(\tau) = \prod_{s=1}^{h} \mu(\tau - c_s)^{\lambda_s}, \quad \Pi_{12}(\tau) = \prod_{l=1}^{p} \mu(\tau - d_l)^{u_l},$$

$$\Pi_{21}(\tau) = \prod_{r=1}^{q} \mu(\tau - e_r)^{\delta_r}, \quad \Pi_{22}(\tau) = \prod_{v=1}^{w} \mu(\tau - t_v)^{n_v}.$$

$$\lambda = \sum_{s=1}^{h} \lambda_s, \quad u = \sum_{l=1}^{p} u_l, \quad \delta = \sum_{r=1}^{q} \delta_r, \quad n = \sum_{v=1}^{w} n_v,$$

where $\mu(\tau) = \dfrac{\sigma(\tau)\sigma(\tau - \omega_1 - \omega_2)}{\sigma(\tau - \omega_1)\sigma(\tau - \omega_2)}$ is doubly periodic, where

$$\sigma(\tau) = z \prod_{k,t}{}' \left(1 - \frac{z}{\Omega_{kt}}\right) \exp\left\{ \frac{z}{\Omega_{kt}} + \frac{z^2}{2\Omega_{kt}^2} \right\}$$

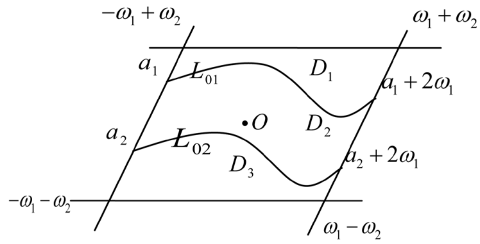

Figure 1. parallel curves in the fundamental period parallelogram **P**.

With k, t and λ_s, u_l, δ_r, n_v being integers. Without loss of generality, we suppose that c_s, $d_l \in L_{01}$ with c_s, $d_l \neq a_1$, $a_1 + 2\omega_1$ $(s = 1, \cdots, h; l = 1, \cdots, p)$ as well as e_r, $f_v \in L_{02}$ and

$$e_r,\ f_v \neq a_1,\ a_2 + 2\omega_1 \quad (r = 1, \cdots, q; v = 1, \cdots, w).$$

Since a_j plays the same roles as other points on L_{0j} $(j = 1, 2)$, it is natural to require that the unknown functions are bounded at $z = a_j$, that is, the unknown functions $F(z)$ and $\Omega(z)$ are both bounded on L_{01}^* and L_{02}^*. And if we allow the solution $\Omega(z)$ has poles of order m at $z = 0$, it is actually to solve problem (1) in DR_m.

3. Preliminary Notes

Since $D_j(\tau) \in H$ with $D_j(\tau) \neq 0 (j = 1, 2)$, by taking logarithm of $\log D_j(\tau)$ for some branch on L_{0j}, we may obtain a continuous single-valued function such as

$$-\frac{1}{2\pi i} \log D_j(a_j) = \alpha_{a_j} + i\beta_{a_j}, \quad j = 1, 2,$$
$$\frac{1}{2\pi i} \log D_j(a_j + 2\omega_j) = -\alpha_{a_j} - i\beta_{a_j}, \quad j = 1, 2.$$

with $0 \le \alpha_{a_j} < 1$. Now we call the integer $\kappa = \kappa_1 + \kappa_2$ the index of problem (1), where κ_j is the integer satisfying

$$0 \le -\alpha_{a_j} - \kappa_j < 1, \quad j = 1, 2.$$

Since κ_j can only be 0 and -1, the index κ can only take $0, -1, -2$.

Set

$$D_* = D_{1*} + D_{2*} = \frac{1}{2\pi i} \int_{L_{01}} \log D_1(\tau) + \frac{1}{2\pi i} \int_{L_{02}} \log D_2(\tau) \mathrm{d}\tau \tag{2}$$

$$\gamma_j(z) = \frac{1}{2\pi i} \int_{L_{0j}} \log D_j(\tau) \xi(\tau - z) \mathrm{d}\tau, \quad z \notin L_{0j}, \quad j = 1, 2 \tag{3}$$

We can easily see that $1/\mathrm{e}^{\gamma_j(z)}$ will have singularities at most less than one order near the endpoints a_j and $a_j + 2\omega_1$ $(j = 1, 2)$. Let

$$\mathrm{e}^{\gamma(z)} = \mathrm{e}^{\gamma_1(z)} \mathrm{e}^{\gamma_2(z)} \tag{4}$$

then we have

$$\mathrm{e}^{\gamma(z + 2\omega_j)} = \mathrm{e}^{-2\eta_j D_*} \mathrm{e}^{\gamma(z)}, \quad j = 1, 2,$$

where $\eta_j = \zeta(\omega_j)$ $(j = 1, 2)$ and $2\omega_2\eta_1 - 2\omega_1\eta_2 = \pi i$. Thus $\mathrm{e}^{\gamma(z)}$ is not doubly periodic generally. In fact, $\mathrm{e}^{\gamma(z)}$ is doubly periodic if and only if

$$\eta_j D_* = k_j \pi i, \quad k_j \text{ is positive integer for } j = 1, 2. \tag{5}$$

Lemma 1. Formula (5) is valid if and only if

$$\eta_1/\eta_2 = k_1/k_2, \quad D_* = 2k_1\omega_2 - 2k_2\omega_1.$$

And if both $D_* = 2l_1\omega_1 + 2l_2\omega_2$ and $\eta_j D_* = k_j \pi i$ are true, then we have $l_1 = -k_2$ and $l_2 = k_1$, where l_j, k_j are all integers.

4. Solution for Problem (1)

Problem (1) can be transferred as

$$\begin{cases} \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}} F^+(\tau) = \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^-(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}} \Omega^-(\tau) + \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}} g_1(\tau), & \tau \in L_{01}, \\ \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}} \Omega^+(\tau) = \dfrac{\Pi_{21}(\tau)\Pi_{11}(\tau)}{\mathrm{e}^{\gamma_2^-(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}} F^-(\tau) + \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^-(\tau)}\mathrm{e}^{\gamma_2^+(\tau)}} g_2(\tau), & \tau \in L_{02}. \end{cases} \tag{6}$$

Case 1. If formula (5) holds, that is, $e^{\gamma(z)}$ is doubly periodic, then by **Lemma 1** we have

$$D_* \equiv 0 \pmod{2\omega_1, 2\omega_2}. \tag{7}$$

The function $1/e^{\gamma_j(z)}$ always has singularities less than one order near the endpoints a_j and $a_j + 2\omega_1$ $(j = 1, 2)$ whatever $\kappa = 0, -1, -2$. And then both

$$\frac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}} g_1(\tau) \quad \text{and} \quad \frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^-(\tau)}e^{\gamma_2^+(\tau)}} g_2(\tau)$$

must belong to class H or class H^* on L_{01} and L_{02}, respectively.

Set

$$\Psi_1(z) = \frac{1}{2\pi i}\int_{L_{01}} \frac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}} g_1(\tau)\left[\zeta(\tau - z) + \zeta(z)\right]\mathrm{d}\tau, \quad z \notin L_{01} \tag{8}$$

$$\Psi_2(z) = \frac{1}{2\pi i}\int_{L_{02}} \frac{\Pi_{22}(\tau)\Pi_{11}(\tau)}{e^{\gamma_1^-(\tau)}e^{\gamma_2^+(\tau)}} g_2(\tau)\left[\zeta(\tau - z) + \zeta(z)\right]\mathrm{d}\tau, \quad z \notin L_{02}, \tag{9}$$

then (6) can be rewritten as

$$\begin{cases} \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}} F^+(\tau) - \Psi_1^+ - \Psi_2^+ = \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^-(\tau)}e^{\gamma_2^-(\tau)}} \Omega^-(\tau) - \Psi_1^- - \Psi_2^+, & \tau \in L_{01}, \\ \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}} \Omega^+(\tau) - \Psi_1^- - \Psi_2^+ = \dfrac{\Pi_{11}(\tau)\Pi_{21}(\tau)}{e^{\gamma_2^-(\tau)}e^{\gamma_1^-(\tau)}} F^-(\tau) - \Psi_1^- - \Psi_2^-, & \tau \in L_{02}, \end{cases} \tag{10}$$

where Ψ_1^+ and Ψ_2^+ (or Ψ_1^- and Ψ_2^-) denote the boundary values of the functions $\Psi_1(z)$ and $\Psi_2(z)$. By the definitions of $\Psi_1(z)$ and $\Psi_2(z)$, we see that

(i) $\Psi_1^+(z)$ has no zeros in domain D_1;

(ii) The part of $\Psi_1^-(z)$ which has zeros in domain $D_1 \cup D_3$ is $\Pi_{22}(z)$;

(iii) The part of $\Psi_2^+(z)$ which has zeros in domain $D_1 \cup D_2$ is $\Pi_{11}(z)$;

(iv) $\Phi_2^-(z)$ has no zeros in domain D_3.

Write

$$\Phi_1(z) = \begin{cases} \dfrac{\Pi_{12}(z)\Pi_{22}(z)}{e^{\gamma(z)}} F^+(z) - \Psi_1^+(z) - \Psi_2^+(z), & z \in D_1, \\ \dfrac{\Pi_{11}(z)\Pi_{22}(z)}{e^{\gamma(z)}} \Omega(z) - \Psi_1^-(z) - \Psi_2^+(z), & z \in D_2, \\ \dfrac{\Pi_{11}(z)\Pi_{21}(z)}{e^{\gamma(z)}} F^-(z) - \Psi_1^-(z) - \Psi_2^-(z), & z \in D_3. \end{cases}$$

When we solve problem (1) in DR_m, the unknown function $\Phi_1(z)$ is n-order at $z = 0$. And now we will meet three kinds of situations in solving problem (1) in DR_m, according to the value of $m - \lambda - n$.

1° When $m - \lambda - n > 0$, problem (1) is solvable without any restrictive conditions and the general solution is given by

$$\begin{cases} F^+(z) = \left[c_0 + c_1\zeta'(z) + \cdots + c_{m-\lambda-n-1}\zeta^{(m-\lambda-n-1)}(z) + \Psi_1^+(z) + \Psi_2^+(z)\right]\dfrac{e^{\gamma(z)}}{\Pi_{12}(z)\Pi_{22}(z)}, & z \in D_1, \\ \Omega(z) = \left[c_0 + c_1\zeta'(z) + \cdots + c_{m-\lambda-n-1}\zeta^{(m-\lambda-n-1)}(z) + \Psi_1^-(z) + \Psi_2^+(z)\right]\dfrac{e^{\gamma(z)}}{\Pi_{11}(z)\Pi_{22}(z)}, & z \in D_2, \\ F^-(z) = \left[c_0 + c_1\zeta'(z) + \cdots + c_{m-\lambda-n-1}\zeta^{(m-\lambda-n-1)}(z) + \Psi_1^-(z) + \Psi_2^-(z)\right]\dfrac{e^{\gamma(z)}}{\Pi_{11}(z)\Pi_{21}(z)}, & z \in D_3, \end{cases} \tag{11}$$

where $c_0, c_1, \cdots, c_{m-\lambda-n-1}$ are arbitrary constants.

2° When $m-\lambda-n=0$, problem (1) is solvable if and only if the restrictive conditions

$$\begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{L_{01}} \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}} g_1(\tau)\,\mathrm{d}\tau = 0, \\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}} \dfrac{\Pi_{22}(\tau)\Pi_{11}(\tau)}{\mathrm{e}^{\gamma_1^-(\tau)}\mathrm{e}^{\gamma_2^+(\tau)}} g_2(\tau)\,\mathrm{d}\tau = 0, \end{cases} \tag{12}$$

are satisfied, and now the solution is given by

$$\begin{cases} F^+(z) = \dfrac{\mathrm{e}^{\gamma(z)}}{\Pi_{12}(\tau)\Pi_{22}(\tau)}\left[c+\Psi_1^+(z)+\Psi_2^+(z)\right], & z \in D_1, \\ \Omega(z) = \dfrac{\mathrm{e}^{\gamma(z)}}{\Pi_{11}(\tau)\Pi_{22}(\tau)}\left[c+\Psi_1^-(z)+\Psi_2^+(z)\right], & z \in D_2, \\ F^-(z) = \dfrac{\mathrm{e}^{\gamma(z)}}{\Pi_{11}(\tau)\Pi_{21}(\tau)}\left[c+\Psi_1^-(z)+\Psi_2^-(z)\right], & z \in D_3, \end{cases} \tag{13}$$

where c is arbitrary constant.

3° When $m-\lambda-n<0$, $z=0$ is the zero point of order $m-\lambda-n$ of the function $\Phi_1(z)$, and due to this the solution for problem (1) has $-m$ order at the point $z=0$. Now the solution for problem (1) can still be given by (13), but the following two restrictive conditions are necessary:

$$c = -\frac{1}{2\pi i}\int_{L_{01}} \frac{g_1(\tau)\Pi_{12}(\tau)\Pi_{22}}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}}\zeta(\tau)\,\mathrm{d}\tau - \frac{1}{2\pi i}\int_{L_{02}} \frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}} g_2(\tau)\zeta(\tau)\,\mathrm{d}\tau = 0, \tag{14}$$

$$\begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{L_{01}} \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}} g_1(\tau)\zeta^{(k)}(\tau)\,\mathrm{d}\tau = 0, \\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}} \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}} g_2(\tau)\zeta^{(k)}(\tau)\,\mathrm{d}\tau = 0, \end{cases} \quad k = 1,2,\cdots,-(m-\lambda-n)-1. \tag{15}$$

(when $m-\lambda-n=-1$, the condition (15) is unnecessary).

Case 2. If formula (5) fails to hold, then by Lemma 1 we see that $D_* \neq 0$. Let

$$h_*(z) = \sigma(z)/\sigma(z-D_*),$$

then the function $\mathrm{e}^{\gamma(z)}h_*(z)$ become doubly periodic, and function $1/\left[\mathrm{e}^{\gamma(z)}h_*(z)\right]$ has singularities at most less than one order near the endpoints a_j and $a_j+2\omega_1$ $(j=1,2)$. Thus now, we can transform (6) to

$$\begin{cases} \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}h_*(\tau)}F^+(\tau) = \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^-(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}h_*(\tau)}\Omega^-(\tau) + \dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}h_*(\tau)}g_1(\tau), & \tau \in L_{01}, \\ \dfrac{\Pi_{22}(\tau)\Pi_{11}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}h_*(\tau)}\Omega^+(\tau) = \dfrac{\Pi_{21}(\tau)\Pi_{11}(\tau)}{\mathrm{e}^{\gamma_2^-(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}h_*(\tau)}F^-(\tau) + \dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}h_*(\tau)}g_2(\tau), & \tau \in L_{02}. \end{cases} \tag{16}$$

When $\kappa = 0,-1,-2$, the two functions $\dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_1^+(\tau)}\mathrm{e}^{\gamma_2^-(\tau)}h_*(\tau)}g_1(\tau)$, $\dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}h_*(\tau)}g_2(\tau)$ belong to class H or class H^* on L_{01} and L_{02}, respectively. Write

$$\Psi_1(z) = \frac{1}{2\pi i}\int_{L_{02}} \frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}h_*(z)} g_2(\tau)\left[\zeta(\tau-z)+\zeta(z)\right]\mathrm{d}\tau, \quad z \notin L_{01}, \tag{17}$$

$$\Psi_2(z) = \frac{1}{2\pi i}\int_{L_{02}} \frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{\mathrm{e}^{\gamma_2^+(\tau)}\mathrm{e}^{\gamma_1^-(\tau)}h_*(z)} g_2(\tau)\left[\zeta(\tau-z)+\zeta(z)\right]\mathrm{d}\tau, \quad z \notin L_{02}. \tag{18}$$

By (17) and (18), we can rewrite (16) as

$$\begin{cases}\dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}F^+(\tau)-\Psi_1^+(\tau)-\Psi_2^+(\tau)=\dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^-(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}\Omega^-(\tau)-\Psi_1^-(\tau)-\Psi_2^+(\tau), & \tau\in L_{01},\\ \dfrac{\Pi_{22}(\tau)\Pi_{11}(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}\Omega^+(\tau)-\Psi_1^-(\tau)-\Psi_2^+(\tau)=\dfrac{\Pi_{21}(\tau)\Pi_{11}(\tau)}{e^{\gamma_2^-(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}F^-(\tau)-\Psi_1^-(\tau)-\Psi_2^-(\tau), & \tau\in L_{02}.\end{cases} \tag{19}$$

Now we will meet two kinds of situations in solving problem (1) in DR_m.

(a) When $D_*\equiv 0\bmod(2\omega_1,2\omega_2)$, the function $h_*(z)$ is an entire function. And we can write it without counting nonzero constant as

$$h_*(z)=\exp\{2(l_1\eta_1+l_2\eta_2)\},$$

where l_1, l_2 are determined by the identity $D_*=2l_1\omega_1+2l_2\omega_2$.

1° When $m-\lambda-n>0$, problem (1) is solvable without any restrictive conditions and the general solution is given by

$$\begin{cases}F^+(z)=\left[c_0+c_1\zeta'(z)+\cdots+c_{m-\lambda-n-1}\zeta^{(m-\lambda-n-1)}(z)+\Psi_1^+(z)+\Psi_2^+(z)\right]\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{12}(z)\Pi_{22}(z)}, & z\in D_1,\\ \Omega(z)=\left[c_0+c_1\zeta'(z)+\cdots+c_{m-\lambda-n-1}\zeta^{(m-\lambda-n-1)}(z)+\Psi_1^-(z)+\Psi_2^+(z)\right]\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{22}(z)}, & z\in D_2,\\ F^-(z)=\left[c_0+c_1\zeta'(z)+\cdots+c_{m-\lambda-n-1}\zeta^{(m-\lambda-n-1)}(z)+\Psi_1^-(z)+\Psi_2^-(z)\right]\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{21}(z)}, & z\in D_3,\end{cases} \tag{20}$$

where $c_0,c_1,\cdots,c_{m-\lambda-n-1}$ are arbitrary constants.

2° When $m-\lambda-n=0$, problem (1) is solvable if and only if the restrictive conditions

$$\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_{01}}\dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}g_1(\tau)\,d\tau=0,\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}}\dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}g_2(\tau)\,d\tau=0,\end{cases} \tag{21}$$

are satisfied, and the general solution is given by

$$\begin{cases}F^+(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{12}(z)\Pi_{22}(z)}\left[c+\Psi_1^+(z)+\Psi_2^+(z)\right], & z\in D_1,\\ \Omega(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{22}(z)}\left[c+\Psi_1^-(z)+\Psi_2^+(z)\right], & z\in D_2,\\ F^-(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{21}(z)}\left[c+\Psi_1^-(z)+\Psi_2^-(z)\right], & z\in D_3,\end{cases} \tag{22}$$

where c is arbitrary constant.

3° When $m-\lambda-n<0$, problem (1) is solvable if and only if the restrictive conditions

$$\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_{02}}\dfrac{\Pi_{12}(\tau)\Pi_{22}(\tau)g_1(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}\zeta^{(k)}(\tau)\,d\tau=0,\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}}\dfrac{\Pi_{11}(\tau)\Pi_{22}(\tau)g_2(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}\zeta^{(k)}(\tau)\,d\tau=0,\end{cases}\quad k=1,2,\cdots,-(m-\lambda-n)-1. \tag{23}$$

are satisfied, and the general solution can still be given by (22) but with

$$c=-\frac{1}{2\pi i}\int_{L_{01}}\frac{\Pi_{12}(\tau)\Pi_{22}(\tau)g_1(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}\xi(\tau)d\tau-\frac{1}{2\pi i}\int_{L_{02}}\frac{\Pi_{11}(\tau)\Pi_{22}(\tau)g_2(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}\xi(\tau)d\tau \tag{24}$$

(b) When $D_*\equiv 0(\bmod 2\omega_1,2\omega_2)$ fails to hold, the function $\frac{1}{e^{\gamma(z)}h_*(z)}$ has singularity of one order at $z = 0$, and has singularities at most less than one order near the endpoints a_j and $a_j+2\omega_1\,(j=1,2)$, has a zero of order one at $z=D_*$.

1° When $m-\lambda-n+1\ge 0$, problem (1) is solvable and the general solution is given by

$$\begin{cases}F^+(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{12}(z)\Pi_{22}(z)}\left[c_0+c_1\zeta'(z)+\cdots+c_{m-\lambda-n}\zeta^{(m-\lambda-n)}(z)+\Psi_1^+(z)+\Psi_2^+(z)\right], & z\in D_1,\\ \Omega(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{22}(z)}\left[c_0+c_1\zeta'(z)+\cdots+c_{m-\lambda-n}\zeta^{(m-\lambda-n)}(z)+\Psi_1^-(z)+\Psi_2^+(z)\right], & z\in D_2,\\ F_1^-(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{21}(z)}\left[c_0+c_1\zeta'(z)+\cdots+c_{m-\lambda-n}\zeta^{(m-\lambda-n)}(z)+\Psi_1^-(z)+\Psi_2^-(z)\right], & z\in D_3,\end{cases} \tag{25}$$

with the restrictive condition that

$$c_0=-c_1\zeta'(D_*)-\cdots-c_{m-\lambda-n}\zeta^{(m-\lambda-n)}(D_*)-\Psi_1(D_*)-\Psi_2(D_*)=0,$$

which is to ensure that the solution be finite at $z=D_*$, where $c_1,c_2,\cdots,c_{m-\lambda-n}$ are arbitrary constants.

2° When $m-\lambda-n+1=-1$, problem (2.1) is solvable if and only if the restrictive conditions

$$\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_{01}}\frac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}g_1(\tau)d\tau=0,\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}}\frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}g_2(\tau)d\tau=0,\end{cases} \tag{26}$$

are satisfied, and now the solution is given by

$$\begin{cases}F^+(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{12}(z)\Pi_{22}(z)}\left[\Psi_1^+(z)+\Psi_2^+(z)-\Psi_1(D_*)-\Psi_2(D_*)\right], & z\in D_1,\\ \Omega(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{22}(z)}\left[\Psi_1^-(z)+\Psi_2^+(z)-\Psi_1(D_*)-\Psi_2(D_*)\right], & z\in D_2,\\ F^-(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{21}(z)}\left[\Psi_1^-(z)+\Psi_2^-(z)-\Psi_1(D_*)-\Psi_2(D_*)\right], & z\in D_3,\end{cases} \tag{27}$$

which is finite at $z=D_*$ owing to its structure.

3° When $m-\lambda-n+1<-1$, if and only if both conditions (26) and the following conditions

$$\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_{01}}\frac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}g_1(\tau)\left[\zeta(\tau)-\zeta(\tau-D_*)\right]d\tau=0,\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}}\frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}g_2(\tau)\left[\zeta(\tau)-\zeta(\tau-D_*)\right]d\tau=0,\end{cases} \tag{28}$$

$$\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{L_{01}}\frac{\Pi_{12}(\tau)\Pi_{22}(\tau)}{e^{\gamma_1^+(\tau)}e^{\gamma_2^-(\tau)}h_*(\tau)}g_1(\tau)\zeta^{(k)}(\tau)d\tau=0,\\ \dfrac{1}{2\pi i}\displaystyle\int_{L_{02}}\frac{\Pi_{11}(\tau)\Pi_{22}(\tau)}{e^{\gamma_2^+(\tau)}e^{\gamma_1^-(\tau)}h_*(\tau)}g_2(\tau)\zeta^{(k)}(\tau)d\tau=0,\end{cases}\quad k=1,2,\cdots,-m+\lambda+n-2 \tag{29}$$

(when $m-\lambda-n+1=-2$, (29) is unnecessary) are satisfied, problem (1) is solvable and the solution is given by

$$\begin{cases} F^{+}(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{12}(z)\Pi_{22}(z)}\left[\Psi_1^{+}(z)+\Psi_2^{+}(z)-\Psi_1(0)-\Psi_2(0)\right], & z\in D_1, \\ \Omega(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{22}(z)}\left[\Psi_1^{-}(z)+\Psi_2^{+}(z)-\Psi_1(0)-\Psi_2(0)\right], & z\in D_2, \\ F^{-}(z)=\dfrac{e^{\gamma(z)}h_*(z)}{\Pi_{11}(z)\Pi_{21}(z)}\left[\Psi_1^{-}(z)+\Psi_2^{-}(z)-\Psi_1(0)-\Psi_2(0)\right], & z\in D_3. \end{cases} \tag{30}$$

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