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Extension of Generalized Bernoulli Learning Models

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Abstract

In this article, we study the generalized Bernoulli learning model based on the probability of success $p_i = \alpha_i / n$ where $i = 1, 2, \cdots, n$ $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \le n$ and n is positive integer. This gives the previous results given by Abdulnasser and Khidr [1], Rashad [2] and EL-Desouky and Mahfouz [3] as special cases, where $p_i = i/n$ $p_i = i^2/n^2$ and $p_i = i^p/n^p$ respectively. The probability function $P(W_n = k)$ of this model is derived, some properties of the model are obtained and the limiting distribution of the model is given.

Keywords

Stirling Numbers, Bernoulli Learning Models, Comtet Numbers, Inclusion-Exclusion Principle

1. Introduction

In industry, training programmes are conducted with the aim of training new workers to do particular job repeatedly every day. It is assumed that a particular trainee will show progress proportional to the number of days he attends the program, otherwise his ability will be different from one day to another, see [1] [4].

Let n be the length of a programme in days and l the number of repetitions of the job per day a trainee has to do. If a trainee is responding to the instructions, it would be reasonable to assume the probability that he will do a single job right, *i.e*. the probability of success on the i^{th} day is $p_i = i/n$, see Abdulnasser and Khidr [1], and hence the probability that he will do x jobs correctly out of l jobs on the i^{th} day is $\binom{l}{x} (i/n)^x$

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$\left(1-i/n\right)^{1-x}$, $x=0,1,\cdots,l$ and $i=0,1,\cdots,n$.

When a trainee is not responding to the instructions, p_i will be a constant p, $0<p<1$. To test whether a trainee is responding or not, we test if p_i is varying or sustaining a constant value p. This can be done by computing the total number of jobs that have been done correctly over the whole period of the program.

Let $X_{n,i}^{l}$ stand for the number of jobs done correctly out of l jobs on i^{th} day, $i=1,2,\cdots,n$, $l=1,2,\cdots$ and $W_n^l=\sum_{i=1}^{n} X_{n,i}^{l}$, $l\le W_n^l\le nl$. In case $p_i=p$, $0<p<1$, the distribution of W_n^l will be $B\left(nl,p\right)$.

In this article, we study a generalization of Bernoulli learning model based on probability of success $p_i=\alpha_i/n$ where n positive integer, α_i are real numbers, $i=1,2,\cdots,n$, and $0<\alpha_1<\alpha_2<\cdots<\alpha_n\le n$ and n is positive integer. This gives the previous results given in [1]-[3] as special cases, where $p_i=i/n$ $p_i=i^2/n^2$ and $p_i=i^p/n^p$ respectively. In Section 2, the probability function $P\left(W_n=k\right)$ of this model and some properties of the model are obtained. In Section 3, we derive the limiting distribution of the model. Finally, in Section 4, we discuss some special cases.

2. The Generalized Bernoulli Learning Model

Theorem 1. *The distribution function of* W_n *is*

$$P\left(W_n=k\right)=\left(-1\right)^k\sum_{m=k}^{n}\frac{s_{\bar{\alpha}}\left(n+1,n+1-m\right)}{m^n}\binom{m}{k}, \tag{1}$$

where $W_n=\sum_{i=1}^{n}X_{n,i}^{1}$, $X_{n,i}^{1}\approx B\left(1,\alpha_i/n\right)$.

Proof. To derive the distribution of Bernoulli learning model based on the sum of the independent random variable $\left\{X_{n,i}^{1}\right\}_{i=1}^{n}$, $i=1,2,\cdots,n$,

where the probability of success is $p_i=\alpha_i/n$ we define the event E_i as the event $X_{n,i}^{1}$, $i=1,2,\cdots,n$ see [5], and the sum

$$\begin{aligned}
P\left(n,k\right)&=\sum_{1\le i_1<\cdots<i_k\le n}P\left(E_{i_1},E_{i_2},\cdots,E_{i_k}\right)\\
&=\sum_{1\le i_1<\cdots<i_k\le n}P\left(X_{n,i_1}^{1}=1,X_{n,i_2}^{1}=1,\cdots,X_{n,i_k}^{1}=1\right)\\
&=\sum_{1\le i_1<\cdots<i_k\le n}\frac{\alpha_{i_1}}{n}\frac{\alpha_{i_2}}{n}\cdots\frac{\alpha_{i_k}}{n}=\frac{1}{n^k}\sum_{1\le i_1<\cdots<i_k\le n}\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}\\
&=\left(-1\right)^k s_{\bar{\alpha}}\left(n+1,n+1-k\right)/n^k,
\end{aligned}$$

where $s_{\bar{\alpha}}\left(n,k\right)$ the generalized Stirling number of the first kind (Comtet numbers), defined by Comtet in [6] [7] as follows

$$\left(x-\alpha_0\right)\left(x-\alpha_1\right)\cdots\left(x-\alpha_{n-1}\right)=\sum_{k=0}^{n}s_{\bar{\alpha}}\left(n,k\right)x^k$$

where $\bar{\alpha}=\left(\alpha_0,\alpha_1,\cdots,\alpha_{n-1}\right)$, for more details, see [8] and [9].

Employing the inclusion-exclusion principle, see [5], we get

$$\begin{aligned}
P\left(W_n\ge k\right)&=\sum_{m=k}^{n}\left(-1\right)^{m-k}\binom{m-1}{k-1}P\left(n,m\right)\\
&=\sum_{m=k}^{n}\left(-1\right)^{m-k}\binom{m-1}{k-1}\frac{\left(-1\right)^m s_{\bar{\alpha}}\left(n+1,n+1-m\right)}{n^m}\\
&=\left(-1\right)^k\sum_{m=k}^{n}\frac{s_{\bar{\alpha}}\left(n+1,n+1-m\right)}{n^m}\binom{m-1}{k-1},
\end{aligned}$$

then

$$P\left(W_n \geq k+1\right)=(-1)^{k+1} \sum_{m=k+1}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\binom{m-1}{k},$$

hence

$$P\left(W_n=k\right)=P\left(W_n \geq k\right)-P\left(W_n \geq k+1\right)$$

$$\begin{aligned}
P\left(W_n=k\right)&=(-1)^k \sum_{m=k}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\binom{m-1}{k-1}-(-1)^{k+1} \sum_{m=k+1}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\binom{m-1}{k} \\
&=(-1)^k\binom{k-1}{k-1} \frac{s_{\bar{\alpha}}(n+1, n+1-k)}{n^k}+(-1)^k \sum_{m=k+1}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\binom{m-1}{k-1} \\
&\quad+(-1)^k \sum_{m=k+1}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\binom{m-1}{k} \\
&=(-1)^k \frac{s_{\bar{\alpha}}(n+1, n+1-k)}{n^k}+(-1)^k \sum_{m=k+1}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\left\{\binom{m-1}{k-1}+\binom{m-1}{k}\right\} \\
&=(-1)^k \frac{s_{\bar{\alpha}}(n+1, n+1-k)}{n^k}+(-1)^k \sum_{m=k+1}^{n} \frac{s_{\bar{\alpha}}(n+1, n+1-m)}{n^m}\binom{m}{k},
\end{aligned}$$

this yields (1). □

Lemma 1.

$$\mu_{W_n}=E\left(W_n\right)=\sum_{i=1}^{n} \frac{\alpha_i}{n}, \tag{2}$$

$$\operatorname{Var}\left(W_n\right)=\frac{1}{n^2}\left(2 \sum_{1 \leq i_1<i_2 \leq n} \alpha_{i_1} \alpha_{i_2}+n \sum_{i=1}^{n} \alpha_i-\left(\sum_{i=1}^{n} \alpha_i\right)^2\right). \tag{3}$$

Proof. Consider the pair of inverse relation, see [10]

$$a_k=\sum_{m=k}\binom{m}{k} b_m, \quad b_k=\sum_{m=k}(-1)^{m+k}\binom{m}{k} a_m. \tag{4}$$

Then using (1), let

$$g_k=P\left(W_n=k\right)=(-1)^k \sum_{m=k}\binom{m}{k} \frac{s_{\bar{\alpha}}(n+1, n+1-k)}{n^m}.$$

Hence from (4), we get

$$s_{\bar{\alpha}}(n+1, n+1-k) n^{-k}=(-1)^k \sum_{m=k}\binom{m}{k} g_m, \tag{5}$$

and setting $k=1$, we have

$$s_{\bar{\alpha}}(n+1, n) n^{-1}=-\sum_{m=k}^{n} m g_m=-E\left(W_n\right). \tag{6}$$

But we have, see [7]

$$s_{\bar{\alpha}}(n, k)=(-1)^{n-k} \sum_{1 \leq i_1<i_2<\cdots<i_{n-k} \leq n} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n-k}}. \tag{7}$$

Thus $s_{\bar{\alpha}}(n+1, n)=-\sum_{i=1}^{n} \alpha_i$ and this yields (2).

If putting $k=2$ in (5), we get

$$s_{\bar{\alpha}}(n+1, n-1) n^{-2}=\sum_{m=2}^{n}\binom{m}{2} g_m=\sum_{m=2}^{n} \frac{m(m-1)}{2} g_m=\frac{1}{2} \sum_{m=2}^{n} m^2 g_m-\frac{1}{2} \sum_{m=2}^{n} m g_m=\frac{1}{2} E\left(\left(W_n\right)^2\right)-\frac{1}{2} E\left(W_n\right),$$

using (7), we have $s_{\bar{\alpha}}(n+1,n-1)=\sum_{1\le i_1<i_2\le n}\alpha_{i_1}\alpha_{i_2}$, then

$$E\left((W_n)^2\right)=2n^{-2}\sum_{1\le i_1<i_2\le n}\alpha_{i_1}\alpha_{i_2}+\sum_{i=1}^{n}\frac{\alpha_i}{n},$$

hence

$$\operatorname{Var}(W_n)=E\left((W_n)^2\right)-\left(E(W_n)\right)^2$$

$$\operatorname{Var}(W_n)=2n^{-2}\sum_{1\le i_1<i_2\le n}\alpha_{i_1}\alpha_{i_2}+\sum_{i=1}^{n}\frac{\alpha_i}{n}-\left(\sum_{i=1}^{n}\frac{\alpha_i}{n}\right)^2,$$

this yields (3). □

3. Limiting Distribution of the Bernoulli Learning Model

In this section we study the limiting distribution of the Bernoulli learning model based on the probability with success α_i/n.

Theorem 2. *Let* $W_n=\sum_{i=1}^{n}X_{n,i}^1$ *where* $X_{n,i}^1\approx B(1,\alpha_i/n)$ *and* $X_{n,i}^1$ *are independent random variables. Then* $\lim_{n\to\infty}M_{Z_n}=\exp(t^2/2)$ *where* $Z_n=\frac{W_n-\mu_{W_n}}{\sigma_{W_n}}$ *i.e.* Z_n *is* $N(0,1)$ *as* $n\to\infty$.

Proof. The moment generating function of Z_n is

$$M_{Z_n}(t)=M_{\frac{W_n-\mu_{W_n}}{\sigma_{W_n}}}=\mathrm{e}^{\left(-\frac{\mu_{W_n}}{\sigma_{W_n}}\right)t}M_{W_n}\left(t/\sigma_{W_n}\right),$$

and the moment generating function of W_n is

$$M_{W_n}\left(t/\sigma_{W_n}\right)=E\left(\mathrm{e}^{W_n\left(t/\sigma_{W_n}\right)}\right),\quad W_n=\sum_{i=1}^{n}X_{n,i}^1,\quad \text{hence}$$

$$M_{W_n}\left(t/\sigma_{W_n}\right)=\prod_{i=1}^{n}E\left(\mathrm{e}^{X_{n,i}^1\left(t/\sigma_{W_n}\right)}\right)=\prod_{i=1}^{n}\left(\sum_{x=0}^{1}\mathrm{e}^{xt/\sigma_{W_n}}\binom{1}{x}\left(\frac{\alpha_i}{n}\right)^x\left(1-\frac{\alpha_i}{n}\right)^{1-x}\right)=\prod_{i=1}^{n}\left(\frac{n-\alpha_i}{n}+\frac{\alpha_i}{n}\mathrm{e}^{t/\sigma_{W_n}}\right),\quad \text{then}$$

$$M_{Z_n}(t)=\mathrm{e}^{\left(-\frac{\mu_{W_n}}{\sigma_{W_n}}\right)t}\prod_{i=1}^{n}\left(1+\frac{\alpha_i}{n}\left(\mathrm{e}^{t/\sigma_{W_n}}-1\right)\right),$$

therefore, we have

$$\begin{aligned}
\ln M_{Z_n}(t)&=\frac{-\mu_{W_n}t}{\sigma_{W_n}}+\sum_{i=1}^{n}\ln\left(1+\frac{\alpha_i}{n}\left(\mathrm{e}^{t/\sigma_{W_n}}-1\right)\right)=\frac{-\mu_{W_n}t}{\sigma_{W_n}}+\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{\alpha_i}{n}\left(\mathrm{e}^t/\sigma_{W_n}-1\right)\right)^k\\
&=\frac{-\mu_{W_n}t}{\sigma_{W_n}}+\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{\alpha_i}{n}\right)^k\left(\sum_{j=1}^{\infty}\frac{\left(t/\sigma_{W_n}\right)^j}{j!}\right)^k\\
&=\frac{-\mu_{W_n}t}{\sigma_{W_n}}+\sum_{i=1}^{n}\frac{\alpha_i}{n}\left(\frac{t}{\sigma_{W_n}}+\frac{1}{2!}\frac{t^2}{\sigma_{W_n}^2}+\frac{1}{3!}\frac{t^3}{\sigma_{W_n}^3}+\cdots\right)-\sum_{i=1}^{n}\frac{\alpha_i^2}{2n^2}\left(\frac{t}{\sigma_{W_n}}+\frac{1}{2!}\frac{t^2}{\sigma_{W_n}^2}+\frac{1}{3!}\frac{t^3}{\sigma_{W_n}^3}+\cdots\right)^2+\cdots\\
&=\frac{-\mu_{W_n}t}{\sigma_{W_n}}+\frac{t}{\sigma_{W_n}}\sum_{i=1}^{n}\frac{\alpha_i}{n}+\frac{1}{2!}\frac{t^2}{\sigma_{W_n}^2}\sum_{i=1}^{n}\left(\frac{\alpha_i}{n}-\frac{\alpha_i^2}{n^2}\right)+O(1/n),
\end{aligned}$$

by using (2) and (3), we obtain

$$\ln M_{Z_n}(t) = \frac{t^2}{2} + O(1/n), \quad \text{hence}$$

$$\lim_{n\to\infty} M_{Z_n}(t) \cong \exp\left(t^2/2\right) \tag{8}$$

which is the moment generating function of standard normal distribution $N(0,1)$. □

4. Some Special Cases

In this section we discuss some special cases as follows.

i) Setting the probability of successes $p_i = \frac{i}{n}$ we have the results derived in [1], as special case

Theorem 3. *The distribution of* W_n^1 *is given by* [1]

$$P\left(W_n^1 = k\right) = (-1)^k \sum_{m=k}^{n} \frac{s(n+1, n+1-m)}{n^m} \binom{m}{k}, \quad k = 1, 2, \cdots, n, \tag{9}$$

where $s(n,k)$ are the usual stirling numbers of the first kind, see [10].

Also, they obtained the limiting distribution of learning model, mean and variance as follows.

Theorem 4. *Let* $W_n^1 = \sum_{i=1}^{n} X_i$, *where* $X_i \approx B(1, i/n)$ *and* X *'s are independent random variables. Then* $\lim_{n\to\infty} M_{Z_n} = \mathrm{e}^{t^2/2}$ *where* $Z_n = \frac{W_n - \mu_{W_n}}{\sigma_{W_n}}$ *i.e.* Z_n *has* $N(0,1)$ *as* $n \to \infty$.

Lemma 2.

$$\mu_{W_n^1} = \frac{n+1}{2}, \quad \sigma^2_{W_n^1} = \frac{n^2 - 1}{6n}. \tag{10}$$

ii) Setting the probability of successes $p_i = \left(\frac{i}{n}\right)^2$ we have the results derived in [2], as special case

Theorem 5. *The distribution of* W_n^1 *is given by* [2]

$$P\left(W_n^1 = k\right) = (-1)^{n+1+k} \sum_{m=k}^{n} \binom{m}{k} \left(\frac{1}{n^{2m}}\right) \sum_{l=0}^{2(n+1-k)} s(n+1, l)\, s\left(n+1, 2(n+1-k) - l\right). \tag{11}$$

Lemma 3.

$$\mu_{W_n} = \frac{(n+1)(2n+1)}{6n}, \quad \text{and} \quad \sigma^2_{W_n} = \frac{4n^4 + 1}{30n^3}.$$

iii) Setting the probability of successes $p_i = \left(\frac{i}{n}\right)^p$ we have the results derived in [3], as special case

Theorem 6.

$$P(W_n = k) = (-1)^k \sum_{m=k}^{n} \binom{m}{k} \frac{s_p(n+1, n+1-m)}{n^{pm}}, \tag{12}$$

where $W_n = \sum_{i=1}^{n} X_{n,i}^1$, $X_{n,i}^1 \approx B\left(1, i^p/n^p\right)$ and $s_p(n,k)$, p-Stirling numbers, see [11] [12].

Theorem 7. *Let* $W_n^1 = \sum_{i=1}^{n} X_{n,i}^1$ *where* $X_{n,i}^1 \approx B\left(1, i^p/n^p\right)$ *and* $X_{n,i}^1$ *are independent random variables. Then* $\lim_{n\to\infty} M_{Z_n} = \mathrm{e}^{t^2/2}$ *where* $Z_n = \frac{W_n - \mu_{W_n}}{\sigma_{W_n}}$ *i.e.* Z_n *has* $N(0,1)$ *as* $n \to \infty$.

Lemma 4.

$$\mu_{W_n} = E(W_n) = \sum_{i=1}^{n} \frac{i^p}{n^p}, \quad \text{and}$$

$$\operatorname{Var}(W_n) = \frac{1}{n^{2p}} \left(2 \sum_{1 \le i_1 < i_2 \le n} i_1^p i_2^p + n^p \sum_{i=1}^{n} i^p - \left(\sum_{i=1}^{n} i^p \right)^2 \right).$$

5. Conclusion

Our main goal of this work is concerned with studying the extension of generalized Bernoulli learning model with probability of success $p_i = \alpha_i / n$ $i = 1, 2, \cdots, n,$ $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \le n$ and n is positive integer. Some previous results, see [1]-[3], are concluded as special cases of our result, that is for $p_i = i/n$ $p_i = i^2/n^2$ and $p_i = i^p/n^p$ respectively. The mean and variance of the model are obtained. Finally, the limiting distribution of the general model is derived. This model has many applications in industry, specially for training programmes.

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