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A Non-Preemptive Priority Queueing System with a Single Server Serving Two Queues M/G/1 and M/D/1 with Optional Server Vacations Based on Exhaustive Service of the Priority Units

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Abstract

We study a vacation queueing system with a single server simultaneously dealing with an M/G/1 and an M/D/1 queue. Two classes of units, priority and non-priority, arrive at the system in two independent Poisson streams. Under a non-preemptive priority rule, the server provides a general service to the priority units and a deterministic service to the non-priority units. We further assume that the server may take a vacation of random length just after serving the last priority unit present in the system. We obtain steady state queue size distribution at a random epoch. Corresponding results for some special cases, including the known results of the M/G/1 and the M/D/1 queues, have been derived.

Keywords: Non Preemptive Priority Queueing System, Modified Server Vacations, Combination of General Service and Deterministic Service, Steady State, Queue Size Distribution

1. Introduction

Several authors including Cobham [1], Phipps [2], Schrage [3], Jaiswal [4], Madan [5], Simon [6], Takagi [7], Choi and Chang [8] have studied priority queues. These authors and several others have studied single server or multi-server queues with two or more priority classes under preemptive or non-preemptive priority rules. All these authors essentially assume the same service time distribution for all classes of units with identical or different service rates. Madan and Abu-Dayyeh [9] deal with a single server queueing system with two classes of units, priority units and non-priority units. Under the non-preemptive queue discipline, they assume that the service time V of a priority unit has a general distribution and that of a non–priority unit is deterministic. Thus their model is a combination of the M/G/1 and M/D/1 queues and the server keeps switching over these two queues depending on the class of units present in the system. For separate references on M/G/1 and M/D/1 queues, the reader is referred to Bhat [10], Levy and Yechiali [11], Kleinrock [12], Cohen [13], Lee [14], Gross and Harris [5], Cox and Miller [16], Tijms [17], Yang and Li [18], Bunday [19] and Madan [20,21]. However, in the present paper, we generalize Madan and Abu-Dayyeh [9] paper by adding a significant assumption to their model that the server may take a vacation of random length but we assume that no vacation is allowed if there is even a single priority unit present in the system. Thus the server may take an optional vacation of a random length just after completing the service of the last priority unit present in the system or else may just continue serving the non-priority units if present in the system.

We use the supplementary variable technique by introducing two supplementary variables, one for the elapsed service time of a priority unit and the other for the elapsed vacation time of the server. Thus, we generalize the results of not only Madan and Abu-Dayyeh [9], but also some other known results of the M/G/1 and the M/D/1 queues as particular cases.

2. Assumptions Underlying the Mathematical Model

Priority and non-priority units arrive at the system in independent Poisson streams with respective mean arrival rates λ_1 and λ_2 and form two queues, if the server is busy. The server must serve all the priority units present in the system before taking up a non-priority unit for service. In other words, there is no priority unit present in the system at the time of starting service of a non-priority unit. Further, we assume that the server follows a non-preemptive priority rule, which means that if one or more priority units arrive during the service time of a non-priority unit, the current service of a non-priority unit is not stopped and a priority unit will be taken up for service only after the current service of a non-priority unit is complete. Units are served one by one, on a 'first-come, first-served' basis within each class of units. We assume that the service time S of a priority unit is general with probability density function $b(s)$ and the distribution function $B(s)$. Let $\mu(x)$ dx be the conditional probability of completion of service of a priority unit during the interval $(x, x+dx]$ given that the elapsed service time of such a unit is x, so that

$$\mu(x) = \frac{b(x)}{1-B(x)} \tag{2.1}$$

and, therefore,

$$b(s) = \mu(s)\exp\left[-\int_0^s \mu(x)\mathrm{d}x\right]. \tag{2.2}$$

The service time of a non-priority unit is deterministic with constant duration d (>0).

We further assume that as soon as the service of the last priority unit present in the system is completed, the server has the option to take a vacation of random length with probability p, in which case the vacation starts immediately or else with probability $(1-p)$ he may decide to continue serving the non-priorty units present in the system, if any. In the later case, if there is no non-priority unit present in the system, the server remains idle in the system waiting for the new units to arrive. The vacation period random variable V is assumed to follow a general probability law with probability density function $a(v)$ and the distribution function $A(v)$. Let $\beta(x)$ dx be the conditional probability of completion of server's vacation during the interval $(x, x+dx]$ given that the elapsed vacation time of the server is x, so that

$$\beta(x) = \frac{a(x)}{1-A(x)} \tag{2.3}$$

and, therefore,

$$a(s) = \beta(s)\exp\left[-\int_0^s \beta(x)\mathrm{d}x\right]. \tag{2.4}$$

3. Definitions and Notations

We define

$P_{m,n}^{(1)}(x,t)$: probability that at time t there are m (≥0) priority units and n (≥0) non-priority units in the queue excluding one priority unit in service with elapsed service time x.

$P_{m,n}^{(1)}(t) = \int_0^\infty P_{m,n}^{(1)}(x,t)\mathrm{d}x$: probability that at time t there are m (≥0) priority units and n (≥0) non-priority units in the queue excluding one priority unit in service without regard to the elapsed service time x of a priority unit.

$V_{m,n}(x,t)$: probability that at time t the server is on vacation with elapsed vacation time x and there are m (≥0) priority units and n (≥0) non-priority units in the queue.

$V_{m,n}(t) = \int_0^\infty V_{m,n}(x,t)\mathrm{d}x$: probability that at time t the server is on vacation and there are m (≥0) priority units and n (≥0) non-priority units in the queue, without regard to the elapsed repair time x.

$P_{0,n}^{(2)}(t)$: probability that at time t there are no priority units in the system and n (≥0) non-priority units in the queue excluding one non-priority unit in service.

$Q(t)$: probability that at time t there is neither a priority unit nor a non-priority unit in the system and the server is idle but available in the system.

r_i : probability that i (= 0, 1, 2, ···) priority units arrive during the constant service time d of a non-priority unit.

k_j : probability that j (= 0, 1, 2, ···) non-priority units arrive during the constant service time d of a non-priority unit.

Then assuming that the steady state exists, let

$$\lim_{t\to\infty} P_{m,n}^{(1)}(x,t) = P_{m,n}^{(1)}(x),$$

$$\lim_{t\to\infty} P_{m,n}^{(1)}(t) = \int_0^\infty P_{m,n}^{(1)}(x)\mathrm{d}x = P_{m,n}^{(1)},$$

$$\lim_{t\to\infty} V_{m,n}(x,t) = V_{m,n}(x),$$

$$\lim_{t\to\infty} V_{m,n}(t) = \int_0^\infty V_{m,n}(x)\mathrm{d}x = V_{m,n}; \quad \lim_{t\to\infty} P_{0,n}^{(2)}(t) = P_{0,n}^{(2)}$$

and $\lim_{t\to\infty} Q(t) = Q$

denote the corresponding steady state probabilities. In addition, we define the following steady state probability generating functions:

$$P_m^{(1)}(x,z_2)=\sum_{n=0}^{\infty}P_{m,n}^{(1)}(x)z_2^n;\ P_n^{(1)}(x,z_1)=\sum_{m=0}^{\infty}P_{m,n}^{(1)}(x)z_1^m, \tag{3.1a}$$

$$P^{(1)}(x,z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P_{m,n}^{(1)}(x)z_1^m z_2^n=\sum_{m=0}^{\infty}P_m^{(1)}(x,z_2)z_1^m=\sum_{n=0}^{\infty}P_n^{(1)}(x,z_1)z_2^n, \tag{3.1b}$$

$$P^{(1)}(z_1,z_2)=\int_0^{\infty}P^{(1)}(x,z_1,z_2)\mathrm{d}x=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P_{m,n}^{(1)}z_1^m z_2^n, \tag{3.1c}$$

$$V_m(x,z_2)=\sum_{n=0}^{\infty}V_{m,n}(x)z_2^n;\ V_n(x,z_1)=\sum_{m=0}^{\infty}V_{m,n}(x)z_1^m, \tag{3.1d}$$

$$V(x,z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}V_{m,n}(x)z_1^m z_2^n=\sum_{m=0}^{\infty}V_m(x,z_2)z_1^m=\sum_{n=0}^{\infty}V_n(x,z_1)z_2^n, \tag{3.1e}$$

$$V(z_1,z_2)=\int_0^{\infty}V(x,z_1,z_2)\mathrm{d}x=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}V_{m,n}z_1^m z_2^n, \tag{3.1f}$$

$$P_0^{(2)}(z_2)=\sum_{n=0}^{\infty}P_{0,n}^{(2)}z_2^n, \tag{3.1g}$$

$$P_0^{(1)}(z_2)=\sum_{n=0}^{\infty}P_{0,n}^{(1)}z_2^n, \tag{3.1h}$$

$$R(z_1)=\sum_{i=0}^{\infty}r_i z_1^i=\sum_{i=0}^{\infty}\left(\frac{\exp(-\lambda_1 d)(\lambda_1 d)^i}{i!}\right)z_1^i=\exp\left[-\lambda_1 d(1-z_1)\right], \tag{3.1i}$$

$$K(z_2)=\sum_{j=0}^{\infty}k_j z_2^j=\sum_{j=0}^{\infty}\left(\frac{\exp(-\lambda_2 d)(\lambda_2 d)^j}{j!}\right)z_2^j=\exp\left[-\lambda_2 d(1-z_2)\right], \tag{3.1j}$$

$$|z_1|\le 1,\ |z_2|\le 1.$$

4. Steady State Equations Governing the System

Usual probability reasoning based on our mathematical model, leads to the following equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{m,n}^{(1)}(x)+(\lambda_1+\lambda_2+\mu(x))P_{m,n}^{(1)}(x)=\lambda_1 P_{m-1,n}^{(1)}(x)+\lambda_2 P_{m,n-1}^{(1)}(x),\ m\ge 1,n\ge 1, \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{m,0}^{(1)}(x)+(\lambda_1+\lambda_2+\mu(x))P_{m,0}^{(1)}(x)=\lambda_1 P_{m-1,0}^{(1)}(x),\ m\ge 1,n=0, \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{0,n}^{(1)}(x)+(\lambda_1+\lambda_2+\mu(x))P_{0,n}^{(1)}(x)=\lambda_2 P_{0,n-1}^{(1)}(x),\ m=0,n\ge 1, \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{0,0}^{(1)}(x)+(\lambda_1+\lambda_2+\mu(x))P_{0,0}^{(1)}(x)=0,\ m=0,n=0, \tag{4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{m,n}(x)+(\lambda_1+\lambda_2+\beta(x))V_{m,n}(x)=\lambda_1 V_{m-1,n}(x)+\lambda_2 V_{m,n-1}(x),\ m\ge 1,n\ge 1, \tag{4.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{m,0}(x)+(\lambda_1+\lambda_2+\beta(x))V_{m,0}(x)=\lambda_1 V_{m-1,0}(x),\ m\ge 1,n=0, \tag{4.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{0,n}(x)+(\lambda_1+\lambda_2+\beta(x))V_{0,n}(x)=\lambda_2 V_{0,n-1}(x),\ m=0,n\ge 1, \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{0,0}(x)+(\lambda_1+\lambda_2+\beta(x))V_{0,0}(x)=0,\ m=0,n=0, \tag{4.8}$$

$$Q=\left(Q+P_{0,0}^{(2)}\right)r_0k_0+(1-p)\int_0^\infty P_{0,0}^{(1)}\mu(x)\mathrm{d}x+\int_0^\infty V_{0,0}(x)\beta(x)\mathrm{d}x, \tag{4.9}$$

$$P_{0,0}^{(2)}=\left(Q+P_{0,0}^{(2)}\right)r_0k_1+P_{0,1}^{(2)}r_0k_0+(1-p)\int_0^\infty P_{0,1}^{(1)}\mu(x)\mathrm{d}x+\int_0^\infty V_{0,1}(x)\beta(x)\mathrm{d}x, \tag{4.10}$$

$$P_{0,n}^{(2)}=\left(Q+P_{0,0}^{(2)}\right)r_0k_{n+1}+\sum_{j=1}^{n+1}P_{0,j}^{(2)}r_0k_{n+1-j}+(1-p)\int_0^\infty P_{0,n+1}^{(1)}\mu(x)\mathrm{d}x+\int_0^\infty V_{0,n+1}(x)\beta(x)\mathrm{d}x,\ n\ge 1. \tag{4.11}$$

The above equations are to be solved subject to the following boundary conditions:

$$P_{m,n}^{(1)}(0)=\int_0^\infty P_{m+1,n}^{(1)}(x)\mu(x)\mathrm{d}x+\sum_{j=0}^{n}P_{0,j}^{(2)}r_{m+1}k_{n-j}+Qr_{m+1}k_n+\int_0^\infty V_{m+1,n}(x)\beta(x)\mathrm{d}x,\ m\ge 1,n\ge 1, \tag{4.12}$$

$$P_{m,0}^{(1)}(0)=\int_0^\infty P_{m+1,0}^{(1)}(x)\mu(x)\mathrm{d}x+P_{0,0}^{(2)}r_{m+1}k_0+Qr_{m+1}k_0+\int_0^\infty V_{m+1,0}(x)\beta(x)\mathrm{d}x,\ m\ge 1,n=0, \tag{4.13}$$

$$P_{0,n}^{(1)}(0)=\int_0^\infty P_{1,n}^{(1)}(x)\mu(x)\mathrm{d}x+\sum_{j=0}^{n}P_{0,j}^{(2)}r_1k_{n-j}+Qr_1k_n+\int_0^\infty V_{1,n}(x)\beta(x)\mathrm{d}x,\ m=0,n\ge 1, \tag{4.14}$$

$$P_{0,0}^{(1)}(0)=\int_0^\infty P_{1,0}^{(1)}(x)\mu(x)\mathrm{d}x+P_{0,0}^{(2)}r_1k_0+Qr_1k_0+\int_0^\infty V_{1,0}(x)\beta(x)\mathrm{d}x,\ m=0=n. \tag{4.15}$$

$$V_{0,n}(0)=p\int_0^\infty P_{0,n}^{(1)}(x)\mu(x)\mathrm{d}x,\ n\ge 0 \tag{4.16}$$

5. Steady State Queue Size Distribution at a Random Epoch

We perform the operations $\sum_{n=1}^{\infty}(4.1)z_2^n+(4.2)$; $\sum_{n=1}^{\infty}(4.3)z_2^n+(4.4)$ and use Equation (3.1). Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}P_m^{(1)}(x,z_2)+(\lambda_1+\lambda_2+\mu(x))P_m^{(1)}(x,z_2)=\lambda_1P_{m-1}^{(1)}(x,z_2)+\lambda_2z_2P_m^{(1)}(x,z_2),\ m\ge 1, \tag{4.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_0^{(1)}(x,z_2)+(\lambda_1+\lambda_2+\mu(x))P_0^{(1)}(x,z_2)=\lambda_2z_2P_0^{(1)}(x,z_2). \tag{4.18}$$

Next, we perform $\sum_{m=1}^{\infty}(4.17)z_1^m+(4.18)$, use (3.1) and simplify. Then we have,

$$\frac{\mathrm{d}}{\mathrm{d}x}P^{(1)}(x,z_1,z_2)+(\lambda_1(1-z_1)+\lambda_2(1-z_2)+\mu(x))P^{(1)}(x,z_1,z_2)=0. \tag{4.19}$$

Similarly, we perform the operations $\sum_{n=1}^{\infty}(4.5)z_2^n+(4.6)$; $\sum_{n=1}^{\infty}(4.7)z_2^n+(4.8)$ and use Equation (3.1). Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}V_m(x,z_2)+(\lambda_1+\lambda_2+\beta(x))V_m(x,z_2)=\lambda_1V_{m-1}(x,z_2)+\lambda_2z_2V_m(x,z_2),\ m\ge 1, \tag{4.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}V_0(x,z_2)+(\lambda_1+\lambda_2+\beta(x))V_0(x,z_2)=\lambda_2z_2V_0(x,z_2). \tag{4.21}$$

Next, we perform $\sum_{m=1}^{\infty}(4.20)z_1^m+(4.21)$, use (3.1) and simplify. Then we have,

$$\frac{\mathrm{d}}{\mathrm{d}x}V(x,z_1,z_2)+\big(\lambda_1(1-z_1)+\lambda_2(1-z_2)+\beta(x)\big)V(x,z_1,z_2)=0. \tag{4.22}$$

Then we perform $(4.9)+(4.10)z_2+\sum_{n=1}^{\infty}(4.11)z_2^{n+1}$, use (3.1) and simplify. Thus we have

$$z_2P_0^{(2)}(z_2)=Qr_0K(z_2)+(1-p)\int_0^{\infty}P_0^{(1)}(x,z_2)\mu(x)\mathrm{d}x+P_0^{(2)}(z_2)r_0K(z_2)-Q+\int_0^{\infty}V_0(x,z_2)\beta(x)\mathrm{d}x, \tag{4.23}$$

which again simplifies to

$$\big(z_2-r_0K(z_2)\big)P_0^{(2)}(z_2)=(1-p)\int_0^{\infty}P_0^{(1)}(x,z_2)\mu(x)\mathrm{d}x+Qr_0K(z_2)-Q+\int_0^{\infty}V_0(x,z_2)\beta(x)\mathrm{d}x. \tag{4.24}$$

Now, we shall consider the boundary conditions (4.12) through (4.16) and perform $\sum_{m=1}^{\infty}(4.12)z_1^{m+1}+(4.14)z_1$; $\sum_{m=1}^{\infty}(4.13)z_1^{m+1}+(4.15)z_1$, use (3.1) and simplify. We then obtain

$$\begin{aligned}z_1P_n^{(1)}(0,z_1)=&\int_0^{\infty}P_n^{(1)}(x,z_1)\mu(x)\mathrm{d}x-\int_0^{\infty}P_{0,n}^{(1)}(x)\mu(x)\mathrm{d}x+\int_0^{\infty}V_n(x,z_1)\beta(x)\mathrm{d}x-\int_0^{\infty}V_{0,n}(x)\beta(x)\mathrm{d}x\\&+\big(R(z_1)-r_0\big)\sum_{j=0}^{n}P_{0,j}^{(2)}k_{n-j}+\big(R(z_1)-r_0\big)\sum_{n=1}^{\infty}Qk_n,\quad n\ge 1,\end{aligned} \tag{4.25}$$

$$z_1P_0^{(1)}(0,z_1)=\int_0^{\infty}P_0^{(1)}(x,z_1)\mu(x)\mathrm{d}x-\int_0^{\infty}P_{0,0}^{(1)}(x)\mu(x)\mathrm{d}x+\int_0^{\infty}V_0(x,z_1)\beta(x)\mathrm{d}x-\int_0^{\infty}V_{0,0}(x)\beta(x)\mathrm{d}x+\big(R(z_1)-r_0\big)\Big(P_{0,0}^{(2)}+Q\Big)k_0. \tag{4.26}$$

And yet again, we perform $\sum_{n=1}^{\infty}(4.25)z_2^{n}+(4.26)$, use (3.1) and simplify. This operation yields

$$\begin{aligned}z_1P^{(1)}(0,z_1,z_2)=&\int_0^{\infty}P^{(1)}(x,z_1,z_2)\mu(x)\mathrm{d}x-\int_0^{\infty}P_0^{(1)}(x,z_2)\mu(x)\mathrm{d}x+\int_0^{\infty}V(x,z_1,z_2)\beta(x)\mathrm{d}x-\int_0^{\infty}V_0(x,z_2)\beta(x)\mathrm{d}x\\&+\big(R(z_1)-r_0\big)\Big(P_0^{(2)}(z_2)+Q\Big)K(z_2).\end{aligned} \tag{4.27}$$

Similarly, on performing $\sum_{n=0}^{\infty}(4.16)z_2^{n}$ and using (3.1), we obtain

$$V_0(0,z_2)=p\int_0^{\infty}P_0^{(1)}(x,z_2)\mu(x)\mathrm{d}x. \tag{4.28}$$

Now, we integrate (4.19) from 0 to x and obtain

$$P^{(1)}(x,z_1,z_2)=P^{(1)}(0,z_1,z_2)\exp\left[-\lambda_1(1-z_1)x-\lambda_2(1-z_2)x-\int_0^{x}\mu(t)\mathrm{d}t\right], \tag{4.29}$$

where $P^{(1)}(0,z_1,z_2)$ is given by (4.27).
Similarly, on integrating, (4.22) gives

$$V(x,z_1,z_2)=V(0,z_1,z_2)\exp\left[-\lambda_1(1-z_1)x-\lambda_2(1-z_2)x-\int_0^{x}\beta(t)\mathrm{d}t\right]. \tag{4.30}$$

However, by its definition, $V(0,z_1,z_2)=V_0(0,z_2)$ and, therefore, (4.30) is re-written as

$$V(x,z_1,z_2)=V_0(0,z_2)\exp\left[-\lambda_1(1-z_1)x-\lambda_2(1-z_2)x-\int_0^{x}\beta(t)\mathrm{d}t\right]. \tag{4.31}$$

where $V_0(0,z_2)$ is given by (4.28).

Once again integrating (4.29) and (4.31) with respect to *x* by parts and using (2.2) and (2.4), we have

$$P^{(1)}(z_1,z_2)=P^{(1)}(0,z_1,z_2)\left[\frac{1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right], \tag{4.32}$$

$$V(z_1,z_2)=V_0(0,z_2)\left[\frac{1-V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right], \tag{4.33}$$

where

$B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]=\int_0^\infty e^{-\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}dB(x)$ is the LST of the service time of a priority unit and

$V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]=\int_0^\infty e^{-\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}dV(x)$ is the LST of the server's vacation time respectively.

Now, Equation (4.18) can be re-written as

$$\frac{d}{dx}P_0^{(1)}(x,z_2)+\left(\lambda_1+\lambda_2(1-z_2)+\mu(x)\right)P_0^{(1)}(x,z_2)=0.$$

which, on integration, gives

$$P_0^{(1)}(x,z_2)=P_0^{(1)}(0,z_2)\exp\left[-\lambda_1x-\lambda_2(1-z_2)x-\int_0^x\mu(t)dt\right]. \tag{4.34}$$

and (4.21) yields

$$V_0(x,z_2)=V_0(0,z_2)\exp\left[-\lambda_1x-\lambda_2(1-z_2)x-\int_0^x\beta(t)dt\right]. \tag{4.35}$$

Next, we shall determine the integrals $\int_0^\infty P^{(1)}(x,z_1,z_2)\mu(x)dx$, $\int_0^\infty P_0^{(1)}(x,z_2)\mu(x)dx$ and $\int_0^\infty V_0(x,z_2)\mu(x)dx$ which appear in the right hand sides of Equations (4.24), (4.27) and (4.28).

Then we multiply Equations (4.29) and (4.34) by $\mu(x)$, integrate by parts with respect to *x* and use equation (2.2). Thus we obtain

$$\int_0^\infty P^{(1)}(x,z_1,z_2)\mu(x)dx=B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]P^{(1)}(0,z_1,z_2), \tag{4.36}$$

$$\int_0^\infty P_0^{(1)}(x,z_2)\mu(x)dx=B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2), \tag{4.37}$$

Similarly, we multiply Equations (4.31) and (4.35) by $\beta(x)$, integrate by parts with respect to *x* and obtain

$$\int_0^\infty V(x,z_1,z_2)\beta(x)dx=V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]V_0(0,z_2) \tag{4.38}$$

$$\int_0^\infty V_0(x,z_2)\beta(x)dx=V^*\left[\lambda_1+\lambda_2(1-z_2)\right]V_0(0,z_2). \tag{4.39}$$

Using Equations (4.36) to (4.39) into Equations (4.24), (4.27) and (4.28), we obtain

$$\left(z_2-r_0K(z_2)\right)P_0^{(2)}(z_2)=(1-p)B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)+Qr_0K(z_2)-Q+V^*\left[\lambda_1+\lambda_2(1-z_2)\right]V_0(0,z_2) \tag{4.40}$$

$$\begin{aligned} z_1 P^{(1)}(0,z_1,z_2) &= B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]P^{(1)}(0,z_1,z_2) - B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2) \\ &+ V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]V_0(0,z_2) - V^*\left[\lambda_1+\lambda_2(1-z_2)\right]V_0(0,z_2) \\ &+ \left(R(z_1)-r_0\right)\left(P_0^{(2)}(z_2)+Q\right)K(z_2). \end{aligned} \tag{4.41}$$

$$V_0(0,z_2) = pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2) \tag{4.42}$$

Next, we substitute the value of $V_0(0,z_2)$ from Equation (4.42) into Equations (4.40) and (4.41), replace $R(z_1)$ by $e^{-\lambda_1 d(1-z_1)}$ and $K(z_2)$ by $e^{-\lambda_2 d(1-z_2)}$ from (3.1f) and (3.1g) and simplify. We obtain

$$\left(z_2 - r_0 e^{-\lambda_2 d(1-z_2)}\right)P_0^{(2)}(z_2) = \left[(1-p)+pV^*\left[\lambda_1+\lambda_2(1-z_2)\right]\right]B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2) + Qr_0 e^{-\lambda_2 d(1-z_2)} - Q \tag{4.43}$$

$$\begin{aligned} &\left[z - B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right]P^{(1)}(0,z_1,z_2) \\ &= -B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2) + pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2) \\ &\quad - pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]V^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2) + \left(e^{-\lambda_{11} d(1-\lambda_{11})} - r_0\right)\left(P_0^{(2)}(z_2)+Q\right)e^{-\lambda_2 d(1-z_2)}. \end{aligned} \tag{4.44}$$

Now, substituting for $P^{(1)}(0,z_1,z_2)$ from Equation (4.44) into Equation (4.32), we have

$$\begin{aligned} P^{(1)}(z_1,z_2) &= \frac{\left(\frac{1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right)\left\langle\left(e^{-\lambda_1 d(1-z_1)}-r_0\right)\left(P_0^{(2)}(z_2)+Q\right)e^{-\lambda_2 d(1-z_2)}\right\rangle}{\left(z_1 - B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right)} \\ &- \frac{\left(\frac{1-B^*[\lambda_1(1-z_1)+\lambda_2(1-z_2)]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right)\left\langle B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle}{\left(z_1 - B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right)} \\ &+ \frac{\left(\frac{1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right)\left\langle pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle}{\left(z_1 - B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right)} \\ &- \frac{\left(\frac{1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right)\left\langle pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]V^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle}{\left(z_1 - B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right)} \end{aligned} \tag{4.45}$$

We have yet to determine the 3 unknowns $P_0^{(1)}(0,z_2)$, $P_0^{(2)}(z_2)$ and Q appearing in the numerator of the right hand side of (4.45). For this purpose, we proceed as follows.

It can be easily shown that the denominator of the right hand side of (4.45) has one zero inside or on the unit circle $|z_1| = 1$. Let this zero be denoted as α. Therefore, the numerator of the right side of (4.45) must vanish for this zero giving

$$\begin{aligned} &\left\langle\left(e^{-\lambda_1 d(1-\alpha)}-r_0\right)\left(P_0^{(2)}(z_2)+Q\right)e^{-\lambda_2 d(1-z_2)}\right\rangle - \left\langle B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle \\ &+ \left\langle pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]V^*\left[\lambda_1(1-\alpha)+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle \\ &- \left\langle pB^*\left[\lambda_1+\lambda_2(1-z_2)\right]V^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle = 0. \end{aligned} \tag{4.46}$$

Now, we solve Equations (4.43) and (4.46) for the two unknowns $P_0^{(1)}(0,z_2)$ and $P_0^{(2)}(z_2)$. Thus we obtain

$$P_0^{(1)}(o,z) = \frac{\left(e^{-\lambda_1 d(1-\alpha)}-r_0\right)e^{-\lambda_2 d(1-z_2)}(1-z_2)Q}{D(z)} \tag{4.47}$$

$$P_0^{(2)}(z_2)=\frac{B^*\left[\lambda_1+\lambda_2(1-z_2)\right]\left\langle \begin{array}{l}\left[(1-p)+pV^*\left[\lambda_1+\lambda_2(1-z_2)\right]\right]\left(r_0-\mathrm{e}^{-\lambda_1 d(1-\alpha)}\right)\mathrm{e}^{-\lambda_2 d(1-z_2)}\\ -\left[(1+pV^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]-pV^*\left[\lambda_1+\lambda_2(1-z_2)\right]\right]\left[1-r_0\mathrm{e}^{-\lambda_2 d(1-z_2)}\right]\end{array}\right\rangle Q}{D(z)} \tag{4.48}$$

where $D(z)$ in Equations (4.47) and (4.48) is the common denominator given by

$$\begin{aligned} D(z)=B^*\left[\lambda_1+\lambda_2(1-z_2)\right]&\Big(\left[(1-p)+pV^*[\lambda_1+\lambda_2(1-z_2)\right]\left[\mathrm{e}^{-\lambda_1 d(1-\alpha)}-r_0\right]\\ &-\left[(1+\mathrm{p}V^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]-pV^*\left[\lambda_1+\lambda_2(1-z_2)\right]\right]\left[z_2-r_0\mathrm{e}^{-\lambda_2 d(1-z_2)}\right]\Big)\end{aligned} \tag{4.49}$$

Then, we substitute for $P_0^{(1)}(0,z_2)$ and $P_0^{(2)}(z_2)$ from (4.47) and (4.48) into Equation (4.45) giving us $P^{(1)}(z_1,z_2)$. Finally, we shall use the normalizing condition $P^{(1)}(1,1)+P_0^{(2)}(1)+Q=1$ to determine the only remaining unknown Q.

Using L' Hopital's rule and proceeding as in Madan and Aby-Dayyeah (2003), we obtain

$$Q=\frac{\left[1-\lambda_1\left((E(S)+pE(V)\right)\right](1-\lambda_2 d)}{\lambda_1 d\left(E(S)+pE(V)\right)(1-\lambda_2 d)+\left[1-\lambda_1\left(E(S)+pE(V)\right)\right]} \tag{4.50}$$

where $E(S)$ is the mean service time of a priority unit and $E(V)$ is the mean vacation time of the server.

Having thus determined the value of Q, the probability that the server is idle, we have completely determined $P^{(1)}(z_1,z_2)$.

Further, system's utilization factor is given by

$$\rho=1-Q=\frac{\lambda_1 d\left((E(S)+pE(V)\right)(1-\lambda_2 d)\left[1-\lambda_1\left((E(S)+pE(V)\right)\right](\lambda_2 d)}{\lambda_1 d\left(E(S)+pE(V)\right)(1-\lambda_2 d)+\left[1-\lambda_1\left(E(S)+pE(V)\right)\right]} \tag{4.51}$$

The stability condition, under which the steady state exists, emerges from (4.50 and (4.51)). This condition is given by

$$0<\frac{\lambda_1 d\left((E(S)+pE(V)\right)(1-\lambda_2 d)\left[1-\lambda_1\left((E(S)+pE(V)\right)\right](\lambda_2 d)}{\lambda_1 d\left(E(S)+pE(V)\right)(1-\lambda_2 d)+\left[1-\lambda_1\left(E(S)+pE(V)\right)\right]}<1. \tag{4.52}$$

Note that (4.52) essentially implies that $\lambda_1\left(E(S)+pE(V)\right)<1$ and $\lambda_2 d<1$ should jointly hold for the steady state to exist. This is also intuitively true.

6. Particular Cases

Case 1: If there are no server vacations, then we let $p = 0$ in the above results (4.45) to (4.52) and obtain

$$\begin{aligned} P^{(1)}(z_1,z_2)=&\frac{\left(\dfrac{1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right)\left\langle\left(\mathrm{e}^{-\lambda_1 d(1-z_1)}-r_0\right)\left(P_0^{(2)}(z_2)+Q\right)\mathrm{e}^{-\lambda_2 d(1-z_2)}\right\rangle}{\left(z_1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right)}\\ &-\frac{\left(\dfrac{1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}\right)\left\langle B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle}{\left(z_1-B^*\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)\right]\right)}\end{aligned} \tag{4.53}$$

$$\left\langle\left(\mathrm{e}^{-\lambda_1 d(1-\alpha)}-r_0\right)\left(P_0^{(2)}(z_2)+Q\right)\mathrm{e}^{-\lambda_2 d(1-z_2)}\right\rangle-\left\langle B^*\left[\lambda_1+\lambda_2(1-z_2)\right]P_0^{(1)}(0,z_2)\right\rangle \tag{4.54}$$

$$P_0^{(1)}(o,z)=\frac{\left(\mathrm{e}^{-\lambda_1 d(1-\alpha)}-r_0\right)\mathrm{e}^{-\lambda_2 d((1-z_2)}(1-z_2)Q}{B^*\left[\lambda_1+\lambda_2(1-z_2)\right]\left(\left[z_2-r_0\mathrm{e}^{-\lambda_2 d(1-z_2)}\right]\right)} \tag{4.55}$$

$$P_0^{(2)}(z_2)=\frac{B^*\left[\lambda_1+\lambda_2(1-z_2)\right]\left\langle\left(r_0-\mathrm{e}^{-\lambda_1 d(1-\alpha)}\right)\mathrm{e}^{-\lambda_2 d(1-z_2)}-\left[1-r_0\mathrm{e}^{-\lambda_2 d(1-z_2)}\right]\right\rangle Q}{B^*\left[\lambda_1+\lambda_2(1-z_2)\right]\left(\left[z_2-r_0\mathrm{e}^{-\lambda_2 d(1-z_2)}\right]\right)} \tag{4.56}$$

We further obtain Q the steady state probability that thw server is idle as

$$Q=\frac{\left[1-\lambda_1 E(S)\right](1-\lambda_2 d)}{\lambda_1 dE(S)(1-\lambda_2 d)+\left[1-\lambda_1 E(S)\right]} \tag{4.57}$$

where E(S) is the mean service time of a priority unit.

The utilization factor of the system is given by

$$\rho=1-Q=\frac{\lambda_1 dE(S)(1-\lambda_2 d)\left[1-\lambda_1 E(S)\right](\lambda_2 d)}{\lambda_1 dE(S)(1-\lambda_2 d)+\left[1-\lambda_1 E(S)\right]} \tag{4.58}$$

The stability condition, under which the steady state exists, emerges from (4.57 and (4.58). This condition is given by

$$0<\frac{\lambda_1 dE(S)(1-\lambda_2 d)\left[1-\lambda_1 E(S)\right](\lambda_2 d)}{\lambda_1 dE(S)(1-\lambda_2 d)+\left[1-\lambda_1 E(S)\right]}<1. \tag{4.59}$$

All results in (4.53) to (4.59) agree with the results of Madan and Abu-Dayyeah [15].

We may point out that with suitable substitutions, the main results of this paper will reduce to many other particular cases including a combination of $M/E_k/1$ and $M/D/1$ queues, a combination of $M/M/1$ and $M/D/1$ queues, the case when no priority units arrive at the system and the case when no non-priority units arrive at the system. Further, with $p = 0$, the results of all the particular cases of this paper agree with the corresponding particular cases of Madan and Abu-Dayyeah [9].

7. References

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