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Portfolio Optimization without the Self-Financing Assumption

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Abstract

In this paper, we relax the assumption of a self-financing strategy in the dynamic investment models. In so doing we provide smooth solutions and constrained viscosity solutions.

Keywords: Portfolio, Investment, Stochastic, Viscosity Solutions, Self Financing

1. Introduction

The literature on dynamic portfolio optimization is vast. However, previous literature on dynamic investment relied on the assumption of a self-financing strategy; that is, the investor cannot add or withdraw funds during the trading horizon. Examples include [1], [2], [3] and [4] among many others. However, this assumption is somewhat restrictive and sometimes unrealistic.

Moreover, even with the assumption of a self-financing strategy, the previous literature usually provided explicit solutions under the assumption of a logarithmic or power utility function. Therefore, the assumption of a self- financing strategy did not offer a significant simplification of the solutions. Therefore, the self-financing assumption needs to be relaxed.

Consequently, the goal of this paper is to relax the assumption of self-financing strategies. In this paper, we show that the assumption of a self-financing strategy can be relaxed without a significant complication of the optimal solutions. In so doing, we present a stochastic-fac- tor incomplete-markets investment model and provide both smooth solutions and constrained viscosity solutions.

2. The Model

We consider an investment model, which includes a risky asset, a risk-free asset and a random external economic factor (see, for example, [5]). We use a three-dimensional standard Brownian motion $\{W_{1s}, W_{2s}, W_{3s}, F_s\}_{t \le s \le T}$ on the probability space (Ω, F_s, P), where $\{F_s\}_{0 \le s \le T}$ is the augmentation of filtration. The risk-free asset price process is $S_0 = e^{\int_t^T r(Y_s)\mathrm{d}s}$, where $r(Y_s)$ ä $C_b^2(R)$ is the rate of return and Y_s is the stochastic economic factor.

The dynamics of the risky asset price are given by

$$\mathrm{d}S_s = S_s\left\{\mu(Y_s)\mathrm{d}s + \sigma_1(Y_s)\mathrm{d}W_{1s}\right\}, \tag{1}$$

where $\mu(Y_s)$ and $\sigma_1(Y_s)$ are the rate of return and the volatility, respectively. The economic factor process is given by

$$\mathrm{d}Y_s = b(Y_s)\mathrm{d}s + \sigma_2(Y_s)\mathrm{d}W_{2s}, Y_t = y, \tag{2}$$

where $\sigma_2(Y_s)$ is its volatility and $b(Y_s)$ ä $C^1(R)$.

The amount of money added to or withdrawn from the investment at time s is denoted by Φ_s, and its dynamics are given by

$$\mathrm{d}\Phi_s = a(Y_s)\mathrm{d}s + \sigma_3(Y_s)\mathrm{d}W_{3s}, \tag{3}$$

where $\sigma_3(Y_s)$ is its volatility and $a(Y_s)b(Y_s)$ ä $C^1(R)$. Thus the wealth process is given by

$$\begin{aligned} X_T^\pi = x &+ \int_t^T a(Y_s)\mathrm{d}s + \int_t^T \sigma_3(Y_s)\mathrm{d}W_{3s} \\ &+ \int_t^T \left\{r(Y_s)X_s^\pi + \left(\mu(Y_s) - r(Y_s)\pi_s\right)\right\}\mathrm{d}s \\ &+ \int_t^T \pi_s\sigma_1(Y_s)\mathrm{d}W_{1s}, \end{aligned} \tag{4}$$

where x is the initial wealth, $\{\pi_s, F_s\}_{t \le s \le T}$ is the portfolio process with $E\int_t^T \sigma_1^2(Y_s)\pi_s^2\mathrm{d}s < \infty$.

The investor's objective is to maximize the expected utility of the terminal wealth

$$V(t,x,y)=\sup_{\pi_t} E\left[u\left(X_T^{\pi}\right)\middle|F_t\right], \tag{5}$$

where $V(.)$ is the value function, $u(.)$ is a differentiable, bounded and concave utility function.

Under regularity conditions, the value function is differentiable and thus satisfies the Hamiltonian-Jacobi-Bellman PDE

$$\begin{aligned}
&V_t+\left[r(y)x+a(y)\right]V_x+b(y)V_y\\
&+\frac{1}{2}\sigma_2^2(y)V_{yy}+\rho_{23}\sigma_2(y)\sigma_3(y)V_{xy}+\frac{1}{2}\sigma_3^2(y)V_{xx}\\
&+\sup_{\pi_t}\Big\{\pi_t\left(\mu(y)-r(y)\right)V_x\\
&\qquad+\left[\frac{1}{2}\pi_t^2\sigma_1^2(y)+\rho_{13}\sigma_1(y)\sigma_3(y)\pi_t\right]V_{xx}\\
&\qquad+\rho_{12}\sigma_1(y)\sigma_2(y)\pi_t V_{xy}\Big\}\\
&=0,
\end{aligned}$$

$$V(T,x,y)=u(x), \tag{6}$$

where ρ_{ij} is the correlation coefficient between the Brownian motions. Hence, the optimal solution is

$$\begin{aligned}
\pi_t^* = &-\frac{\left(\mu(y)-r(y)\right)V_x+\rho_{12}\sigma_1(y)\sigma_2(y)V_{xy}}{\sigma_1^2(y)V_{xx}}\\
&-\rho_{13}\sigma_1^{-1}(y)\sigma_3(y).
\end{aligned} \tag{7}$$

Similar to the previous literature, an explicit solution can be obtained for specific forms of utility such as a logarithmic utility function.

3. Viscosity Solutions

We can apply the constrained viscosity solutions to (6), given the HJB is degenerate elliptic and monotone increasing in V (see, for example, [6]).

Consider this HJB

$$\begin{aligned}
H\left(x,V(x),V_x(x),V_{xx}(x)\right)=0, x \in \Omega,\\
V(x)=g(x), x \in \partial\Omega,
\end{aligned} \tag{8}$$

where Ω is a bounded open set.

Definition 1 A continuous function $V(x)$ is a viscosity subsolution of (6) if

$$\begin{aligned}
&H\left(x,V(x),P,X\right)\le 0, \forall P \in D^{+}V(x),\\
&\forall X \in J^{+}V(x), \forall x \in \Omega
\end{aligned} \tag{9}$$

A continuous function $V(x)$ is a viscosity supersolution of (8) if

$$\begin{aligned}
&H\left(x,V(x),P,X\right)\ge 0, \forall P \in D^{-}V(x),\\
&\forall X \in J^{-}V(x), \forall x \in \Omega,
\end{aligned} \tag{10}$$

where

$$D^{+}V(x)=\left\{P:\limsup_{y\to x}\frac{V(y)-V(x)-\langle P,y-x\rangle}{|y-x|}\le 0\right\}, \tag{11}$$

$$D^{-}V(x)=\left\{P:\liminf_{y\to x}\frac{V(y)-V(x)-\langle P,y-x\rangle}{|y-x|}\ge 0\right\}, \tag{12}$$

are the super-differential and sub-differential, respectively; and

$$\begin{aligned}
&J^{2+}V(x)=\Big\{(P,X):\\
&\limsup_{y\to x}\frac{V(y)-V(x)-\langle P,y-x\rangle-\frac{1}{2}\langle X(y-x),y-x\rangle}{|y-x|}\\
&\le 0\Big\},
\end{aligned} \tag{13}$$

$$\begin{aligned}
&J^{2-}V(x)=\Big\{(P,X):\\
&\liminf_{y\to x}\frac{V(y)-V(x)-\langle P,y-x\rangle-\frac{1}{2}\langle X(y-x),y-x\rangle}{|y-x|}\\
&\ge 0\Big\},
\end{aligned} \tag{14}$$

are the superject and subject, respectively. A function $V(x)$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Proposition 1 $V(x)$ is the unique constrained viscosity solution of (6).

Proof Let $V \in C(\partial\Omega)$ and let $s(V)$ and $i(V)$ be the upper and lower semicontinuous envelopes of V, respectively, where

$$s(V)=\sup\left\{u(x):u_1\le u\le u_2\right\},$$

$$i(V)=\inf\left\{u(x):u_1\le u\le u_2\right\},$$

where u_1 and u_2 are sub-solution and super-solution, respectively.

Thus $s(V)\in USC(\bar{\Omega})$ and $i(V)\in LSC(\bar{\Omega})$ are a viscosity subsolution and supersolution, respectively. At the boundary we have

$$V(x)=s(V)=i(V),$$

by the comparison principle

$$s(V)\le i(V) \text{ in } \Omega.$$

By definition $s(V)\ge i(V)$ and

thus

$$V(x) = s(V) = i(V) \text{ in } \overline{\Omega}$$

is the unique viscosity solution.

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