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Symmetry Reduction and Explicit Solutions of the (2 + 1)-Dimensional DLW Equation

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Abstract

Utilizing the Clarkson-Kruskal direct method, the symmetry of the (2 + 1)-dimensional dispersive long wave equation is derived. From which, through solving the characteristic equations, four types of the explicit reduction solutions that related the hyperbolic tangent function are obtained. Finally, several soliton excitations are depicted from one of the solutions.

Keywords

Dispersive Long Wave Equation, Symmetry Reduction, Explicit Solution, Soliton Excitation

1. Introduction

Soliton theory, one of the typical topics in nonlinear science, has been widely applied in optics of nonlinear media, photonics, plasmas, mean-field theory of Bose-Einstein condensates, condensed matter physics, and many other fields. For describing these nonlinear physical phenomena, the study of symmetry is a very important approach, especially in the integrable nonlinear partial differential equations (NPDEs) for the sake of the existence of symmetries in infinity. Traditionally, there are three powerful methods to find the symmetry structure of the NPDEs, that is, the Lie group method of infinitesimal transformations, the nonclassical Lie group method and the Clarkson and Kruskal (CK) direct method [1]-[5]. Among them, the classical Lie symmetries of the partial differential equations (PDEs) can be obtained through the Lie group method of infinitesimal transformations. Using the basic prolongation method and the infinitesimal criterion of invariance, one can find some particular Lie point symmetries group of the NPDEs.

In Section 2 of this paper, a (2 + 1)-dimensional dispersive long wave (DLW) equation is taken to illustrate the symmetry reduction related the CK direct method. Section 3 is a direct result which the explicit reduction solutions are solved and the soliton excitations are depicted. Section 4 is the conclusion.

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2. Symmetry Structure through the Direct Approach

In the following of this paper, we fucus on the (2 + 1)-dimensional dispersive long wave (DLW) equation

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \quad v_t + u_x v + u v_x + u_{xxy} = 0 \tag{1}$$

The system was first derived by Boiti et al. as a compatibility for a weak Lax pair [6]. In Ref. [7], Paquin and Winternitz showed that the symmetry algebra of Equation (1) is infinite-dimensional and a Kac-Moody-Virasoro structure. The more general symmetry algebra, W_∞ symmetry algebra, was given in Ref. [8]. In Ref. [9], Lou gave out nine types of two dimensional similarity reductions and thirteen types of ordinary differential equation reductions. In Ref. [10], Lou showed that Equation (1) has no Painléve property, though the system is Lax or 1ST integrable. Abundant propagating localized excitations were also derived by Lou [11] [12] with the help of Painlvé-Bäcklund transformation and a multilinear variable separation approach. With the aid of a projective Riccati equation approach and by introducing appropriate lower-dimensional localized patterns, abundant coherent soliton excitations, that is, solitons, chaos and fractals were derived by ours [13]-[15].

According to the Clarkson and Kruskal (CK) direct method [4] [5], we first seek the similarity reduction of Equation (1) in the form of

$$u = \alpha_1 + \beta_1 U(\xi,\eta,\tau), \quad v = \alpha_2 + \beta_2 V(\xi,\eta,\tau) \tag{2}$$

where $\alpha_i \equiv \alpha_i(x,y,t)$, $\beta_i \equiv \beta_i(x,y,t)$ $(i=1,2)$ and $\xi \equiv \xi(x,y,t)$, $\eta \equiv \eta(x,y,t)$, $\tau \equiv \tau(x,y,t)$ are all differentiable functions to be determined, $U \equiv U(\xi,\eta,\tau)$, $V \equiv V(\xi,\eta,\tau)$ satisfy the following DLW equation as Equation (1)

$$U_{\eta\tau} + V_{\xi\xi} + U_\xi U_\eta + U U_{\xi\eta} = 0, \quad V_\tau + U_\xi V + U V_\xi + U_{\xi\xi\eta} = 0 \tag{3}$$

The result of the symbolic computation, one can deduce

$$\begin{aligned} &\alpha_1 = -\frac{F_{1,t}(t)x + F_{2,t}(t)}{F_1(t)}, \quad \alpha_2 = 0, \quad \beta_1 = F_1(t), \quad \beta_2 = F_1(t)F_{3,y}(y), \\ &\xi = F_1(t)x + F_2(t), \quad \eta = F_3(y), \quad \tau = \int_0^t F_1^2(s)\,\mathrm{d}s, \end{aligned} \tag{4}$$

where $F_1(t)$, $F_2(t)$ and $F_3(y)$ are arbitrary functions of their own variables.

Utilizing the variable separation solution of Equation (3) which derived in Refs. [13] [14]

$$U = \frac{\chi_{\xi\xi}(\xi,\tau) - \chi_\tau(\xi,\tau)}{\chi_\xi(\xi,\tau)} - \frac{2\chi_\xi(\xi,\tau)}{\chi(\xi,\tau)+\psi(\eta)}, \quad V = -\frac{2\chi_\xi(\xi,\tau)\psi_\eta(\eta)}{\left[\chi(\xi,\tau)+\psi(\eta)\right]^2} \tag{5}$$

the similarity solution of Equation (1) can be written

$$u = -\frac{F_{1,t}(t)x + F_{2,t}(t)}{F_1(t)} + F_1(t)\left[\frac{\chi_{\xi\xi}(\xi,\tau) - \chi_\tau(\xi,\tau)}{\chi_\xi(\xi,\tau)} - \frac{2\chi_\xi(\xi,\tau)}{\chi(\xi,\tau)+\psi(\eta)}\right], \quad v = -\frac{2F_1(t)F_{3,y}(y)\chi_\xi(\xi,\tau)\psi_\eta(\eta)}{\left[\chi(\xi,\tau)+\psi(\eta)\right]^2} \tag{6}$$

where $\chi(\xi,\tau)$ and $\psi(\eta)$ are two arbitrary variable separation functions of (ξ,τ) and η, respectively.

Second, under the transformation

$$F_1(t) = 1 + \epsilon f_{1,t}(t), \quad F_2(t) = \epsilon f_2(t), \quad F_3(y) = y + \epsilon f_3(y) \tag{7}$$

with the infinitesimal parameter ϵ, we have

$$\xi = x + \epsilon\left[x f_{1,t}(t) + f_2(t)\right], \quad \eta = y + \epsilon f_3(y), \quad \tau = t + 2\epsilon f_1(t) \tag{8}$$

$$\alpha_1 = -\epsilon\left[f_{1,tt}(t)x + f_{2,t}(t)\right], \quad \alpha_2 = 0, \quad \beta_1 = 1 + \epsilon f_{1,t}(t), \quad \beta_2 = 1 + \epsilon\left[f_{3,y}(y) + f_{1,t}(t)\right] \tag{9}$$

then Equation (2) can reduce to

$$u \to \epsilon\sigma(u) + u, \quad v \to \epsilon\sigma(v) + v \tag{10}$$

$$\sigma(u)=\left[xf_{1,t}(t)+f_2(t)\right]u_x+f_3(y)u_y+2f_1(t)u_t+f_{1,t}(t)u-f_{1,tt}(t)x-f_{2,t}(t) \tag{11}$$

$$\sigma(v)=\left[xf_{1,t}(t)+f_2(t)\right]v_x+f_3(y)v_y+2f_1(t)v_t+\left[f_{3,y}(y)+f_{1,t}(t)\right]v \tag{12}$$

The equivalent vector expression of the above symmetry is

$$V=\left[xf_{1,t}(t)+f_2(t)\right]\partial_x+f_3(y)\partial_y+2f_1(t)\partial_t+\left[f_{1,t}(t)u-f_{1,tt}(t)x-f_{2,t}(t)\right]\partial_u+\left[f_{3,y}(y)+f_{1,t}(t)\right]\partial_v \tag{13}$$

When taking

$$f_1(t)=m_1t+m_2,\quad f_2(t)=m_3t+m_4,\quad f_3(y)=m_5y+m_6 \tag{14}$$

from Equation (13), the following six operators are obtained

$$K_1=\partial_x,\ K_2=\partial_y,\ K_3=2\partial_t,\ K_4=t\partial_x+\partial_u,\ K_5=y\partial_y-v\partial_v,\ K_6=x\partial_x+2t\partial_t-u\partial_u-v\partial_v \tag{15}$$

Hence, we obtain the commutator table listed in **Table 1** with the (i,j)-th entry indicating $\left[K_i,K_j\right]$ according to the commutator operators $\left[K_p,K_q\right]=K_pK_q-K_qK_p$.

3. Reduction Solutions

Solving the following characteristic equations

$$\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}y}{Y}=\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}u}{\Phi_1}=\frac{\mathrm{d}v}{\Phi_2} \tag{16}$$

with $X=xf_{1,t}(t)+f_2(t)$, $Y=f_3(y)$, $T=2f_1(t)$, $\Phi_1=f_{1,t}(t)U-f_{1,tt}(t)x-f_{2,t}(t)$ and $\Phi_2=f_{3,y}(y)+f_{1,t}(t)$, we can obtain four types of similarity reductions of Equation (1).

1) Taking $f_1(t)=c_1,\ f_2(t)=c_2,$ we get the similarity reduction of Equation (1) through solving Equations (16)

$$\begin{aligned}u_1&=-\frac{B}{A}+2A\tanh\left[A\left(x-c_2\Delta_1\right)+B\left(t-2c_1\Delta_1\right)\right],\\ v_1&=\frac{2B\left(2Ac_1+Bc_2\right)\left\{\tanh^2\left[A\left(x-c_2\Delta_1\right)+B\left(t-2c_1\Delta_1\right)\right]-1\right\}}{f_3(y)},\end{aligned} \tag{17}$$

where $\Delta_1=\int_0^y\frac{\mathrm{d}s}{f_3(s)}$.

2) Taking $f_1(t)=c_1,\ f_3(y)=0,$ we get the similarity reduction of Equation (1) through solving Equations (16)

Table 1. Lie Bracket.

$\left[K_i,K_j\right]$	K_1	K_2	K_3	K_4	K_5	K_6
K_1	0	0	0	0	0	K_1
K_2	0	0	0	0	K_2	0
K_3	0	0	0	$2K_1$	0	$2K_2$
K_4	0	3	$-2K_1$	0	0	$-K_4$
K_5	0	$-K_2$	0	0	0	0
K_6	$-K_1$	0	$-2K_2$	K_4	0	0

$$u_2=\frac{f_2(t)-4Ac_1\tanh\left[A\left(x-\frac{\Delta_2}{2c_1}\right)+By\right]}{2c_1},\quad v_2=2AB\left\{1-\tanh^2\left[A\left(x-\frac{\Delta_2}{2c_1}\right)+By\right]\right\},\tag{18}$$

where $\Delta_2=\int_0^t f_2(s)\mathrm{d}s$

3) Taking $f_1(t)=c_1,\ f_2(t)=0,\ f_3(y)=c_3,$ we get the similarity reduction of Equation (1) through solving Equations (16)

$$u_3=-\frac{B}{A}+2A\tanh\left[Ax+\frac{B(c_3t-2c_1y)}{c_3}\right],\quad v_3=\frac{4ABc_1\left\{\tanh^2\left[Ax+\frac{B(c_3t-2c_1y)}{c_3}\right]-1\right\}}{c_3}.\tag{19}$$

4) Taking $f_1(t)=0,\ f_3(y)=0$ we get the similarity reduction of Equation (1) through solving Equations (16)

$$u_4=\frac{xf_{2,t}(t)}{f_2(t)}+\omega_2(t)+\frac{\omega_3(y)}{f_2(t)},\qquad v_4=\frac{\omega_1(y)}{f_2(t)}\tag{20}$$

As we all know, to derive soliton structures of a explicit solution is a meaningful task fo a nonlinear physical equation. Now, when taking $V=-|v_1|$ (v_1 is a solution in Equation (17)), a fundmental soliton, that is a dromion-like structure, is found if the constants $A=c_1=c_2=1$, $B=0$, the function $f_3(y)=\frac{1-0.1^3\tanh^2(1+y)}{\operatorname{sech}(1+y)}$ and the fixed time $t=-3$ (**Figure 1(a)**). A further considering is, when the constants $A=B=c_1=c_2=1$ and the function $f_3(y)=1$, $V=-|v_1|=-\left|6\left[\tanh^2(x-3y+t-1)-1\right]\right|$. For this time, a typical line soliton is derived for the fixed time $t=-3$ (**Figure 1(b)**).

The solitoff is another special type of coherernt structure for a nonlinear equation, where the wave fields decays exponentially in all directions except for a preferred direction [16]. For the field $V=-|v_1|$, the solitoff structure can also be constructed. **Figure 2(a)** shows a two-solitoff solution when taking the constants $A=c_1=c_2=1$, $B=0$, the function $f_3(y)=\frac{1}{\tanh(y)}$ and $V=-|v_1|=-2\left|\left\{\tanh^2\left[x-\ln(\cosh(y))\right]-1\right\}\tanh(y)\right|$. After adjusting the constants $A=B=c_1=c_2=1$ and the

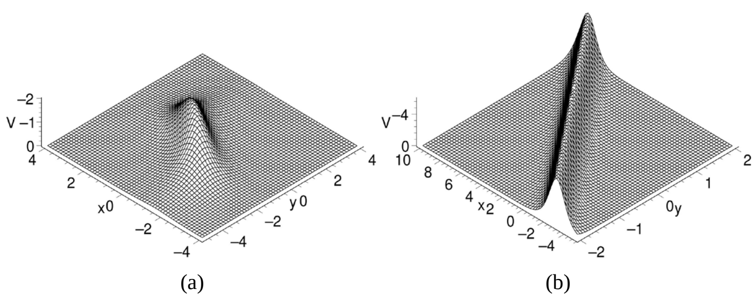
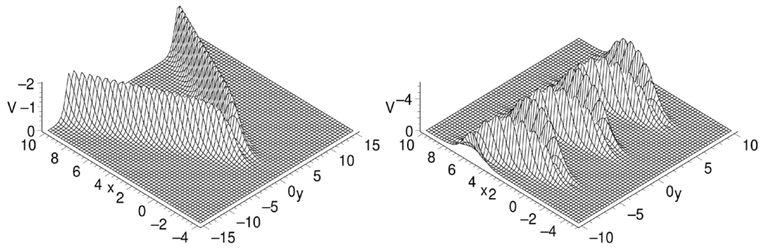


Figure 1. (a) A dromion structure of the solution (17) when taking the constants $A=c_1=c_2=1$, $B=0$ and the function $f_3(y)=\frac{1-0.1^3\tanh^2(1+y)}{\operatorname{sech}(1+y)}$; (b) The line soliton structure of the solution (17) when taking the constants $A=B=c_1=c_2=1$ and the function $f_3(y)=1$.



(a) (b)

Figure 2. (a) A two-solitoff soliton of the solution (17) when taking the constants $A = c_1 = c_2 = 1$, $B = 0$ and the function $f_3(y) = \frac{1}{\tanh(y)}$; (b) The periodic dromion solitons of the solution (17) when taking the constants $A = B = c_1 = c_2 = 1$ and the function $f_3(y) = \frac{1}{\sin(y)}$.

function $f_3(y) = \frac{1}{\sin(y)}$, the field $V = -|v_1| = -6\left|\left\{\tanh^2\left[3\cos(y) + x + t\right] - 1\right\}\sin(y)\right|$, the periodic dromion solitons is depicted (**Figure 2(b)**).

4. Summary and Conclusion

In summary, we have obtained the symmetry reduction with the aid of the CK direct method and some explicit solutions through solving the characteristic equations of the (2 + 1)-dimensional DLW equation. These obtained solutions contain several free functions of variables x, y and t, which provide us with more chose of these functions to generate the abundant soliton structures. Here, we chose several types of elementary functions to exhibit these soliton propagations related to the obtained solutions. These solutions may provide more information to further study the nonlinear physical system.

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References

[1] Bluman, G.W. and Cole, J.D. (1974) Similarity Methods for Differential Equations. Springer-Verlag, Berlin. http://dx.doi.org/10.1007/978-1-4612-6394-4

[2] Bluman, G.W. and Kumei, S. (1989) Symmetries and Differential Equations. Springer-Verlag, Berlin. http://dx.doi.org/10.1007/978-1-4757-4307-4

[3] Olver, P.J. (1993) Applications of Lie Groups to Differential Equations. Springer-Verlag, New York. http://dx.doi.org/10.1007/978-1-4612-4350-2

[4] Clarkson, P.A. and Kruskal, M.D. (1989) New Similarity Reductions of the Boussinesq Equation. *Journal of Mathematical Physics*, **30**, 2201-2213. http://dx.doi.org/10.1063/1.528613

[5] Clarkson, P.A. and Mansfield, E.L. (1994) Algorithms for the Nonclassical Method of Symmetry Reductions. *SIAM Journal on Applied Mathematics*, **54**, 1693-1719. http://dx.doi.org/10.1137/S0036139993251846

[6] Boiti, M., Leon, J.J.P. and Pempinelli, F. (1987) Integrable Two-Dimensional Generalisation of the Sine and Sinh-Gordon Equations. *Inverse Problems*, **3**, 37-50. http://dx.doi.org/10.1088/0266-5611/3/1/009

[7] Paquin, G. and Winternitz, P. (1990) Group Theoretical Analysis of Dispersive Long Wave Equations in Two Space

Dimensions. *Physica D*, **46**, 122-138. http://dx.doi.org/10.1016/0167-2789(90)90115-6

[8] Lou, S.Y. (1994) Symmetries and Algebras of the Integrable Dispersive Long Wave Equations in 2+1-Dimensional Spaces. *Journal of Physics A*: *Mathematical and General*, **27**, 3235-3243. http://dx.doi.org/10.1088/0305-4470/27/9/033

[9] Lou, S.Y. (1995) Similarity Solutions of Dispersive Long Wave Equations in Two Space Dimensions. *Mathematical Methods in the Applied Sciences*, **18**, 789-802. http://dx.doi.org/10.1002/mma.1670181004

[10] Lou, S.Y. (1993) Painlevé Test for the Integrable Dispersive Long Wave Equations in Two Space Dimensions. *Physics Letters A*, **176**, 96-100. http://dx.doi.org/10.1016/0375-9601(93)90322-Q

[11] Tang, X.Y. and Lou, S.Y. (2003) Extended Multilinear Variable Separation Approach and Multivalued Localized Excitations for Some (2 + 1)-Dimensional Integrable Systems. *Journal of Mathematical Physics*, **44**, 4000-4025. http://dx.doi.org/10.1063/1.1598619

[12] Tang, X.Y., Lou, S.Y. and Zhang, Y. (2002) Localized Excitations in (2 + 1)-Dimensional Systems. *Physical Review E*, **66**, Article ID: 046601. http://dx.doi.org/10.1103/PhysRevE.66.046601

[13] Ma, Z.Y., Liu, Y.L., Lu, Z.M. and Zheng, C.L. (2006) Solitons and Waves in (2 + 1)-Dimensional Dispersive Long-Wave Equation. *Communications in Theoretical Physics*, **46**, 799-803. http://dx.doi.org/10.1088/0253-6102/46/5/006

[14] Ma, Z.Y. and Hu, Y.H. (2007) Solitons, Chaos and Fractals in the (2 + 1)-Dimensional Dispersive Long Wave Equation. *Chaos*, *Solitons & Fractals*, **34**, 1667-1676. http://dx.doi.org/10.1016/j.chaos.2006.04.073

[15] Ma, Z.Y. (2007) The Projective Riccati Equation Expansion Method and Variable Separation Solutions for the Nonlinear Physical Differential Equation in Physics. *Chinese Physics B*, **16**, 1848-1854. http://dx.doi.org/10.1088/1009-1963/16/7/007

[16] Ruan, H.Y. (2001) Study of Solitons Interaction in Integrable Models. *Acta Physica Sinica*, **50**, 369-376.