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A New Method of Estimating the Asset Rate of Return

Moawia Alghalith, Tracy Polius
Economics, University of the West Indies, St Augustine, USA
E-mail: malghalith@gmail.com, tracy.polius@sta.uwi.edu
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Abstract

We present a new consumption-based method of estimating the asset rate of return.

Keywords: Return, Investment, Portfolio, Asset, Stochastic, Consumption CAPM

In this note, we present a new model that links the stock/portfolio rate of return to consumption. Our approach is more general than the existing models such as the consumption-CAPM models, that are based on very restrictive assumptions [1]. In so doing, we utilize a more advanced and appropriate theoretical and empirical framework than the ones used by previous literature. It is worth noting that previous literature mainly used simple linear regressions without a rigorous theoretical basis.

We use a stochastic factor model, which includes a risky asset (portfolio, a risk-free asset and a stochastic external economic factor [2,3]. Thus, we have a two-dimensional standard Brownian motion $\left\{\left(W_{1s}, W_{2s}\right), \mathcal{F}_s\right\}_{t \le s \le T}$ on the probability space $\left(\Omega, \mathcal{F}_s, P\right)$, where $\left\{\mathcal{F}_s\right\}_{t \le s \le T}$ is the augmentation of filtration. The risk-free asset price process is $S_0 = \mathrm{e}^{\int_t^T r(Z_s)\mathrm{d}s}$, where $r\left(Z_s\right) \in C_b^2\left(R\right)$, is the rate of return and Z_s is the stochastic economic factor.

The dynamics of the risky asset price is given by

$$\mathrm{d}S_s = S_t\left\{\mu\left(Z_s\right)\mathrm{d}t + \sigma\left(Z_s\right)\mathrm{d}W_s^1\right\}, \tag{1}$$

where $\mu\left(Z_t\right)$ and $\sigma\left(Z_t\right)$ are the rate of return and the volatility, respectively. The stochastic economic factor process is defined as

$$\mathrm{d}Z_s = a\left(z_s\right)\mathrm{d}t + \rho \mathrm{d}W_{1s} + \gamma\sqrt{1-\rho^2}\,\mathrm{d}W_{2s}, Z_t = z, \tag{2}$$

where $\left|\rho\right| < 1$ is the correlation factor between the two Brownian motions, γ is a parameter, and $a\left(Z_s\right) \in C^1\left(R\right)$ has a bounded derivative.

The wealth process is given by

$$X_T^{\pi,c} = x + \int_t^T \left\{r\left(Y_s\right)X_s^{\pi,c} + \left(\mu\left(Y_s\right) - r\left(Y_s\right)\pi_s\right) - c_s\right\}\mathrm{d}s + \int_t^T \pi_s\sigma\left(Y_s\right)\mathrm{d}W_s^1, \tag{3}$$

where x is the initial wealth, $\left\{\pi_t, \mathcal{F}_s\right\}_{t \le s \le T}$ is the portfolio process and $\left\{c_t, \mathcal{F}_s\right\}_{t \le s \le T}$ is the consumption process, with $\int_t^T \pi_s^2 \mathrm{d}s < \infty$, $\int_t^T c_s \mathrm{d}s < \infty$ and $c \ge 0$. The trading strategy $\left(\pi_s, c_s\right) \in A\left(x, y\right)$ is admissible.

The investor's objective is to maximize the expected utility of terminal wealth and consumption

$$v\left(t, x, y\right) = \sup_{\pi_t, c_t} E\left[U_1\left(X_T^{\pi,c}\right) + \int_t^T U_2\left(c_s\right)\mathrm{d}s \middle| \mathcal{F}_t\right], \tag{4}$$

where $v(.)$ is the (smooth) value function, $U(.)$, bounded and strictly concave utility function.

The value function satisfies the Hamiltonian-Jacobi-Bellman PDE

$$\begin{gathered} v_t + r\left(y\right)xv_x + a\left(z\right)v_y + \frac{1}{2}v_{yy} \\ + \sup_{\pi_t, c}\left\{\frac{1}{2}\pi_t^2\sigma^2\left(z\right)v_{xx} + \left[\pi_t\left(\mu\left(z\right) - r\left(z\right)\right) - c_t\right]v_x \right. \\ \left. + \gamma\rho\sigma\left(z\right)\pi_t v_{xy} + u_2\left(c_t\right)\right\} = 0, \\ v\left(T, x, z\right) = U\left(x\right), \end{gathered} \tag{5}$$

Hence, the optimal solutions are

$$\pi_t^* = -\frac{\left(\mu\left(z\right) - r\left(z\right)\right)v_x}{\sigma^2\left(z\right)v_{xx}} - \frac{\gamma\rho v_{xy}}{\sigma\left(z\right)v_{xx}}, \tag{6}$$

$$U_2'\left(c_t^*\right)=v_x. \tag{7}$$

Using the result of Alghalith [3], the optimal portfolio can be expressed as

$$\pi_t^*=-\frac{\left(\mu(z)-r(z)\right)\left(\alpha_1+\alpha_2 c_t^*\right)}{\sigma^2(z)}-\frac{\alpha_3\rho}{\sigma(z)}, \tag{8}$$

and thus

$$\mu(z)=r-\frac{\pi_t^*\sigma^2(z)+\alpha_3\rho\sigma(z)}{\left(\alpha_1+\alpha_2 c_t^*\right)}, \tag{9}$$

where α_i is a constant that can be estimated. Moreover, this formula allows us to determine the impact of consumption on the rate of the return of the portfolio, as follows

$$\frac{\partial\mu(z)}{\partial c_t^*}=\frac{\alpha_2\left(\mu(z)-r(z)\right)\left(\alpha_1+\alpha_2 c_t^*\right)+\alpha_2\left(\pi_t^*\sigma^2(z)+\alpha_3\rho\sigma(z)\right)}{\left(\alpha_1+\alpha_2 c_t^*\right)^2}. \tag{10}$$

1. Empirical Example

We used quarterly data for Jamaica for the period March 1998 to June 2010 for real private aggregate consumption (in millions of dollars), stock index (JSI) and the Treasury bill rate (r), and GDP (as the stochastic factor Z_s). We also computed the volatility of the index and the correlation factor between GDP and the JSI.

Using $(\ )-(\ )$, we estimated each of the following non-linear equations

$$\pi_t^*=-\frac{\left(\mu(z)-r(z)\right)\left(\beta_1+\beta_2 c_t^*\right)}{\sigma^2(z)}-\frac{\beta_3\rho}{\sigma(z)}+\varepsilon_1, \tag{11}$$

$$\mu(z)=r-\frac{\pi_t^*\sigma^2(z)+\beta_4\rho\sigma(z)}{\beta_5+\beta_6 c_t^*}+\varepsilon_3, \tag{12}$$

where $\beta_i s$ are the parameter to be estimated, while the other variables are observed data, and ε is the estimation error. Using the estimated values $\hat{\beta}_i s$, we obtain the following comparative statics

$$\frac{\partial\mu(z)}{\partial c_t^*}=\frac{\hat{\beta}_2\left(\mu(z)-r(z)\right)\left(\hat{\beta}_5+\hat{\beta}_6 c_t^*\right)+\hat{\beta}_6\left(\pi_t^*\sigma^2(z)+\hat{\beta}_4\rho\sigma(z)\right)}{\left(\hat{\beta}_5+\hat{\beta}_6 c_t^*\right)^2}. \tag{13}$$

In contrast, to previous studies that used simple linear regressions , the results support the existence of a very weak relationship between private consumption in Jamaica and the rate of return of the stock index (see **Table 1**).

Table 1. Empirical results.

$\hat{\beta}_2$	502 072.4	$\hat{\beta}_6$	−36 904.06
$\hat{\beta}_4$	−1.67E+12		
$\hat{\beta}_5$	29 470 778	$\frac{\partial\mu(z)}{\partial c_t^*}$	3.78E−11

2. References

[1] J. Cvitanic and F. Zapatero, "Introduction to the Economics and Mathematics of Financial Markets," MIT Press, Cambridge, 2004.

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[3] M. Alghalith, "General Closed-Form Solutions to the Dynamic Optimization Problem in Incomplete Markets," 2011. http://mpra.ub.uni-muenchen.de/21950/

