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Solution of Stochastic Non-Homogeneous Linear First-Order Difference Equations

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Abstract

In this paper, the closed form solution of the non-homogeneous linear first-order difference equation is given. The studied equation is in the form: $x_n = x_0 + b^n$, where the initial value x_0 and b, are random variables.

Keywords

Stochastic Linear Difference Equations, Random Variables, Closed Form Solution, Direct Transformation Technique

1. Introduction

A difference equation or dynamical system describes the evolution of some variable over time. The value of this variable in period t is denoted by x_n. The time index n takes on discrete values and typically runs over all integer numbers Z, e.g. $t = \cdots, -2, -1, 0, 1, 2, \cdots$. By interpreting t as the time index, we have automatically introduced the notion of past, present and future [1].

Many problems in Probability give rise to difference equations. Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics. Anyone who has made a study of differential equations will know that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with.

Aside from Probability, Computer Scientists take an interest in difference equations for a number of reasons. For example, difference equations frequently arise while determining the cost of an algorithm in big-O notation [2]. In engineering, difference equations arise in control engineering, digital signal processing, electrical networks, etc. In social sciences, difference equations arise to study the national income of a country and then its

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variation with time, Cobweb phenomenon in economics [3], etc. Similar to differential equation, difference equation is the most powerful instrument for the treatment of discrete processes.

Stochastic difference equations are difference equations with random parameters. Such equations arise in various disciplines, for example economics [4], physics, nuclear technology, biology and sociology. In this paper, we present a new technique based on the direct transformation technique to express analytically the probability density function of the general solution of stochastic linear 1st order difference equations.

2. Linear 1st Order Difference Equations

A typical linear non-homogeneous first order difference equation is given by:

$$x_n = a_n x_{n-1} + b_n, \quad x_0 = \alpha, \quad n \geq 0 \tag{1}$$

where $a_n \neq 0$, a_n and b_n are real-valued functions. The unique solution of (1) may be found as follows:

$$x_1 = a_1 x_0 + b_1$$

$$x_2 = a_2 x_1 + b_2 = a_2\left(a_1 x_0 + b_1\right) + b_2 = a_2 a_1 x_0 + a_2 b_1 + b_2$$

Now by induction, we can show:

$$x_n = \prod_{i=0}^{n-1} a_i x_0 + \sum_{r=0}^{n-1}\left(\prod_{i=r+1}^{n-1} a_i\right) b_r \tag{2}$$

To establish this, assume that (2) holds for order n. Then from (1) $x_{n+1} = a_{n+1} x_n + b_{n+1}$, hence (2) gives:

$$x_{n+1} = a_{n+1}\left(\prod_{i=0}^{n-1} a_i x_0 + \sum_{r=0}^{n-1}\left(\prod_{i=r+1}^{n-1} a_i\right) b_r\right) + b_{n+1} = \prod_{i=0}^{n} a_i x_0 + \sum_{r=0}^{n}\left(\prod_{i=r+1}^{n} a_i\right) b_r$$

Thus (2) holds for all $n \geq 0$.

There is a special case of (1) that is important in many engineering and business applications, is given by:

$$x_n = x_0 + bn \tag{3}$$

This case is studied throughout this paper where x_0 and b, are random variables.

3. Direct Transformation Technique

There are several versions of the Direct Transformation Technique in the literature [5]-[9]; we will provide a simple one for 1-dimension variable then for *N*-dimension variables.

1-*dimension variable*

Let X be a continuous random variable with probability density function (pdf), $f_X(x)$ and defined over an interval I. Let g: $I \to R$ be a continuous one-to-one function, *i.e.* its inverse h: $J \to I$ $(J \subset R)$. Let $Y = g(X)$, then the probability density function of Y, $f_Y(y)$ is given by:

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right) \times \left|\dfrac{\mathrm{d}}{\mathrm{dy}} g^{-1}(y)\right|, & \text{for } y \in J \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Example: X has probability density function given by $f_X(x) = \dfrac{\alpha}{x^\beta}$ (where x, α, and β are strictly positive real numbers). Let $Y = -2\mathrm{e}^x$, we want to find the probability density function of Y, $f_Y(y)$. Here $I = R_+^*$, $J = R_-^*$. Using (4), we have:

$$f_Y(y) = \begin{cases} \dfrac{\alpha}{\left[\ln\left(\dfrac{y}{-2}\right)\right]^\beta} \times \left|-\dfrac{1}{2y}\right|, & \text{for } y \in J \\ 0 & \text{otherwise} \end{cases}.$$

N-dimension variables

Let $(X_1, X_2, \cdots, X_n)$ n continuous random variable with given joint probability density function $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n)$. Consider a one-to-one transformation function $g: R^n \to R^n$ where $(Y_1, Y_2, \cdots, Y_n) = g(X_1, X_2, \cdots, X_n)$ is continuous with respect to every variable and its partial derivatives is continuous as well. Then the joint probability density function of $(Y_1, Y_2, \cdots, Y_n)$ is given by:

$$f_{Y_1 Y_2 \cdots Y_n}(y_1, y_2, \cdots, y_n) = f_{X_1 X_2 \cdots X_n}\left(g^{-1}(y_1, y_2, \cdots, y_n)\right) \left\| \begin{matrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{matrix} \right\|. \tag{5}$$

4. Application

In this section, we will solve the stochastic difference equation $x_n = x_0 + bn$ (given in (3)), where x_0 and b are random variables with the known joint probability density function, by finding the probability density function of x_n using (5). So in this stochastic difference equation we have two random variables (x_0, b) as input and one random variable (x_n) as output. For example, let $f(x_0, b) = \mathrm{e}^{-x_0 - b}$, $x_0 \geq 0, b \geq 0$. By using (5), let $y_1 = x_0$ and $y_2 = x_0 + bn$ with inverse $x_0 = y_1$ and $b = \frac{y_2 - y_1}{n}$. The joint probability density function of y_1 and y_2 (**Figure 1**) is given by:

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_0, B}\left(y_1, \frac{y_2 - y_1}{n}\right) \left\| \begin{matrix} 1 & 0 \\ \frac{-1}{n} & \frac{1}{n} \end{matrix} \right\| = \mathrm{e}^{-y_1 - \frac{y_2 - y_1}{n}} \times \frac{1}{n}.$$

Now to find the probability density function of x_n, *i.e.* y_2 we must find the marginal probability density function of Y_2 (**Figure 2**) by integrating out of Y_1:

$$f_{Y_2}(y_2 = x_n) = \int_0^{y_2} \mathrm{e}^{-y_1 - \frac{y_2 - y_1}{n}} \times \frac{1}{n} \mathrm{d}y_1 = \frac{1}{1-n}\left[\mathrm{e}^{-y_2} - \mathrm{e}^{-\frac{y_2}{n}}\right]$$

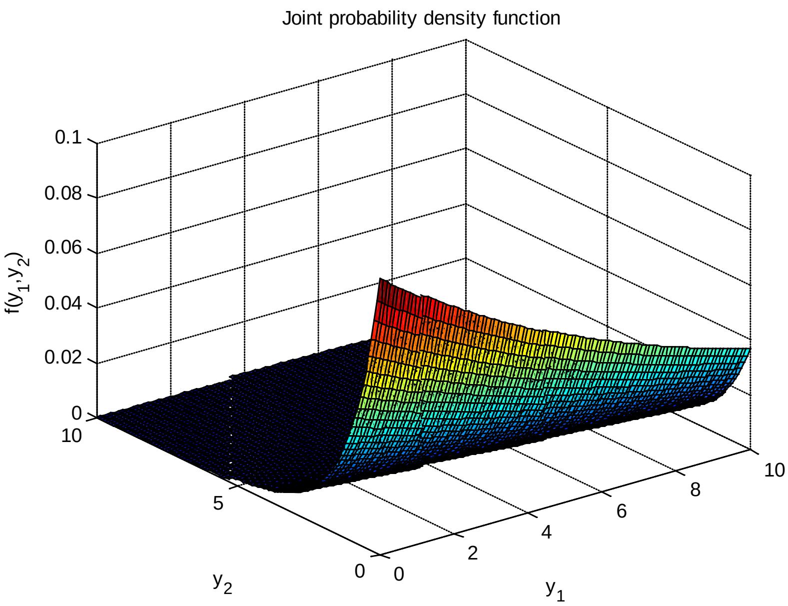

Figure 1. Joint probability density function of y_1 and y_2.

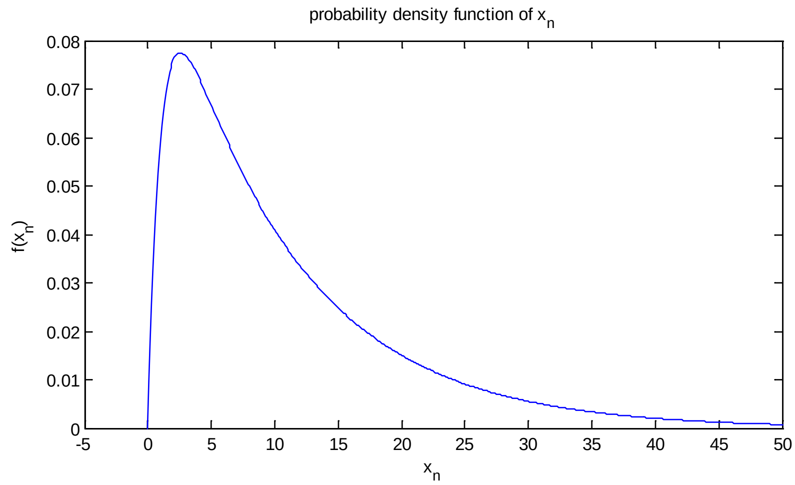

Figure 2. Probability density function of x_n.

We can show easily that $\int_0^\infty f(y_2)\,\mathrm{d}y_2 = 1$.

References

[1] Elaydi, S. (2004) An Introduction to Difference Equations. 3rd Edition, Springer-Verlag, New York.

[2] King, F. (2005) Difference Equations [PDF Document]. Retrieved from Lecture Notes Online Web Site: http://www.cl.cam.ac.uk/teaching/2004/Probability/

[3] Chui, C.K. and Chen, G. (1987) Kalman Filtering with Real-Time Applications. 2nd Edition, Springer-Verlag, New York. http://dx.doi.org/10.1007/978-3-662-02508-6

[4] Novak, S.Y. (2011) Extreme Value Methods with Applications to Finance. Chapman & Hall/CRC Press, London. http://dx.doi.org/10.1201/b11537

[5] Soong, T.T. (1973) Random Differential Equations in Science and Engineering. Academic Press, New York.

[6] Kadry, S.A. (2007) Solution of Linear Stochastic Differential Equation. USA: WSEAS Transactions on Mathematics, April 2007, 618.

[7] Kadry, S. (2012) Exact Solution of the Stochastic System of Difference Equations. *Journal of Mathematical Control Science and Applications* (*JMCSA*), **5**, 67-70.

[8] Kadry, S. and Younes, R. (2005) Étude Probabiliste d'un Système Mécanique à Paramètres Incertains par une Technique Basée sur la Méthode de transformation. *Proceedings of the* 20*th Canadian Congress of Applied Mechanics* (*CANCAM*' 05), Montreal, 30 May, 490-491.

[9] Kadry, S. (2012) Probabilistic Solution of Rational Difference Equations System with Random Parameters. *ISRN Applied Mathematics*, **2012**, Article ID: 290186. http://dx.doi.org/10.5402/2012/290186