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Banach Λ-Frames for Operator Spaces

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Abstract

The Banach frame for a Banach space $\mathcal{X}$ can reconstruct each vector in $\mathcal{X}$ by the pre-frame operator or the reconstruction operator. The Banach Λ-frame for operator spaces was introduced by Kaushik, Vashisht and Khattar [Reconstruction Property and Frames in Banach Spaces, *Palestine Journal of Mathematics*, 3(1), 2014, 11-26]. In this paper we give necessary and sufficient conditions for the existence of the Banach Λ-frames. A Paley-Wiener type stability theorem for Λ-Banach frames is discussed.

1. Introduction

Duffin and Schaeffer in [1] while working in nonharmonic Fourier series developed an abstract framework for the idea of time-frequency atomic decomposition by Gabor [2] and defined frames for Hilbert spaces. In 1986, Daubechies, Grossmann and Meyer [3] found new applications to wavelets and Gabor transforms in which frames played an important role.

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space with inner product $\langle .,. \rangle$. A system $\{f_k\} \subset \mathcal{H}$ is a *frame* (Hilbert) for $\mathcal{H}$ if there exist positive constants A and B such that

$$A\|f\|^2 \leq \left\| \{\langle f, f_k \rangle\} \right\|_{\ell^2}^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The positive constants A and B are called the *lower* and *upper bounds* of the frame $\{f_k\}$, respectively. They are not unique. The inequality (1.1) is called the *frame inequality* of the frame.

Gröchenig in [4] generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [5] [6] related to the *atomic decompositions*. Atomic decompositions appeared in the field of applied mathematics providing many applications [7].

An atomic decomposition allows a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call atoms. On the other hand Banach frames for a Banach space ensure reconstruction via a bounded linear operator or the *synthesis operator*.

Definition 1.1. [4]. Let $\mathcal{X}$ be a Banach space, $\mathcal{X}^*$ the conjugate space of $\mathcal{X}$ and let $\mathcal{X}_d$ be an associated Banach space of scalar valued sequences. A pair $\left(\{f_k^*\},\Theta\right)$ $\left(\{f_k^*\}\subset\mathcal{X}^*,\Theta:\mathcal{X}_d\to\mathcal{X}\right)$ is called a Banach frame for $\mathcal{X}$ with respect to an associated sequence space $\mathcal{X}_d$ if

1) $\left\{f_k^*(f)\right\}\in\mathcal{X}_d$, for each $f\in\mathcal{X}$.

2) There exist positive constants $\left(0 < A_0 \leq B_0 < \infty\right)$ such that

$$A\|f\| \leq \left\|\left\{f_k^*(f)\right\}\right\|_{\mathcal{X}_d} \leq B\|f\|, \text{ for each } f\in\mathcal{X}.$$

3) Θ is a bounded linear operator operator such that $\Theta\left(\left\{f_k^*(f)\right\}\right) = f, f\in\mathcal{X}$.

In the later half of twentieth century, Coifman and Weiss in [8] introduced the notion of atomic decomposition for function spaces. Later, Feichtinger and Gröchenig [5] [6] extended this idea to Banach spaces. This concept was further generalized by Gröchenig [4], who introduced the notion of Banach frames for Banach spaces. Casazza, Han and Larson [9] also carried out a study of atomic decompositions and Banach frames. For recent development in frames for Banach spaces one may refer to [10]-[17]. Recently, various generalizations of frames in Banach spaces have been introduced and studied. Han and Larson [18] defined a Schauder frame for a Banach space $\mathcal{X}$ to be an inner direct summand (*i.e.* a compression) of a Schauder basis of $\mathcal{X}$. The reconstruction property in Banach spaces was introduced and studied by Casazza and Christensen in [19] and further studied in [20]-[23]. The basic theory of frames can be found in [24]-[26].

Definition 1.2. [19]. Let $\mathcal{X}$ be a separable Banach space. A sequence $\{f_k^*\}\subset\mathcal{X}^*$ has the reconstruction property for $\mathcal{X}$ with respect to a sequence $\{f_k\}\subset\mathcal{X}$ if

$$f = \sum_{n=1}^{\infty} f_n^*(f) f_n, \text{ for all } f\in\mathcal{X}. \tag{1.2}$$

In short, we will say that the pair $\left(\{f_k\},\{f_k^*\}\right)$ has the reconstruction property for $\mathcal{X}$. More precisely, we say that $\left(\{f_k\},\{f_k^*\}\right)$ is a *reconstruction system* or the *reconstruction property* for $\mathcal{X}$.

The reconstruction property is an important tool in several areas of mathematics and engineering. The reconstruction property is also used to study the geometry of Banach spaces. In fact, it is related to the bounded approximated property as observed in [9] [27].

Recently, Kaushik *et al.* in [20] introduced Banach Λ-frame for operator spaces while working in the reconstruction property in Banach spaces. In this paper we give necessary and sufficient conditions for the existence of Banach Λ-frames for operator spaces. A Paley-Wiener type stability theorem for Λ-Banach frames is discussed.

2. Banach Λ-Frames

The reconstruction property in Banach spaces is a source of other *redundant systems*! For example, if $\{f_k^*\}$ has the reconstruction property for $\mathcal{X}$ with respect to $\{f_k\}\subset\mathcal{X}$. Then, we can find a reconstruction operator Θ such that $\mathcal{F}\equiv\left(\{f_k^*\},\Theta\right)$ is a Banach frame for $\mathcal{X}$. The Banach frame $\mathcal{F}$ is called the associated Banach frame for the underlying space. Similarly we can find a reconstruction operator associated with the system $\{f_k\}$. It is natural to ask whether we can find Banach frames for a large class of spaces associated with a given reconstruction system. In this direction the Banach Λ-frames for the operator spaces introduced in [20]. First recall that the family of all bounded linear operator from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X},\mathcal{Y})$. If $\mathcal{X}=\mathcal{Y}$, then we write $\mathcal{B}(\mathcal{X},\mathcal{Y})=\mathcal{B}(\mathcal{X})$. An operator $T\in\mathcal{B}(\mathcal{X},\mathcal{Y})$ is said to be coercive if there exists $a > 0$ such that $\|T(f)\| \geq a\|f\|$, for all $f\in\mathcal{X}$.

Definition 2.1. [20]. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{Y}_d$ be a sequence space associated with $\mathcal{Y}$. A sequence $\{f_k\}\subset\mathcal{X}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ if there exist positive constants $0 < A_0 \leq B_0 < \infty$ such that

$$A_0\|\Lambda\| \leq \left\|\left\{\Lambda(f_k)\right\}\right\|_{\mathcal{Y}_d} \leq B_0\|\Lambda\|, \text{ for all } \Lambda\in\mathcal{B}(\mathcal{X},\mathcal{Y}). \tag{2.1}$$

If upper inequality in (2.1) is satisfied, then $\{f_k\}$ is called a Λ-*Bessel sequence* for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ with Bessel bound B_0. The operator $S:\mathcal{Y}_d \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ given by $S(\{\Lambda(f_k)\})=\Lambda$, $\Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is called the *pre-frame operator* and the *analysis operator* $T:\mathcal{B}(\mathcal{X},\mathcal{Y})\to \mathcal{Y}_d$ is given by

$$T(\Lambda)=\{\Lambda(f_k)\}, \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}).$$

The positive constants A_0, B_0 are called the lower and *upper frame bounds* of the Banach Λ-frame, respectively. If the removal of any f_j from the Banach Λ-frame renders the collection $\{f_k\}_{j\neq k}$ to be a Banach Λ-frame for the underlying space, then $\{f_k\}$ is said to be an exact Banach Λ-frame.

Remark 2.2. If $\mathcal{Y}=\mathbb{K}$, then $\mathcal{B}(\mathcal{X},\mathcal{Y})\equiv \mathcal{X}^*$. Therefore, $\{\pi[\Lambda(f_k)]\}$ becomes a Banach frame for $\mathcal{X}^*$ with respect to the associated Banach space $\mathcal{Z}_0$.

Suppose that $\{f_k^*\}\subset \mathcal{X}^*$ has the reconstruction property for $\mathcal{X}$ with respect to $\{f_k\}$ where $\{f_k^*\}\subset \mathcal{X}^* \setminus \{0\}$. Let $\mathcal{Z}$ be a Banach space and let

$$\mathcal{Z}_d = \left\{ \{\xi_k\}\subset \mathcal{Z}: \sup_n \sup_{\substack{f\in \mathcal{X} \\ \|f\|\leq 1}} \left\| \sum_{k=1}^n f_k^*(f)\xi_k \right\| < \infty \right\}$$

be its associated Banach space of sequences with the norm given by

$$\|\{\xi_k\}\|_{\mathcal{Z}_d} = \sup_n \sup_{\substack{f\in \mathcal{X} \\ \|f\|\leq 1}} \left\| \sum_{k=1}^n f_k^*(f)\xi_k \right\|_{\mathcal{Z}}.$$

Then, $\{f_k\}$ is a Banach Λ-frame for the operator space $\mathcal{B}(\mathcal{X},\mathcal{Z})$ with respect to $\mathcal{Z}_d$. There may be other sequence spaces with respect to which $\{f_k\}$ form a Banach frame for the underlying space. The following theorem provides existence of the Banach Λ-frame for the operator spaces (see [20]). We give the proof for the completeness.

Theorem 2.3. [20]. *Suppose that* $\{f_k^*\}\subset \mathcal{X}^* \setminus \{0\}$ *has the reconstruction property for* $\mathcal{X}$ *with respect to* $\{f_k\}\subset \mathcal{X}$. *Then,* $\{f_k\}$ *is a Banach Λ-frame for the operator space* $\mathcal{B}(\mathcal{X},\mathcal{Y})$ *with respect to* $\mathcal{Z}_d$.

Proof. Let $\Lambda \in B(\mathcal{X},\mathcal{Z})$ be arbitrary. For each $n\in \mathbb{N}$, define $\Lambda_n:\mathcal{X}\to \mathcal{V}$ by

$$\Lambda_n(f)=\sum_{k=1}^n f_k^*(f)\Lambda(f_k), f\in \mathcal{X}.$$

Then

$$\lim_{n\to\infty}\Lambda_n(f)=\lim_{n\to\infty}\sum_{k=1}^n f_k^*(f)\Lambda(f_k)=\Lambda\left(\sum_{k=1}^\infty f_k^*(f)f_k\right)=\Lambda(f).$$

Thus, $\sup_{1\leq n<\infty}\|\Lambda_n(f)\|<\infty$, for all $f\in \mathcal{X}$. Therefore, by using the Banach-Steinhaus Theorem, we have $\sup_{1\leq n<\infty}\|\Lambda_n\|<\infty$.

Fix $\Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Z})$. Then,

$$\|\Lambda\| = \sup_{\substack{f\in \mathcal{X} \\ \|f\|\leq 1}}\|\Lambda(f)\| = \sup_{\substack{f\in \mathcal{X} \\ \|f\|\leq 1}}\left\|\Lambda\left(\sum_{k=1}^\infty f_k^*(f)f_k\right)\right\| = \sup_{\substack{f\in \mathcal{X} \\ \|f\|\leq 1}}\left\|\lim_{n\to\infty}\sum_{k=1}^n f_k^*(f)\Lambda(f_k)\right\| \leq \sup_{1\leq n<\infty}\|\Lambda_n\| = \|\{\Lambda(f_n)\}\|_{\mathcal{Z}_d}. \qquad (2.2)$$

Also for all $f\in \mathcal{X}$, we have

$$\left\|\sum_{k=1}^n f_k^*(f)\Lambda(f_k)\right\| \leq \|\Lambda\|\left\|\sum_{k=1}^n f_k^*(f)f_k\right\| = \|\Lambda\|\|P_n(f)\| \leq B\|\Lambda\|\|f\|,$$

for all $\Lambda \in B(\mathcal{X},\mathcal{Z})$, where $B=\sup_{1\leq n<\infty}\|P_n\|$. $\qquad (2.3)$

Therefore, by using (2.3) we obtain $\sup_{\substack{f\in \mathcal{X} \\ \|f\|\leq 1}}\left\|\sum_{k=1}^n f_k^*(f)\Lambda(f_k)\right\| \leq B\|\Lambda\|.$

This gives

$$\left\|\left\{\Lambda\left(f_k\right)\right\}\right\|_{\mathcal{Z}_d} = \sup_n \sup_{\substack{f \in \mathcal{X} \\ \|f\| \le 1}} \left\|\sum_{k=1}^{n} f_k^*(f)\Lambda\left(f_k\right)\right\|_{\mathcal{Z}} \le B\|\Lambda\|, \text{ for all } \Lambda \in B(\mathcal{X}, \mathcal{Z}). \tag{2.4}$$

By using (2.2) and (2.4) with $A = 1$, we have

$$A\|\Lambda\| \le \left\|\left\{\Lambda\left(f_k\right)\right\}\right\|_{\mathcal{Z}_d} \le B\|\Lambda\|, \text{ for all } \Lambda \in B(\mathcal{X}, \mathcal{Z}).$$

Hence $\left\{\Lambda\left(f_k\right)\right\}$ is a Banach Λ-frame for the operator space $\mathcal{B}(\mathcal{X}, \mathcal{Z})$ with respect to $\mathcal{Z}_d$. This completes the proof. □

The following theorem gives necessary and sufficient conditions for $\{f_k\} \subset \mathcal{X}$ to be a Λ-Banach frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with respect to an associated Banach space of scalar valued sequences $\mathcal{Y}_d$.

Theorem 2.4. A sequence $\{f_k\} \subset \mathcal{X}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with respect to $\mathcal{Y}_d$ which is generated by $\left\{\Lambda\left(f_k\right)\right\}$ if and only if $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is isomorphic to a closed subspace of $\mathcal{Y}_d$.

Proof. Assume that $\{f_k\}$ is Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with respect to $\mathcal{Y}_d$. Then, there exist positive constants A, B such that

$$A\|\Lambda\| \le \left\|\left\{\Lambda\left(f_k\right)\right\}\right\|_{\mathcal{Y}_d} \le B\|\Lambda\|, \text{ for all } \Lambda \in B(\mathcal{X}, \mathcal{V}). \tag{2.5}$$

By using lower frame inequality in (2.5), the analysis operator T of $\{f_k\}$ is coercive. Thus T is injective and has close range. From the Inverse Mapping Theorem, $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is isomorphic to the range $T(\mathcal{B}(\mathcal{X}, \mathcal{Y}))$, which is a subspace of $\mathcal{Y}_d$. For the reverse part, assume that M is a closed subspace of $\mathcal{Y}_d$ and U is an isomorphic from $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ onto M. Let $\{P_i\}$ be the sequence coordinate operators on $\mathcal{Y}_d$, then $P_i(\{y_i\}) = y_i$ for all $i \in \mathbb{N}$.

Choose $\Lambda(f_k) = P_k U(\Lambda)$, $k \in \mathbb{N}$. Then, for all $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ we have

$$\|\Lambda\| = \left\|U^{-1}U\Lambda\right\| \le \left\|U^{-1}\right\|\|U(\Lambda)\|.$$

Therefore

$$\frac{\|\Lambda\|}{\left\|U^{-1}\right\|} \le \left\|\left\{\Lambda\left(f_k\right)\right\}\right\|_{\mathcal{Y}_d} = \left\|\left\{P_k U(\Lambda)\right\}\right\|_{\mathcal{Y}_d} = \|U(\Lambda)\| \le \|U\|\|\Lambda\| \text{ for all } \Lambda \in B(\mathcal{X}, \mathcal{Y}).$$

Hence $\{f_k\}$ is Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with respect to $\mathcal{Y}_d$. □

Theorem 2.5. A sequence $\{f_k\} \subset \mathcal{X}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ if and only if $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is isomorphic to a complemented subspace of $\mathcal{Y}_d$ which is generated by $\left\{\Lambda\left(f_k\right)\right\}$.

Proof. Assume first that $\{f_k\}$ is Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and let T is the analysis operator and S is the synthesis operator for the Banach Λ-frame $\{f_k\}$. Then, $ST = I_{\mathcal{B}(\mathcal{X}, \mathcal{Y})}$ is the identity operator on $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. Choose $P = TS$. Then, $P^2 = P$ and $\text{Ran}(P) = \text{Ran}(T)$. Therefore, P is the projection from $\mathcal{Y}_d$ to the range of T. Thus, $T : \mathcal{B}(\mathcal{X}, \mathcal{Y}) \to \text{Ran}(T)$ is an isomorphism and $\text{Ran}(T)$ is complemented subspace of $\mathcal{Y}_d$.

For the reverse part, if $U : \mathcal{B}(\mathcal{X}, \mathcal{Y}) \to M$ is an isomorphism, where M is the complemented subspace of $\mathcal{Y}_d$. Then, by Theorem 2.4, the sequence $\{f_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. □

2.1. Construction of Banach Λ-Frames from Operators on $\mathcal{Y}_d$

Let $\{f_k\} \subset \mathcal{X}$ ba a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and let $\{g_k\} \subset \mathcal{X}$. Let $\Theta \in \mathcal{B}(\mathcal{Y}_d)$ be such that $\Theta\left(\left\{\Lambda\left(f_k\right)\right\}\right) = \left\{\Lambda\left(g_k\right)\right\}$, $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then, $\{g_k\}$ is a Λ-Bessel sequence for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, but in general, not a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

The following theorem provides necessary and sufficient conditions for the construction of a Banach Λ-frame from a bounded linear operator on $\mathcal{Y}_d$.

Theorem 2.6. Let $\{f_k\} \subset X$ ba a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and let $\Theta \in \mathcal{B}(\mathcal{Y}_d)$ be such that $\Theta\left(\left\{\Lambda\left(f_k\right)\right\}\right) = \left\{\Lambda\left(g_k\right)\right\}$, where $\{g_k\} \subset \mathcal{X}$. Then, $\{g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ if and only if

$$\left\|\Theta\left(\Lambda\left(\{f_k\}\right)\right)\right\|_{\mathcal{Y}_d} \ge \gamma \left\|Q\left(\left\{\Lambda\left(g_k\right)\right\}\right)\right\|_{\mathcal{Y}_d}, \Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y}),$$

where γ is a positive constant and $Q \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is such that $Q(\{\Lambda(g_k)\}) = \{\Lambda(f_k)\}$, $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Proof. Assume first that $\{g_k\} \subset \mathcal{X}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with bounds a_0, b_0. Let S and T be the pre-frame operator and analysis operator associated with $\{f_k\}$, respectively. Choose $Q = TS$. Then, $Q \in \mathcal{B}(\mathcal{Y}_d)$ is such that $Q(\{\Lambda(g_k)\}) = \{\Lambda(f_k)\}$, $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Let S^0 be the pre-frame operator associated with Banach Λ-frame $\{g_k\}$. Choose $\gamma = \dfrac{\|S^0\|^{-1}}{\|T\|}$. Then, for all $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ we have

$$\left\| \Theta\left(\Lambda(\{f_k\}) \right) \right\|_{\mathcal{Y}_d} = \left\| \{\Lambda(g_k)\} \right\|_{\mathcal{Y}_d} \geq \|S^0\|^{-1} \|\Lambda\| \geq \gamma \left\| Q(\{\Lambda(g_k)\}) \right\|_{\mathcal{Y}_d}.$$

For the reverse part, we compute

$$\gamma \|S\|^{-1} \|\Lambda\| \leq \gamma \left\| \{\Lambda(f_k)\} \right\|_{\mathcal{Y}_d} = \gamma \left\| Q(\{\Lambda(g_k)\}) \right\|_{\mathcal{Y}_d} \leq \left\| \Theta\left(\Lambda(\{f_k\}) \right) \right\|_{\mathcal{Y}_d} \left(= \left\| \{\Lambda(g_k)\} \right\|_{\mathcal{Y}_d} \right) \leq \|\Theta\| \|T\| \|\Lambda\|,$$

for all $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Hence $\{g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with bounds $\gamma \|S\|^{-1}$ and $\|\Theta\| \|T\|$. □

The following theorem gives the better Λ-Bessel bound for the sum of two Banach Λ-frames.

Theorem 2.7. Let $\{f_k\}$ and $\{g_k\}$ be Banach Λ-frames for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with respect to $\mathcal{Y}_d$ and let $\Theta \in \mathcal{B}(\mathcal{Y}_d)$ be an invertible operator such that $\Theta(\{\Lambda(f_k)\}) = \{\Lambda(g_k)\}$, $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then, $\{f_k + g_k\}$ is a Λ-Bessel sequence with bound

$$\alpha = \min \left\{ \|T\| \|I + \Theta\|, \|T^0\| \|I + \Theta^{-1}\| \right\},$$

where T, T^0 are the analysis operators associated with $\{f_k\}$ and $\{g_k\}$, respectively and I is the identity operator on $\mathcal{Y}_d$.

Proof. For all $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, we have

$$\left\| \{\Lambda(f_k + g_k)\} \right\|_{\mathcal{Y}_d} = \left\| (I + \Theta)(\{\Lambda(f_k)\}) \right\|_{\mathcal{Y}_d} \leq \|I + \Theta\| \|T\| \|\Lambda\|.$$

Similarly, we can show that

$$\left\| \{\Lambda(f_k + g_k)\} \right\|_{\mathcal{Y}_d} \leq \|I + \Theta^{-1}\| \|T^0\| \|T\| \|\Lambda\|, \text{ for all } \Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y}).$$

Hence $\{f_k + g_k\}$ is a Λ-Bessel sequence with required Bessel bound. □

Remark 2.8. The Λ-Bessel sequence $\{f_k + g_k\}$ in Theorem 2.7, in general, not a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. If the analysis operator associated with the Λ-Bessel sequence is coercive, then a Λ-Bessel sequence turns out to be a Banach Λ-frame for the underlying space. This is summarized in the following lemma.

Lemma 2.9. Let $\{h_k\} \subset \mathcal{X}$ be Λ-Bessel sequence for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then $\{h_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ if and only if its analysis operator is coercive.

The following theorem gives a relation between the bounds of a Banach Λ-frame $\{f_k\}$ and Bessel bound for a Λ-Bessel sequence $\{g_k\}$ such that $\{f_k \pm g_k\}$ becomes a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Theorem 2.10. Let $\{f_k\} \subset \mathcal{X}$ be a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with bounds A, B and let $\{g_k\}$ be a Λ-Bessel sequence for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ with bound $M < A$, then $\{f_k \pm g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Proof. Suppose that T and R are analysis operators associated with $\{f_k\}$ and $\{g_k\}$ for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. For any $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, we have

$$\left\| \{\Lambda(f_k \pm g_k)\} \right\|_{\mathcal{Y}_d} = \left\| \{\Lambda(f_k)\} \pm \{\Lambda(g_k)\} \right\|_{\mathcal{Y}_d} = \|T(\Lambda) \pm R(\Lambda)\|_{\mathcal{Y}_d} \leq \|T(\Lambda)\|_{\mathcal{Y}_d} + \|R(\Lambda)\|_{\mathcal{Y}_d} \leq (B + M) \|\Lambda\|.$$

Thus, $\{f_k \pm g_k\}$ is a Λ-Bessel sequence for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Now

$$\left\| \{\Lambda(f_k \pm g_k)\} \right\|_{\mathcal{Y}_d} = \|(T \pm R)\Lambda\|_{\mathcal{Y}_d} = \|T(\Lambda) \pm R(\Lambda)\|_{\mathcal{Y}_d} \geq \|T(\Lambda)\|_{\mathcal{Y}_d} - \|R(\Lambda)\|_{\mathcal{Y}_d} \geq (A - M) \|\Lambda\|,$$

for all $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Hence $\{f_k \pm g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. □

Given a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$, we now give an estimate of the Bessel bound for $\{f_k + g_k\}$ such that $\{g_k\}$ becomes a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$. This is given in the following proposition.

Proposition 2.11. Assume that $\{f_k\} \subset \mathcal{X}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ with respect to $\mathcal{Y}_d$. Let $\{g_k\} \subset \mathcal{X}$ be a sequence such that $\{f_k + g_k\}$ is a Λ-Bessel sequence for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ with respect to $\mathcal{Y}_d$ with Bessel bound $\delta < \|S\|^{-1}$, where S is the pre-frame operator associated with $\{f_k\}$. Then, $\{g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$.

Proof. We compute

$$\left(\|S\|^{-1} - \delta\right)\|\Lambda\| \leq \left\|\{\Lambda(f_k)\}\right\|_{\mathcal{Y}_d} - \left\|\{\Lambda(f_k + g_k)\}\right\|_{\mathcal{Y}_d} \leq \left\|\{\Lambda(g_k)\}\right\|_{\mathcal{Y}_d} \leq \left\|\{\Lambda(f_k)\}\right\|_{\mathcal{Y}_d} + \left\|\{\Lambda(f_k + g_k)\}\right\|_{\mathcal{Y}_d}$$

$$\leq \left(\|T\| + \delta\right)\|\Lambda\|, \ \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}).$$

Hence $\{g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$. $\square$

2.2. Perturbation of Λ-Banach Frames

Perturbation theory is a very important tool in various areas of applied mathematics [7] [19] [28]. In frame theory, it began with the fundamental perturbation result of Paley and Wiener. The basic of Paley and Wiener is that a system that is sufficient close to an orthonormal system (basis) in a Hilbert space also forms an orthonormal system (basis). Since then, a number of variations and generalization of this perturbation to the setting of Banach space and then to perturbation of the atomic decompositions, frames (Hilbert)and Banach frames, the reconstruction property in Banach spaces [19] [20]. The following theorem gives a Paley-Wiener type perturbation (in Banach space setting) for Λ-Banach frames.

Theorem 2.12. Let $\{f_k\} \subset \mathcal{X}$ be a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ with bounds A, B and let $\{g_k\} \subset \mathcal{X}$. Assume that $\lambda, \mu, \nu \geq 0$ are non-negative real number such that $\max\left(\lambda + \dfrac{\nu}{A}\right) < 1$ and

$$\left\|(T-R)(\Lambda)\right\|_{\mathcal{Y}_d} \leq \lambda\|T\Lambda\|_{\mathcal{Y}_d} + \mu\|R(\Lambda)\|_{\mathcal{Y}_d} + \nu\|\Lambda\|, \ \text{for all } \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}) \tag{2.6}$$

where T and R are the analysis operators associated with $\{f_k\}$ and $\{g_k\}$, respectively. Then, $\{g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ with bounds $\dfrac{(1-\lambda)A - \nu}{1+\mu}$ and $\dfrac{(1+\lambda)B + \nu}{1-\mu}$.

Proof. For any $\Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, we have

$$A\|\Lambda\| \leq \left\|\{\Lambda(f_k)\}\right\|_{\mathcal{Y}_d} \leq B\|\Lambda\|, \ \text{for all } \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}).$$

Since

$$\left\|(T-R)(\Lambda)\right\|_{\mathcal{Y}_d} \geq \|R(\Lambda)\|_{\mathcal{Y}_d} - \|T(\Lambda)\|_{\mathcal{Y}_d}. \tag{2.7}$$

By using (2.6) and (2.7), we have

$$\|R(\Lambda)\|_{\mathcal{Y}_d} \leq \frac{1+\lambda}{1-\mu}\|T(\Lambda)\|_{\mathcal{Y}_d} + \frac{\nu}{1-\mu}\|\Lambda\| \leq \frac{(1+\lambda)B + \nu}{1-\mu}\|\Lambda\|, \ \text{for all } \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}). \tag{2.8}$$

Now

$$\left\|(T-R)(\Lambda)\right\|_{\mathcal{Y}_d} \geq \|T(\Lambda)\|_{\mathcal{Y}_d} - \|R(\Lambda)\|_{\mathcal{Y}_d}, \ \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}). \tag{2.9}$$

By using (2.6) and (2.9), we have

$$\|R(\Lambda)\|_{\mathcal{Y}_d} \geq \frac{1-\lambda}{1+\mu}\|T(\Lambda)\|_{\mathcal{Y}_d} - \frac{\nu}{1+\mu}\|\Lambda\| \geq \frac{(1-\lambda)A - \nu}{1+\mu}\|\Lambda\|, \ \text{for all } \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}). \tag{2.10}$$

Therefore, by using (2.8) and (2.10) we conclude that $\{g_k\}$ is a Banach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$ with desired frame bounds. $\square$

Remark 2.13. For other types of perturbation results one may refer to [11], which can be generalized to Ba-

nach Λ-frame for $\mathcal{B}(\mathcal{X},\mathcal{Y})$.

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