Advances in Pure Mathematics, 2014, 4, 326-340
Published Online July 2014 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2014.47044

Some Hermite-Hadamard Type Inequalities for Differentiable Co-Ordinated Convex Functions and Applications*

Kai-Chen Hsu

Department of Applied Mathematics, Aletheia University, Tamsui, New Taipei City, Taiwan
Email: victor8@mail.au.edu.tw, hsukaichen@gmail.com

Received 20 May 2014; revised 18 June 2014; accepted 3 July 2014

Abstract

In this paper, we shall establish an inequality for differentiable co-ordinated convex functions on a rectangle from the plane. It is connected with the left side and right side of extended Hermite-Hadamard inequality in two variables. In addition, six other inequalities are derived from it for some refinements. Finally, this paper shows some examples that these inequalities are able to be applied to some special means.

Keywords

Hermite-Hadamard's Inequality, Convex Function, Co-Ordintaed Convex Function, Hölder's Inequality

1. Introduction

Throughout this paper, let $\Delta := [a,b]\times[c,d]$ be double intervals with $a<b$, $c<d$ in $\mathbb{R}^2$, and a partial derivative of second order $\frac{\partial^2 f}{\partial y \partial x}$ is denoted by f_{xy} for brevity.

The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}, \tag{1}$$

which holds for all convex functions $f:[a,b]\to\mathbb{R}$ is known as Hermite-Hadamard's inequality [1] or simply Hadamard's inequality.

*2000 Mathematics Subject Classification. Primary 26D15. Secondary 26A51.

How to cite this paper: Hsu, K.-C. (2014) Some Hermite-Hadamard Type Inequalities for Differentiable Co-Ordinated Convex Functions and Applications. *Advances in Pure Mathematics*, **4**, 326-340. http://dx.doi.org/10.4236/apm.2014.47044

For some results which generalize, improve, and extend the Inequality (1), please refer to [2]-[17].

Based on the convex functions on Δ, Dragomir proposed the concept of co-ordinated convex functions in [3], defined as follows:

Definition 1. *A function* $f:\Delta\to\mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the partial mappings*

$$f(u,y):[a,b]\to\mathbb{R},\forall y\in[c,d],\ f(x,v):[c,d]\to\mathbb{R},\forall x\in[a,b]$$

are convex.

Definition 2. *A function* $f:\Delta\to\mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the inequality*

$$f\left(tx+(1-t)y,su+(1-s)w\right)\le tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w), \tag{2}$$

holds for all $t,s\in[0,1]$ *and* (x,u), (x,w), (y,u), $(y,w)\in\Delta$.

Clearly, we can observe that every convex function $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates, but in some special cases, some co-ordinated convex functions are not convex (please refer to [3]). For more relevant co-ordinated convex functions, please refer to [5] [6] [8]-[10] [12].

The following extended Hadamard's inequality for co-ordinated convex functions on Δ in two variables was proved in [3]:

Theorem 1. *Suppose that* $f:\Delta\to\mathbb{R}$ *is co-ordinated convex on* Δ. *Then the following inequalities hold*:

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\ &\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y\\ &\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}x+\frac{1}{b-a}\int_a^b f(x,d)\,\mathrm{d}x+\frac{1}{d-c}\int_c^d f(a,y)\,\mathrm{d}y+\frac{1}{d-c}\int_c^d f(b,y)\,\mathrm{d}y\right]\\ &\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}\tag{3}$$

The above inequalities are sharp.

In [10], Latif and Dragomir established the following Hadamard-type inequalities that gave an estimate of the difference in the left side of the Inequalities (3) for differentiable co-ordinated convex functions on Δ.

Theorem 2. *Let* $f:\Delta\to\mathbb{R}$ *be a partial differentiable mapping on* Δ.

(1) *If* $\left|f_{xy}\right|$ *is convex on the co-ordinates on* Δ, *then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\gamma\right|\\ &\le\frac{(b-a)(d-c)}{16}\left[\frac{\left|f_{xy}(a,c)\right|+\left|f_{xy}(a,d)\right|+\left|f_{xy}(b,c)\right|+\left|f_{xy}(b,d)\right|}{4}\right]; \end{aligned}\tag{4}$$

(2) *If* $\left|f_{xy}\right|^q$ *is convex on the co-ordinates on* Δ *and* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds*:

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\gamma\right|\\ &\le\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}; \end{aligned}\tag{5}$$

(3) *If* $\left|f_{xy}\right|^q$ *is convex on the co-ordinates on* Δ *and* $q\ge1$, *then the following inequality holds*:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\gamma\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left[\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}\right]^{\frac{1}{q}},\tag{6}$$

where

$$\gamma=\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y.$$

Remark 1. *The Inequality* (6) *shows the result of giving the Inequality* (5) *an improved and simplified constant.*

In [12], Sarikaya *et al*. established the following results that gave an estimate of the difference in the right side of the Inequalities (3) for differentiable co-ordinated convex functions on Δ.

Theorem 3. *Let* $f:\Delta\to\mathbb{R}$ *be a partial differentiable mapping on* Δ.

(1) *If* $\left|f_{xy}\right|^q$ *is convex on the co-ordinates on* Δ, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\delta\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left[\frac{\left|f_{xy}(a,c)\right|+\left|f_{xy}(a,d)\right|+\left|f_{xy}(b,c)\right|+\left|f_{xy}(b,d)\right|}{4}\right];\tag{7}$$

(2) *If* $\left|f_{xy}\right|^q$ *is convex on the co-ordinates on* Δ *and* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds*:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\delta\right|$$
$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}\right]^{\frac{1}{q}};\tag{8}$$

(3) *If* $\left|f_{xy}\right|^q$ *is convex on the co-ordinates on* Δ *and* $q\geq 1$, *then the following inequality holds*:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\delta\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left[\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}\right]^{\frac{1}{q}},\tag{9}$$

where

$$\delta=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f(x,c)+f(x,d)\right]\mathrm{d}x+\frac{1}{d-c}\int_c^d\left[f(a,y)+f(b,y)\right]\mathrm{d}y\right].$$

Remark 2. *The Inequality* (9) *shows the result of giving the Inequality* (8) *an improved and simplified constant.*

The goal of this paper is to establish an inequality which could be connected with the left side and right side of the extended Hadamard's Inequality (3) and improve and generalize the Theorem 2 and Theorem 3. Also, the paper aims to note some consequent applications to special means.

In order to show our main results, we need the following identities (I)-(VI):

(I) For $a \le A \le e_1$, $c \le C \le e_2$, the following four identities hold:

$$\int_c^{e_2}\int_a^{e_1}\left|(x-A)(y-C)\right|^p(e_1-x)(e_2-y)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]$$
$$\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right],$$

$$\int_c^{e_2}\int_a^{e_1}\left|(x-A)(y-C)\right|^p(e_1-x)(y-c)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]$$
$$\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right],$$

$$\int_c^{e_2}\int_a^{e_1}\left|(x-A)(y-C)\right|^p(x-a)(e_2-y)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]$$
$$\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right],$$

$$\int_c^{e_2}\int_a^{e_1}\left|(x-A)(y-C)\right|^p(x-a)(y-c)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]$$
$$\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right].$$

(II) For $a \le A \le e_1$, $e_2 \le D \le d$, the following four identities hold:

$$\int_{e_2}^{d}\int_a^{e_1}\left|(x-A)(y-D)\right|^p(e_1-x)(d-y)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]$$
$$\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-D)^{p+2}\right],$$

$$\int_{e_2}^{d}\int_a^{e_1}\left|(x-A)(y-D)\right|^p(e_1-x)(y-e_2)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]$$
$$\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right],$$

$$\int_{e_2}^{d}\int_a^{e_1}\left|(x-A)(y-D)\right|^p(x-a)(d-y)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]$$
$$\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-D)^{p+2}\right],$$

$$\int_{e_2}^{d}\int_a^{e_1}\left|(x-A)(y-D)\right|^p(x-a)(y-e_2)\mathrm{d}x\mathrm{d}y=\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]$$
$$\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right].$$

(III) For $e_1 \le B \le b$, $c \le C \le e_2$, the following four identities hold:

$$\int_c^{e_2}\int_{e_1}^{b}\left|(x-B)(y-C)\right|^p(b-x)(e_2-y)\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right],$$

$$\int_c^{e_2}\int_{e_1}^{b}\left|(x-B)(y-C)\right|^p(b-x)(y-c)\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right],$$

$$\int_c^{e_2}\int_{e_1}^{b}\left|(x-B)(y-C)\right|^p(x-e_1)(e_2-y)\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right],$$

$$\int_c^{e_2}\int_{e_1}^{b}\left|(x-B)(y-C)\right|^p(x-e_1)(y-c)\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right].$$

(IV) For $e_1 \le B \le b$, $e_2 \le D \le d$, the following four identities hold:

$$\int_{e_2}^{d}\int_{e_1}^{b}\left|(x-B)(y-D)\right|^p(b-x)(d-y)\,\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-D)^{p+2}\right],$$

$$\int_{e_2}^{d}\int_{e_1}^{b}\left|(x-B)(y-D)\right|^p(b-x)(y-e_2)\,\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right],$$

$$\int_{e_2}^{d}\int_{e_1}^{b}\left|(x-B)(y-D)\right|^p(x-e_1)(d-y)\,\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-D)^{p+2}\right],$$

$$\int_{e_2}^{d}\int_{e_1}^{b}\left|(x-B)(y-D)\right|^p(x-e_1)(y-e_2)\,\mathrm{d}x\mathrm{d}y=\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right].$$

2. Main Results

In this section, let the mapping $S(x,y)$ for all $(x,y)\in\Delta$ be defined as follows:

$$S(x,y):=\begin{cases}(x-A)(y-C), & (x,y)\in[a,e_1]\times[c,e_2]\\(x-A)(y-D), & (x,y)\in[a,e_1]\times(e_2,d]\\(x-B)(y-C), & (x,y)\in(e_1,b]\times[c,e_2]\\(x-B)(y-D), & (x,y)\in(e_1,b]\times(e_2,d]\end{cases}. \tag{10}$$

In order to prove our main results, we need the following lemma:

Lemma 1. *Let* $f:\Delta\to\mathbb{R}$ *be a partial differentiable mapping on* Δ. *Then the following inequality holds*:

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b S(x,y)\cdot f_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y\\&=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y+P-N\end{aligned} \tag{11}$$

where

$$\begin{aligned}P=\frac{1}{(b-a)(d-c)}\Big[&(B-A)(D-C)f(e_1,e_2)+(A-a)(D-C)f(a,e_2)+(B-A)(C-c)f(e_1,c)\\&+(B-A)(d-D)f(e_1,d)+(b-B)(D-C)f(b,e_2)+(A-a)(C-c)f(a,c)+(A-a)(d-D)f(a,d)\\&+(b-B)(C-c)f(b,c)+(b-B)(d-D)f(b,d)\Big]\end{aligned}$$

$$\begin{aligned}N=&\frac{1}{b-a}\int_a^b\left[\frac{C-c}{d-c}f(x,c)+\frac{D-C}{d-c}f(x,e_2)+\frac{d-D}{d-c}f(x,d)\right]\mathrm{d}x+\frac{1}{d-c}\int_c^d\left[\frac{A-a}{b-a}f(a,y)+\frac{B-A}{b-a}f(e_1,y)\right.\\&\left.+\frac{b-B}{b-a}f(b,y)\right]\mathrm{d}y.\end{aligned}$$

Proof. It suffices to note that

$$\begin{aligned}\int_c^d\int_a^b S(x,y)\cdot f_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y&=\int_c^{e_2}\int_a^{e_1}(x-A)(y-C)f_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y+\int_{e_2}^{d}\int_a^{e_1}(x-A)(y-D)f_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y\\&\quad+\int_c^{e_2}\int_{e_1}^{b}(x-B)(y-C)f_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y+\int_{e_2}^{d}\int_{e_1}^{b}(x-B)(y-D)f_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y\\&=I_1+I_2+I_3+I_4.\end{aligned} \tag{12}$$

Integration by parts, we have

$$
\begin{aligned}
I_1 &= (e_2 - C)\left[(e_1 - A) f(e_1, e_2) + (A - a) f(a, e_2)\right] + (C - c)\left[(e_1 - A) f(e_1, c) + (A - a) f(a, c)\right] \\
&\quad - \int_a^{e_1} \left[(e_2 - C) f(x, e_2) + (C - c) f(x, c)\right] \mathrm{d}x - \int_c^{e_2} \left[(e_1 - A) f(e_1, y) + (A - a) f(a, y)\right] \mathrm{d}y \\
&\quad + \int_c^{e_2} \int_a^{e_1} f(x, y) \mathrm{d}x\mathrm{d}y, \\
I_2 &= (d - D)\left[(e_1 - A) f(e_1, d) + (A - a) f(a, d)\right] + (D - e_2)\left[(e_1 - A) f(e_1, e_2) + (A - a) f(a, e_2)\right] \\
&\quad - \int_a^{e_1} \left[(d - D) f(x, d) + (D - e_2) f(x, e_2)\right] \mathrm{d}x - \int_{e_2}^{d} \left[(e_1 - A) f(e_1, y) + (A - a) f(a, y)\right] \mathrm{d}y \\
&\quad + \int_{e_2}^{d} \int_a^{e_1} f(x, y) \mathrm{d}x\mathrm{d}y, \\
I_3 &= (e_2 - C)\left[(b - B) f(b, e_2) + (B - e_1) f(e_1, e_2)\right] + (C - c)\left[(b - B) f(b, c) + (B - e_1) f(e_1, c)\right] \\
&\quad - \int_{e_1}^{b} \left[(e_2 - C) f(x, e_2) + (C - c) f(x, c)\right] \mathrm{d}x - \int_c^{e_2} \left[(b - B) f(b, y) + (B - e_1) f(e_1, y)\right] \mathrm{d}y \\
&\quad + \int_c^{e_2} \int_a^{b} f(x, y) \mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= (d - D)\left[(b - B) f(b, d) + (B - e_1) f(e_1, d)\right] + (D - e_2)\left[(b - B) f(b, e_2) + (B - e_1) f(e_1, e_2)\right] \\
&\quad - \int_{e_1}^{b} \left[(d - D) f(x, d) + (D - e_2) f(x, e_2)\right] \mathrm{d}x - \int_{e_2}^{d} \left[(b - B) f(b, y) + (B - e_1) f(e_1, y)\right] \mathrm{d}y \\
&\quad + \int_{e_2}^{d} \int_{e_1}^{b} f(x, y) \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

By summing the above four identities I_1, I_2, I_3 and I_4 and simplifying the result, it follows that

$$
\begin{aligned}
&\int_c^d \int_a^b f(x, y) \mathrm{d}x\mathrm{d}y + (B - A)(D - C) f(e_1, e_2) + (A - a)(D - C) f(a, e_2) + (B - A)(C - c) f(e_1, c) \\
&+ (B - A)(d - D) f(e_1, d) + (b - B)(D - C) f(b, e_2) \\
&+ (A - a)(C - c) f(a, c) + (A - a)(d - D) f(a, d) \\
&+ (b - B)(C - c) f(b, c) + (b - B)(d - D) f(b, d) \\
&- \int_a^b \left[(C - c) f(x, c) + (D - C) f(x, e_2) + (d - D) f(x, d)\right] \mathrm{d}x \\
&- \int_c^d \left[(A - a) f(a, y) + (B - A) f(e_1, y) + (b - B) f(b, y)\right] \mathrm{d}y.
\end{aligned}
\tag{13}
$$

Then, multiply both sides by $(b - a)(d - c)$ in (12). From (12) and (13), we get the equations P and N. This proof of the identity 11 is complete. □

Now, we are ready to state and prove the main results.

Theorem 1. *Let* f *be defined as Lemma* 1. *If* $q \geq 1$ *and the mapping* $\left|f_{xy}\right|^q$ *is convex on the co-ordinates on* Δ, *then*

$$
\begin{aligned}
&\left| \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) \mathrm{d}x\mathrm{d}y + P - N \right| \\
&\leq \begin{cases}
\left[\dfrac{R(A, B, e_1; C, D, e_2)}{(b - a)(d - c)}\right]^{\frac{q-1}{q}} \left[\dfrac{E(A, B, e_1; C, D, e_2)}{(b - a)(d - c)}\right]^{\frac{1}{q}}, & \text{if } q > 1, 0 \leq p \leq q, \\
\dfrac{E(A, B, e_1; C, D, e_2)}{(b - a)(d - c)}, & \text{if } p = q = 1,
\end{cases}
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
&R\left(A,B,e_1;C,D,e_2\right)\\
&=\left(\frac{q-1}{2q-p-1}\right)^2\left[\left(A-a\right)^{(2q-p-1)/(q-1)}+\left(e_1-A\right)^{(2q-p-1)/(q-1)}+\left(B-e_1\right)^{(2q-p-1)/(q-1)}+\left(b-B\right)^{(2q-p-1)/(q-1)}\right]\\
&\quad\cdot\left[\left(C-c\right)^{(2q-p-1)/(q-1)}+\left(e_2-C\right)^{(2q-p-1)/(q-1)}+\left(D-e_2\right)^{(2q-p-1)/(q-1)}+\left(d-D\right)^{(2q-p-1)/(q-1)}\right].
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left(A,B,e_1;C,D,e_2\right)\\
&=\frac{1}{(p+1)^2(p+2)^2(e_1-a)(e_2-c)}\\
&\times\Big\{\left|f_{xy}(a,c)\right|^q\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]\\
&\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right]\\
&+\left|f_{xy}(a,e_2)\right|^q\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]\\
&\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right]\\
&+\left|f_{xy}(e_1,c)\right|^q\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]\\
&\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right]\\
&+\left|f_{xy}(e_1,e_2)\right|^q\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]\\
&\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right]\Big\}\\
&+\frac{1}{(p+1)^2(p+2)^2(e_1-a)(d-e_2)}\\
&\times\Big\{\left|f_{xy}(a,e_2)\right|^q\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]\\
&\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-B)^{p+2}\right]\\
&+\left|f_{xy}(a,d)\right|^q\left[(A-a)^{p+1}\left((p+2)e_1-(p+1)a-A\right)+(e_1-A)^{p+2}\right]\\
&\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right]\\
&+\left|f_{xy}(e_1,e_2)\right|^q\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]\\
&\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-B)^{p+2}\right]\\
&+\left|f_{xy}(e_1,d)\right|^q\left[(A-a)^{p+2}+(e_1-A)^{p+1}\left((p+1)e_1-(p+2)a+A\right)\right]\\
&\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right]\Big\}.\\
&+\frac{1}{(p+1)^2(p+2)^2(b-e_1)(e_2-c)}\\
&\times\Big\{\left|f_{xy}(e_1,c)\right|^q\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]\\
&\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right]
\end{aligned}
$$

$$+\left|f_{xy}(e_1,e_2)\right|^q\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right]$$
$$+\left|f_{xy}(b,c)\right|^q\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(C-c)^{p+1}\left((p+2)e_2-(p+1)c-C\right)+(e_2-C)^{p+2}\right]$$
$$+\left|f_{xy}(b,e_2)\right|^q\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(C-c)^{p+2}+(e_2-C)^{p+1}\left((p+1)e_2-(p+2)c+C\right)\right]\Big\}$$
$$+\frac{1}{(p+1)^2(p+2)^2(b-e_1)(d-e_2)}$$
$$\times\Big\{\left|f_{xy}(e_1,e_2)\right|^q\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-D)^{p+2}\right]$$
$$+\left|f_{xy}(e_1,d)\right|^q\left[(B-e_1)^{p+1}\left((p+2)b-(p+1)e_1-B\right)+(b-B)^{p+2}\right]$$
$$\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right]$$
$$+\left|f_{xy}(b,e_2)\right|^q\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(D-e_2)^{p+1}\left((p+2)d-(p+1)e_2-D\right)+(d-D)^{p+2}\right]$$
$$+\left|f_{xy}(b,d)\right|^q\left[(B-e_1)^{p+2}+(b-B)^{p+1}\left((p+1)b-(p+2)e_1+B\right)\right]$$
$$\times\left[(D-e_2)^{p+2}+(d-D)^{p+1}\left((p+1)d-(p+2)e_2+D\right)\right]\Big\}.$$

Proof. By using the identity (11), we have

$$\left|\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y+P-N\right|\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\left|S(x,y)\right|\left|f_{xy}(x,y)\right|\mathrm{d}x\mathrm{d}y.$$

If $q>1$, $0\le p\le q$, it follows from the power mean inequality that

$$\int_c^d\int_a^b\left|S(x,y)\right|\left|f_{xy}(x,y)\right|\mathrm{d}x\mathrm{d}y\le\left(\int_c^d\int_a^b\left|S(x,y)\right|^{(q-p)/(q-1)}\mathrm{d}x\mathrm{d}y\right)^{(q-1)/q}\times\left(\int_c^d\int_a^b\left|S(x,y)\right|^p\left|f_{xy}(x,y)\right|^q\mathrm{d}x\mathrm{d}y\right)^{1/q}. \quad (15)$$

We denote $R(A,B,e_1;C,D,e_2)$ and $E(A,B,e_1;C,D,e_2)$ by

$$R(A,B,e_1;C,D,e_2)=\int_c^d\int_a^b\left|S(x,y)\right|^{(q-p)/(q-1)}\mathrm{d}x\mathrm{d}y$$

and

$$E(A,B,e_1;C,D,e_2)=\int_c^d\int_a^b\left|S(x,y)\right|^p\left|f_{xy}(x,y)\right|^q\mathrm{d}x\mathrm{d}y$$

respectively, and then

$$\begin{aligned}R(A,B,e_1;C,D,e_2)&=\int_c^{e_2}\int_a^{e_1}\left|(x-A)(y-C)\right|^{(q-p)/(q-1)}\mathrm{d}x\mathrm{d}y+\int_{e_2}^d\int_a^{e_1}\left|(x-A)(y-D)\right|^{(q-p)/(q-1)}\mathrm{d}x\mathrm{d}y\\&\quad+\int_c^{e_2}\int_{e_1}^b\left|(x-B)(y-C)\right|^{(q-p)/(q-1)}\mathrm{d}x\mathrm{d}y+\int_{e_2}^d\int_{e_1}^b\left|(x-B)(y-D)\right|^{(q-p)/(q-1)}\mathrm{d}x\mathrm{d}y\\&=J_1+J_2+J_3+J_4.\end{aligned} \quad (16)$$

By using the integration techniques, we have

$$J_1=\int_c^C\int_a^A\left[(A-x)(C-y)\right]^{(q-p)/(q-1)}\,\mathrm{d}x\mathrm{d}y+\int_C^{e_2}\int_a^A\left[(A-x)(y-C)\right]^{(q-p)/(q-1)}\,\mathrm{d}x\mathrm{d}y$$
$$+\int_c^C\int_A^{e_1}\left[(x-A)(C-y)\right]^{(q-p)/(q-1)}\,\mathrm{d}x\mathrm{d}y+\int_C^{e_2}\int_A^{e_1}\left[(x-A)(y-C)\right]^{(q-p)/(q-1)}\,\mathrm{d}x\mathrm{d}y$$
$$=\left(\frac{q-1}{2q-p-1}\right)^2\left[(A-a)^{(2q-p-1)/(q-1)}+(e_1-A)^{(2q-p-1)/(q-1)}\right]\times\left[(C-c)^{(2q-p-1)/(q-1)}+(e_2-C)^{(2q-p-1)/(q-1)}\right],$$

and similary we get,

$$J_2=\left(\frac{q-1}{2q-p-1}\right)^2\left[(A-a)^{(2q-p-1)/(q-1)}+(e_1-A)^{(2q-p-1)/(q-1)}\right]\times\left[(D-e_2)^{(2q-p-1)/(q-1)}+(d-D)^{(2q-p-1)/(q-1)}\right],$$

$$J_3=\left(\frac{q-1}{2q-p-1}\right)^2\left[(B-e_1)^{(2q-p-1)/(q-1)}+(b-B)^{(2q-p-1)/(q-1)}\right]\times\left[(C-c)^{(2q-p-1)/(q-1)}+(e_2-C)^{(2q-p-1)/(q-1)}\right],$$

and

$$J_4=\left(\frac{q-1}{2q-p-1}\right)^2\left[(B-e_1)^{(2q-p-1)/(q-1)}+(b-B)^{(2q-p-1)/(q-1)}\right]\times\left[(D-e_2)^{(2q-p-1)/(q-1)}+(d-D)^{(2q-p-1)/(q-1)}\right].$$

By summing the above four identities J_1, J_2, J_3 and J_4 and simplifying the result. Then according to (16), we get the estimated bound $R(A,B,e_1;C,D,e_2)$.

On the other hand, by using the identity mappings

$$I(x)=\begin{cases}\dfrac{e_1-x}{e_1-a}a+\dfrac{x-a}{e_1-a}e_1, & \text{if } a\le x\le e_1\\[2mm] \dfrac{b-x}{b-e_1}e_1+\dfrac{x-e_1}{b-e_1}b, & \text{if } e_1\le x\le b\end{cases}$$

and

$$I(y)=\begin{cases}\dfrac{e_2-y}{e_2-c}c+\dfrac{y-c}{e_2-c}e_2, & \text{if } c\le x\le e_2\\[2mm] \dfrac{d-y}{d-e_2}e_2+\dfrac{y-e_2}{d-e_2}d, & \text{if } e_2\le x\le d\end{cases}$$

we have

$$\begin{aligned}&\int_c^d\int_a^b|S(x,y)|^p\left|f_{xy}(I(x),I(y))\right|^q\,\mathrm{d}x\mathrm{d}y\\
&=\int_c^{e_2}\int_a^{e_1}|(x-A)(y-C)|^p\times\left|f_{xy}\left(\frac{e_1-x}{e_1-a}a+\frac{x-a}{e_1-a}e_1,\frac{e_2-y}{e_2-c}c+\frac{y-c}{e_2-c}e_2\right)\right|^q\mathrm{d}x\mathrm{d}y\\
&+\int_{e_2}^d\int_a^{e_1}|(x-A)(y-D)|^p\times\left|f_{xy}\left(\frac{e_1-x}{e_1-a}a+\frac{x-a}{e_1-a}e_1,\frac{d-y}{d-e_2}e_2+\frac{y-e_2}{d-e_2}d\right)\right|^q\mathrm{d}x\mathrm{d}y\\
&+\int_c^{e_2}\int_{e_1}^b|(x-B)(y-C)|^p\times\left|f_{xy}\left(\frac{b-x}{b-e_1}e_1+\frac{x-e_1}{b-e_1}b,\frac{e_2-y}{e_2-c}c+\frac{y-c}{e_2-c}e_2\right)\right|^q\mathrm{d}x\mathrm{d}y\\
&+\int_{e_2}^d\int_{e_1}^b|(x-B)(y-D)|^p\times\left|f_{xy}\left(\frac{b-x}{b-e_1}e_1+\frac{x-e_1}{b-e_1}b,\frac{d-y}{d-e_2}e_2+\frac{y-e_2}{d-e_2}d\right)\right|^q\mathrm{d}x\mathrm{d}y\\
&=J_5+J_6+J_7+J_8.\end{aligned}\tag{17}$$

By the convexit3 of $\left|f_{xy}\right|^q$ on the co-ordinates on Δ and the Inequality (2) in J_5, J_6, J_7 and J_8, then we have

$$J_5 \le \frac{1}{(e_1-a)(e_2-c)} \int_c^{e_2}\int_a^{e_1} |(x-A)(y-C)|^p \times \Big[(e_1-x)(e_2-y)|f_{xy}(a,c)|^q + (e_1-x)(y-c)|f_{xy}(a,e_2)|^q$$
$$+ (x-a)(e_2-y)|f_{xy}(e_1,c)|^q + (x-a)(y-c)|f_{xy}(e_1,e_2)|^q \Big] \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{(e_1-a)(e_2-c)} \times \Big[|f_{xy}(a,c)|^q \int_c^{e_2}\int_a^{e_1} |(x-A)(y-C)|^p (e_1-x)(e_2-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(a,e_2)|^q \int_c^{e_2}\int_a^{e_1} |(x-A)(y-C)|^p (e_1-x)(y-c)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(e_1,c)|^q \int_c^{e_2}\int_a^{e_1} |(x-A)(y-C)|^p (x-a)(e_2-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(e_1,e_2)|^q \int_c^{e_2}\int_a^{e_1} |(x-A)(y-C)|^p (x-a)(y-c)\mathrm{d}x\mathrm{d}y \Big],$$

$$J_6 \le \frac{1}{(e_1-a)(d-e_2)} \int_{e_2}^{d}\int_a^{e_1} |(x-A)(y-D)|^p \times \Big[(e_1-x)(d-y)|f_{xy}(a,e_2)|^q + (e_1-x)(y-e_2)|f_{xy}(a,d)|^q$$
$$+ (x-a)(d-y)|f_{xy}(e_1,e_2)|^q + (x-a)(y-e_2)|f_{xy}(e_1,d)|^q \Big] \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{(e_1-a)(d-e_2)} \times \Big[|f_{xy}(a,e_2)|^q \int_{e_2}^{d}\int_a^{e_1} |(x-A)(y-D)|^p (e_1-x)(d-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(a,d)|^q \int_{e_2}^{d}\int_a^{e_1} |(x-A)(y-D)|^p (e_1-x)(y-e_2)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(e_1,e_2)|^q \int_{e_2}^{d}\int_a^{e_1} |(x-A)(y-D)|^p (x-a)(d-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(e_1,d)|^q \int_{e_2}^{d}\int_a^{e_1} |(x-A)(y-D)|^p (x-a)(y-e_2)\mathrm{d}x\mathrm{d}y \Big],$$

$$J_7 \le \frac{1}{(b-e_1)(e_2-c)} \int_c^{e_2}\int_{e_1}^{b} |(x-B)(y-C)|^p \times \Big[(b-x)(e_2-y)|f_{xy}(e_1,c)|^q + (b-x)(y-c)|f_{xy}(e_1,e_2)|^q$$
$$+ (x-e_1)(e_2-y)|f_{xy}(b,c)|^q + (x-e_1)(y-c)|f_{xy}(b,e_2)|^q \Big] \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{(b-e_1)(e_2-c)} \times \Big[|f_{xy}(e_1,c)|^q \int_c^{e_2}\int_{e_1}^{b} |(x-B)(y-C)|^p (b-x)(e_2-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(e_1,e_2)|^q \int_c^{e_2}\int_{e_1}^{b} |(x-B)(y-C)|^p (b-x)(y-c)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(b,c)|^q \int_c^{e_2}\int_{e_1}^{b} |(x-B)(y-C)|^p (x-e_1)(e_2-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(b,e_2)|^q \int_c^{e_2}\int_{e_1}^{b} |(x-B)(y-C)|^p (x-e_1)(y-c)\mathrm{d}x\mathrm{d}y \Big],$$

and

$$J_8 \le \frac{1}{(b-e_1)(d-e_2)} \int_{e_2}^{d}\int_{e_1}^{b} |(x-B)(y-D)|^p \times \Big[(b-x)(d-y)|f_{xy}(e_1,e_2)|^q + (b-x)(y-e_2)|f_{xy}(e_1,d)|^q$$
$$+ (x-e_1)(d-y)|f_{xy}(b,e_2)|^q + (x-e_1)(y-e_2)|f_{xy}(b,d)|^q \Big] \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{(b-e_1)(d-e_2)} \times \Big[|f_{xy}(e_1,e_2)|^q \int_{e_2}^{d}\int_{e_1}^{b} |(x-B)(y-D)|^p (b-x)(d-y)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(e_1,d)|^q \int_{e_2}^{d}\int_{e_1}^{b} |(x-B)(y-D)|^p (b-x)(y-e_2)\mathrm{d}x\mathrm{d}y$$
$$+ |f_{xy}(b,e_2)|^q \int_{e_2}^{d}\int_{e_1}^{b} |(x-B)(y-D)|^p (x-e_1)(d-y)\mathrm{d}x\mathrm{d}y$$

$$+\left|f_{xy}(b,d)\right|^q \int_{e_2}^{d}\int_{e_1}^{b}\left|(x-B)(y-D)\right|^p (x-e_1)(y-e_2)\,\mathrm{d}x\mathrm{d}y\Big].$$

By applying the identities (I), (II), (III) and (IV) to the above four inequalities and then simplifying the results, we get the estimated bound $E(A,B,e_1;C,D,e_2)$ and the Inequality (14) for $q>1$. If $q=p=1$, then the Inequality (14) follows from (15) and (17). The proof of the Inequality (14) is complete. □

Corollary 1. *Under the assumptions of Theorem* 1 *with* $A=a$, $B=b$, $e_1=(a+b)/2$, $C=c$, $D=d$ *and* $e_2=(c+d)/2$, *we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\gamma\right|$$
$$\le\left[\frac{R\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{q-1}{q}}\left[\frac{E\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{1}{q}} \tag{18}$$

where γ is as given in Theorem 2,

$$\frac{R\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}:=\left(\frac{q-1}{2q-p-1}\right)^2\left[\frac{(b-a)(d-c)}{4}\right]^{\frac{q-p}{q-1}}$$

and

$$\frac{E\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}:=\frac{(b-a)^p(d-c)^p}{2^{2p}(p+1)^2(p+2)^2}\times\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}$$
$$+\frac{(b-a)^p(d-c)^p}{2^{2p+1}(p+1)(p+2)^2}\left[\left|f_{xy}\left(\frac{a+b}{2},c\right)\right|^q+\left|f_{xy}\left(b,\frac{c+d}{2}\right)\right|^q+\left|f_{xy}\left(\frac{a+b}{2},d\right)\right|^q\right.$$
$$\left.+2(p+1)\left|f_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|f_{xy}\left(a,\frac{c+d}{2}\right)\right|^q\right].$$

The Corollary 1 shows that we get the new estimated bound of the Inequality (6).

Corollary 2. *Under the assumptions of Corollary* 1 *with* $p=1$, *we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\gamma\right|$$
$$\le\left[\frac{R\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{q-1}{q}}\left[\frac{E\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{1}{q}} \tag{19}$$

where γ is as given in Theorem 2,

$$\frac{R\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)}{(b-a)(d-c)}:=\frac{(b-a)(d-c)}{16}$$

and

$$E\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)\Big/\left[(b-a)(d-c)\right]$$

$$:=\frac{(b-a)(d-c)}{144}\times\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}$$
$$+\frac{(b-a)(d-c)}{144}\left[\left|f_{xy}\left(\frac{a+b}{2},c\right)\right|^q+\left|f_{xy}\left(b,\frac{c+d}{2}\right)\right|^q\right.$$
$$\left.+\left|f_{xy}\left(\frac{a+b}{2},d\right)\right|^q+4\left|f_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|f_{xy}\left(a,\frac{c+d}{2}\right)\right|^q\right].$$

Remark 3. *By using the convexity of* $\left|f_{xy}\right|^q$ *on the co-ordinates on* Δ, *we have the inequality*

$$\left|f_{xy}\left(\frac{a+b}{2},c\right)\right|^q+\left|f_{xy}\left(a,\frac{c+d}{2}\right)\right|^q+\left|f_{xy}\left(b,\frac{c+d}{2}\right)\right|^q+4\left|f_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|f_{xy}\left(\frac{a+b}{2},d\right)\right|^q$$
$$\le 2\left(\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,d)\right|^q\right),$$

and then

$$E\left(a,b,\frac{a+b}{2};c,d,\frac{c+d}{2}\right)\Big/\left[(b-a)(d-c)\right]$$
$$\le\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}\times\frac{(b-a)(d-c)}{16}.$$

Hence the Inequality (19) improves the Inequality (6).

Remark 4. *Under the assumptions of Theorem* 1 *with* $A=a$, $B=b$, $e_1=(a+b)/2$, $C=c$, $D=d$, $e_2=(c+d)/2$ *and* $p=q=1$, *we get the new estimated bound and it could improve the Inequality* (4).

Corollary 3. *Under the assumptions of Theorem* 1 *with* $A=B=e_1=(a+b)/2$, $C=D=e_2=(c+d)/2$, *we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\delta\right|$$
$$\le\left[\frac{R\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{q-1}{q}}\times\left[\frac{E\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{1}{q}}\quad(20)$$

where δ is as given in Theorem 3,

$$\frac{R\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)}{(b-a)(d-c)}:=\left(\frac{q-1}{2q-p-1}\right)^2\left[\frac{(b-a)(d-c)}{4}\right]^{\frac{q-p}{q-1}}$$

and

$$E\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)\Big/\left[(b-a)(d-c)\right]$$
$$:=\frac{(b-a)^p(d-c)^p}{2^{2p}(p+2)^2}\times\frac{\left|f_{xy}(a,c)\right|^q+\left|f_{xy}(b,c)\right|^q+\left|f_{xy}(a,d)\right|^q+\left|f_{xy}(b,d)\right|^q}{4}$$
$$+\frac{(b-a)^p(d-c)^p}{2^{2p+1}(p+1)(p+2)^2}\left[\left|f_{xy}\left(\frac{a+b}{2},c\right)\right|^q+\left|f_{xy}\left(b,\frac{c+d}{2}\right)\right|^q\right.$$

$$+\left|f_{xy}\left(\frac{a+b}{2},d\right)\right|^{q}+\frac{2}{p+1}\left|f_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^{q}+\left|f_{xy}\left(a,\frac{c+d}{2}\right)\right|^{q}\Bigg].$$

The Corollary 3 shows that we get the new estimated bound of the Inequality (9).

Corollary 4. *Under the assumptions of Corollary* 3 *with* $p=1$, *we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\delta\right|$$
$$\le\left[\frac{R\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{q-1}{q}}\times\left[\frac{E\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)}{(b-a)(d-c)}\right]^{\frac{1}{q}},\qquad(21)$$

where δ is as given in Theorem 3,

$$\frac{R\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)}{(b-a)(d-c)}:=\frac{(b-a)(d-c)}{16}$$

and

$$E\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)\Big/\left[(b-a)(d-c)\right]$$
$$:=\frac{(b-a)(d-c)}{36}\times\frac{|f_{xy}(a,c)|^{q}+|f_{xy}(b,c)|^{q}+|f_{xy}(a,d)|^{q}+|f_{xy}(b,d)|^{q}}{4}$$
$$+\frac{(b-a)(d-c)}{144}\Bigg[\left|f_{xy}\left(\frac{a+b}{2},c\right)\right|^{q}+\left|f_{xy}\left(b,\frac{c+d}{2}\right)\right|^{q}$$
$$+\left|f_{xy}\left(\frac{a+b}{2},d\right)\right|^{q}+\left|f_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^{q}+\left|f_{xy}\left(a,\frac{c+d}{2}\right)\right|^{q}\Bigg].$$

Remark 5. *By using the convexity of* $|f_{xy}|^{q}$ *on the co-ordinates on* Δ, *we have the inequality*

$$\left|f_{xy}\left(\frac{a+b}{2},c\right)\right|^{q}+\left|f_{xy}\left(a,\frac{c+d}{2}\right)\right|^{q}+\left|f_{xy}\left(b,\frac{c+d}{2}\right)\right|^{q}+\left|f_{xy}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^{q}+\left|f_{xy}\left(\frac{a+b}{2},d\right)\right|^{q}$$
$$\le 5\left(\frac{|f_{xy}(a,c)|^{q}+|f_{xy}(b,c)|^{q}+|f_{xy}(a,d)|^{q}+|f_{xy}(b,d)|^{q}}{4}\right),$$

and then

$$E\left(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2};\frac{c+d}{2},\frac{c+d}{2},\frac{c+d}{2}\right)\Big/\left[(b-a)(d-c)\right]$$
$$\le\frac{|f_{xy}(a,c)|^{q}+|f_{xy}(b,c)|^{q}+|f_{xy}(a,d)|^{q}+|f_{xy}(b,d)|^{q}}{4}\times\frac{(b-a)(d-c)}{16}.$$

Hence the Inequality (21) improves the Inequality (9).

Remark 6. *Under the assumptions of Theorem* 1 *with* $A=B=e_1=(a+b)/2$, $C=D=e_2=(c+d)/2$ *and* $p=q=1$, *we get the new estimated bound and it could improve the Inequality* (7).

Example 1. *Let the function* $f(x,y)$ *be* $x^{(q+2)/q}y^{(q+2)/q}$, $(x,y)\in[0,1]^2$. *Then the result of the right-hand* *side of* (6) *or* (9) *is* $(1/16)(1/4)^{1/q}(1+(2/q))^2$, *whereas the right-hand side of* (19) *and* (21) *are* $(1/16)(1/9)^{1/q}(1+(2/q))^2$ *and* $(1/16)(25/144)^{1/q}(1+(2/q))^2$, *respectively*.

3. Some Applications to Special Means

As in [11] we shall consider extensions of arithmetic, logarithmic and generalized logarithmic means from positive real numbers. We take

$$A(\alpha_1,\alpha_2,\cdots,\alpha_n)=\frac{1}{n}\sum_{i=1}^{n}\alpha_i,\qquad \alpha_i\in\mathbb{R},\qquad i=1,2,\cdots,n,$$

$$L(\alpha,\beta)=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},\qquad |\alpha|\neq|\beta|,\qquad \alpha\beta\neq 0,$$

$$L_m(\alpha,\beta)=\left[\frac{\beta^{m+1}-\alpha^{m+1}}{(m+1)(\beta-\alpha)}\right]^{\frac{1}{m}},\qquad m\in\mathbb{Z}\setminus\{-1,0\},\qquad \alpha,\beta\in 0,\alpha\neq\beta,$$

where $\mathbb{Z}$ is the set of integers.

Proposition 1. *Let* a, b, c, $d\in\mathbb{R}$, $a<b$, $c<d$, $0\notin[a,b]$, $0\notin[c,d]$, *and* m, $n\in\mathbb{Z}$, $|m|\geq 2$ *and* $|n|\geq 2$. *Then, for* $q\in[1,\infty)$, *we have*

$$\begin{aligned}&\left|\left(A(a^m,b^m)-L_m(a,b)^m\right)\left(A(c^n,d^n)-L_n(c,d)^n\right)\right|\\&\leq\frac{|m||n|(b-a)(d-c)}{16}\times\Bigg[A\Bigg(|a|^{(m-1)q}|c|^{(n-1)q},|b|^{(m-1)q}|c|^{(n-1)q},|a|^{(m-1)q}|d|^{(n-1)q},|b|^{(m-1)q}|d|^{(n-1)q},\\&\left|\frac{a+b}{2}\right|^{(m-1)q}|c|^{(n-1)q},|a|^{(m-1)q}\left|\frac{c+d}{2}\right|^{(n-1)q},|b|^{(m-1)q}\left|\frac{c+d}{2}\right|^{(n-1)q},\left|\frac{a+b}{2}\right|^{(m-1)q}\left|\frac{c+d}{2}\right|^{(n-1)q},\left|\frac{a+b}{2}\right|^{(m-1)q}|d|^{(n-1)q}\Bigg)\Bigg]^{\frac{1}{q}},\end{aligned}\tag{22}$$

Proof. The proof is immediate from Corollary 4 with $f(x,y)=x^my^n$, $x,y\in\mathbb{R}$, $m,n\in\mathbb{Z}$, $|m|\geq 2$, $|n|\geq 2$.

Proposition 2. *Suppose* a, b, c, $d\in\mathbb{R}$, $a<b$, $c<d$, $0\notin[a,b]$, $0\notin[c,d]$. *Then, for* $q\in[1,\infty)$, *we have*

$$\begin{aligned}&\left|\left(A(a^{-1},b^{-1})-L^{-1}(a,b)\right)\left(A(c^{-1},d^{-1})-L^{-1}(c,d)\right)\right|\leq\frac{(b-a)(d-c)}{16}\\&\times\left[A\left(|ac|^{-2q},|bc|^{-2q},|ad|^{-2q},|bd|^{-2q},\left|\frac{ac+ad+bc+bd}{4}\right|^{-2q},\left|\frac{ac+bc}{2}\right|^{-2q},\left|\frac{ac+ad}{2}\right|^{-2q},\left|\frac{bc+bd}{2}\right|^{-2q},\left|\frac{ad+bd}{2}\right|^{-2q}\right)\right]^{\frac{1}{q}}.\end{aligned}\tag{23}$$

Proof. The result follows from Corollary 4 with $f(x,y)=1/(xy)$. □

Remark 7. *The Corollary* 2 *could also be applied to some special means*.

Acknowledgements

The author is very grateful to the reviewers for carefully reading this paper and giving constructive comments.

References

[1] Hadamard, J. (1893) Étude sur les Propriétés des Fonctions Entières en Particulier d'une Fonction Considérée par Riemann. *Journal de Mathématiques Pures et Appliquées*, **58**, 171-215.

[2] Dragomir, S.S. (1992) Two Mappings in Connection to Hadamard's Inequalities. *Journal of Mathematical Analysis and Applications*, **167**, 49-56. http://dx.doi.org/10.1016/0022-247X(92)90233-4

[3] Dragomir, S.S. (2001) On the Hadamard's Inequality for Convex on the Co-Ordinates in a Rectangle from the Plane. *Taiwanese Journal of Mathematics*, **5**, 775-788.

[4] Dragomir, S.S. and Agarwal, R.P. (1998) Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula. *Applied Mathematics Letters*, **11**, 91-95. http://dx.doi.org/10.1016/S0893-9659(98)00086-X

[5] Dragomir, S.S. and Pearce, C.E.M. (2000) Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University. http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html

[6] Hsu, K.-C. (2014) Refinements of Hermite-Hadamard Type Inequalities for Differentiable Co-Ordinated Convex Functions and Applications. *Taiwanese Journal of Mathematics*, (In press). http://dx.doi.org/10.1142/9261

[7] Hwang, D.-Y., Tseng, K.-L. and Hsu, K.-C. (2012) Hadamard-Type and Bullen-Type Inequalities for Lipschitzian Functions and their Applications. *Computers Mathematics with Applications*, **64**, 651-660. http://dx.doi.org/10.1016/j.camwa.2011.12.076

[8] Hwang, D.-Y., Hsu, K.-C. and Tseng, K.-L. (2013) Hadamard-Type Inequalities for Lipschitzian Functions in One and Two Variables with Applications. *Journal of Mathematical Analysis and Applications*, **405**, 546-554. http://dx.doi.org/10.1016/j.jmaa.2013.04.032

[9] Latif, M.A. and Alomari, M. (2009) Hadamard-Type Inequalities for Product Two Convex Functions on the Co-ordinetes. *International Mathematical Forum*, **4**, 2327-2338. http://www.m-hikari.com/imf-password2009/45-48-2009/index.html

[10] Latif, M.A. and Dragomir, S.S. (2012) On Some New Inequalities for Differentiable Co-Ordinated Convex Functions. *Journal of Inequalities and Applications*, **2012**, 28. http://dx.doi.org/10.1186/1029-242X-2012-28

[11] Pearce, C.E.M. and Pečarić, J. (2000) Inequalities for Differentiable Mappings with Application to Special Means and Quadrature Formula. *Applied Mathematics Letters*, **13**, 51-55. http://dx.doi.org/10.1016/S0893-9659(99)00164-0

[12] Sarikaya, M.Z., Set, E., Ozdemir, M.E. and Dragomir, S.S. (2010) New Some Hadamard's Type Inequalities for Co-ordinated Convex Functions. *Tamsui Oxford Journal of Information and Mathematical Sciences*, **28**, 137-152. arXiv:1005.0700v1 [math.CA]. http://www1.au.edu.tw/ox_view/edu/tojms/vol-28-2.htm

[13] Tseng, K.-L., Yang, G.-S. and Hsu, K.-C. (2009) On Some Inequalities of Hadamard's Type and Applications. *Taiwanese Journal of Mathematics*, **13**, 1929-1948.

[14] Yang, G.-S. and Tseng, K.-L. (1999) On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities. *Journal of Mathematical Analysis and Applications*, **239**, 180-187. http://dx.doi.org/10.1006/jmaa.1999.6506

[15] Yang, G.-S. and Tseng, K.-L. (2001) Inequalities of Hadamard's Type for Lipschitzian Mappings. *Journal of Mathematical Analysis and Applications*, **260**, 230-238. http://dx.doi.org/10.1006/jmaa.2000.7460

[16] Yang, G.-S. and Tseng, K.-L. (2003) Inequalities of Hermite-Hadamard-Fejér Type for Convex Functions and Lipschitzian Functions. *Taiwanese Journal of Mathematics*, **7** (3), 433-440.

[17] Yang, G.-S., Hwang, D.-Y. and Tseng, K.-L. (2004) Some Inequalities for Differentiable Convex and Concave Mappings. *Computers Mathematics with Applications*, **47**, 207-216. http://dx.doi.org/10.1016/S0898-1221(04)90017-X