Open Journal of Discrete Mathematics, 2014, 4, 60-66
Published Online July 2014 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2014.43009

Some Results on Prime Labeling*

U. M. Prajapati[1], S. J. Gajjar[2]

[1]St. Xaviers College, Ahmedabad, India
[2]Government Polytechnic, Himmatnagar, India
Email: udayan64@yahoo.com, gjr.sachin@gmail.com

Received 7 May 2014; revised 5 June 2014; accepted 24 June 2014

Abstract

In the present work we investigate some classes of graphs and disjoint union of some classes of graphs which admit prime labeling. We also investigate prime labeling of a graph obtained by identifying two vertices of two graphs. We also investigate prime labeling of a graph obtained by identifying two edges of two graphs. Prime labeling of a prism graph is also discussed. We show that a wheel graph of odd order is switching invariant. A necessary and sufficient condition for the complement of W_n to be a prime graph is investigated.

Keywords

Graph Labeling, Prime Labeling, Switching of a Vertex, Switching Invariance

1. Introduction

We begin with simple, finite, undirected and non-trivial graph $G=(V,E)$ with the vertex set V and the edge set E. The number of elements of V, denoted as $|V|$ is called the order of the graph G while the number of elements of E, denoted as $|E|$ is called the size of the graph G. In the present work C_n denotes the cycle with n vertices and P_n denotes the path of n vertices. In the wheel $W_n = C_n + K_1$ the vertex corresponding to K_1 is called the apex vertex and the vertices corresponding to C_n are called the rim vertices. For various graph theoretic notations and terminology we follow Gross and Yellen [1] whereas for number theory we follow D. M. Burton [2]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

For latest survey on graph labeling we refer to J. A. Gallian [3]. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades. For any graph labeling problem following three features are really noteworthy:

*AMS subject classification number: 05C78.

How to cite this paper: Prajapati, U.M. and Gajjar, S.J. (2014) Some Results on Prime Labeling. *Open Journal of Discrete Mathematics*, **4**, 60-66. http://dx.doi.org/10.4236/ojdm.2014.43009

- a set of numbers from which vertex labels are chosen;
- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

The present work is aimed to discuss one such labeling known as prime labeling.

Definition 1.2: A *prime labeling* of a graph G of order n is an injective function $f:V\rightarrow\{1,2,\cdots,n\}$ such that for every pair of adjacent vertices u and v, $gcd\left(f(u),f(v)\right)=1$. The graph which admits prime labeling is called a prime graph.

The notion of prime labeling was originated by Entringer and was discussed in A.Tout [4]. Many researchers have studied prime graphs. It has been proved by H. L. Fu and C. K. Huang [5] that P_n is a prime graph. It has been proved by S. M. Lee [6] that wheel graph W_n is a prime graph if and only if n is even. T. Deretsky [7] has proved that cycle C_n is a prime graph.

Definition 1.3: A *vertex switching* G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining to every other vertex which is not adjacent to v in G.

Definition 1.4: A prime graph is said to be *switching invariant* if for every vertex v of G, the graph G_v obtained by switching the vertex v in G is also a prime graph.

Definition 1.5: For two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ their *cartesian product* $G_1\times G_2$ is defined as the graph whose vertex set is $V_1\times V_2$ and two vertices (u_1,v_1) and (u_2,v_2) in $G_1\times G_2$ are adjacent if $u_1=u_2$ and v_1 is adjacent to v_2 or u_1 is adjacent to u_2 and $v_1=v_2$.

Definition 1.6: $C_n\times P_2$ is called *prism graph*.

Bertrand's Postulate 1.7: For every positive integer $n>1$ there is a prime p such that $n<p<2n$.

2. Some Results on Prime Labeling

Theorem 2.1: If G_1 is a prime graph with order n, where n is even and G_2 is a graph with order 3 then disjoint union of G_1 and G_2 is a prime graph.

Proof: Let $u_1,u_2,u_3,\cdots,u_n$ be the vertices of G_1 and v_1,v_2,v_3 be the vertices of G_2. Let G be a disjoint union of G_1 and G_2. Now G_1 is a prime graph, so there is an injective function $f:\{u_1,u_2,u_3,\cdots,u_n\}\rightarrow\{1,2,\cdots,n\}$ such that for any arbitrary edge $e=u_iu_j$, we have $gcd\left(f(u_i),f(u_j)\right)=1$.

Define a function $g:\{u_1,u_2,u_3,\cdots,u_n,v_1,v_2,v_3\}\rightarrow\{1,2,\cdots,n,n+1,n+2,n+3\}$ as follows:

$$g(u)=\begin{cases} f(u_i) & \text{for } u=u_i, i=1,2,\cdots,n; \\ n+i & \text{for } u=v_i, i=1,2,3. \end{cases}$$

Clearly g is an injective function.

If $e=u_iu_j$ is any edge of G then $gcd\left(g(u_i),g(u_j)\right)=gcd\left(f(u_i),f(u_j)\right)=1$. If $e=v_1v_2$ then $gcd\left(g(v_1),g(v_2)\right)=gcd(n+1,n+2)=1$. If $e=v_2v_3$ then $gcd\left(g(v_2),g(v_3)\right)=gcd(n+2,n+3)=1$. If $e=v_1v_3$ then $gcd\left(g(v_1),g(v_3)\right)=gcd(n+1,n+3)=1$ as n is even.

Thus G admits a prime labeling. So G is a prime graph.

Theorem 2.2: If G_1 is a prime graph with order n, where n is divisible by 6 and G_2 is a prime graph with order 5 then disjoint union of G_1 and G_2 is a prime graph.

Proof: Let $u_1,u_2,u_3,\cdots,u_n$ be the vertices of G_1 and v_1,v_2,v_3,v_4,v_5 be the vertices of G_2. Let G be the disjoint union of G_1 and G_2. Now G_1 is a prime graph, so there exists an injective function $f:\{u_1,u_2,u_3,\cdots,u_n\}\rightarrow\{1,2,\cdots,n\}$ such that for any arbitrary edge $e=u_iu_j$ of G_1, $gcd\left(f(u_i),f(u_j)\right)=1$. Also G_2 is a prime graph, so there is an injective function $g:\{v_1,v_2,v_3,v_4,v_5\}\rightarrow\{1,2,3,4,5\}$ such that for any arbitrary edge $e=v_iv_j$ of G_2, $gcd\left(g(v_i),g(v_j)\right)=1$. Define a function $h:\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_5\}\rightarrow\{1,2,\cdots,n+4,n+5\}$ as follows:

$$h(u)=\begin{cases} f(u_i) & \text{for } u=u_i, i=1,2,\cdots,n; \\ n+g(v_i) & \text{for } u=v_i, i=1,2,3,4,5. \end{cases}$$

Clearly h is an injective function. To prove h is a prime labeling of G we have the following cases:

Case 1: If $e = u_i u_j$ is any edge of G_1 then $gcd\left(h(u_i), h(u_j)\right) = gcd\left(f(u_i), f(u_j)\right) = 1$.

Case 2: Suppose $e = v_i v_j$ is any edge of G_2 and $gcd\left(h(v_i), h(v_j)\right) = gcd\left(n + g(v_i), n + g(v_j)\right) = d$. Here d is an odd natural number as n is even and at least one of $g(v_i)$ and $g(v_j)$ is odd. As $d \mid \left(n + g(v_i)\right)$ and $d \mid \left(n + g(v_j)\right)$ so $d \mid \left(g(v_i) - g(v_j)\right)$. But possible values of $\left|g(v_i) - g(v_j)\right|$ are 1, 2, 3 and 4, and d is odd. So $d = 1$ or $d = 3$. If $d = 3$ then $3 \mid \left(n + g(v_i)\right)$ and $3 \mid \left(n + g(v_j)\right)$. Also $3 \mid n$, therefore $3 \mid g(v_i)$ and $3 \mid g(v_j)$, which is not possible as $gcd\left(g(v_i), g(v_j)\right) = 1$. Thus $d = 1$, hence $gcd\left(h(v_i), h(v_j)\right) = 1$.

Thus G admits prime labeling. So G is a prime graph.

S. K. Vaidya and U. M. Prajapati [8] has proved that if $n_1 \geq 4$ is an even integer and n_2 is a natural number, then the graph obtained by identifying any of the rim vertices of a wheel W_{n_1} with an end vertex of a path graph P_{n_2} is a prime graph. But in the following theorem we have prove that if n_1 is odd then also it is prime.

Theorem 2.3: If $n_1 + n_2 = p$, where p is prime then the graph obtained by identifying one of the rim vertices of W_{n_1} with an end vertex of P_{n_2} is prime.

Proof: Let u_0 be an apex vertex of W_{n_1} and $u_1, u_2, u_3, \cdots, u_{n_1}$ be consecutive rim vertices of W_{n_1} and $v_1, v_2, v_3, \cdots, v_{n_2}$ are consecutive vertices of P_{n_2}. Without loss of generality we can assume that $G(V, E)$ be the graph obtained by identifying a rim vertex u_1 of W_{n_1} with an end vertex v_1 of P_{n_2}. Define $f : V \to \{1, 2, \cdots, |V|\}$ as follows:

$$f(u) = \begin{cases} p & \text{for } u = u_0; \\ i & \text{for } u = u_i,\ i = 1, 2, \cdots, n_1; \\ n_1 + j - 1 & \text{for } u = v_j,\ j = 2, 3, \cdots, n_2. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

Case 1: If $e = u_0 u_i$ then $gcd\left(f(u_0), f(u_i)\right) = gcd(p, i) = 1$, $\forall i = 1, 2, \cdots, n_1$.

Case 2: If $e = u_i u_{i+1}$ then $gcd\left(f(u_i), f(u_{i+1})\right) = gcd(i, i+1) = 1$, $\forall i = 1, 2, \cdots, n_1 - 1$.

Case 3: If $e = v_j v_{j+1}$ then $gcd\left(f(v_j), f(v_{j+1})\right) = gcd(n_1 + j - 1, n_1 + j) = 1$, $\forall j = 2, \cdots, n_2 - 1$.

Case 4: If $e = u_1 v_2$ then $gcd\left(f(u_1), f(v_2)\right) = gcd(1, n_1 + 1) = 1$.

Case 5: If $e = u_{n_1} u_1$ then $gcd\left(f(u_{n_1}), f(u_1)\right) = gcd(n_1, 1) = 1$.

Thus G admits a prime labeling. So G is a prime graph.

Theorem 2.4: A path P_{m+1} and m copies of cycle C_n are given, then the graph obtained by identifying each edge of P_m with an edge of a corresponding copy of the cycle C_n is prime.

Proof: Let $v_1, v_2, v_3, \cdots, v_{m+1}$ be the vertex of P_{m+1} and $u_{1,i}, u_{2,i}, u_{3,i}, \cdots, u_{n,i}$ be the vertices of i^{th} copy of cycle C_n where $i = 1, 2, \cdots, m$. Let G be a graph obtained by identifying an edge $u_{1,i} u_{n,i}$ of i^{th} copy of cycle C_n with an edge $v_i v_{i+1}$ of path P_m, where $i = 1, 2, \cdots, m$. Let V be the set of vertices of G then $|V| = m(n-1) + 1$. Define a function $f : V \to \{1, 2, \cdots, |V|\}$ as follows:

$$f(u) = \begin{cases} 1 + (i-1)(n-1) & \text{for } u = v_i,\ i = 1, 2, \cdots, m+1; \\ (i-1)(n-1) + j & \text{for } u = u_{j,i},\ i = 1, 2, \cdots, m,\ j = 2, 3, \cdots, n-1. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

Case 1: If $e = v_i v_{i+1}$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\left(1 + (i-1)(n-1), 1 + i(n-1)\right) = 1$, for $i = 1, 2, \cdots, m$.

Case 2: If $e = u_{j,i} u_{(j+1),i}$ then $gcd\left(f(u_{j,i}), f(u_{(j+1),i})\right) = gcd\left((i-1)(n-1) + j, (i-1)(n-1) + j + 1\right) = 1$, for $j = 2, 3, \cdots, n-2$ and $i = 1, 2, \cdots, m$.

Case 3: If $e = v_i u_{2,i}$ then $gcd\left(f(v_i), f(u_{2,i})\right) = gcd\left(1 + (i-1)(n-1), 2 + (i-1)(n-1)\right) = 1$.

Case 4: If $e = u_{(n-1),i}v_{i+1}$ then $gcd\left(f\left(u_{(n-1),i}\right), f\left(v_{i+1}\right)\right) = gcd\left(i(n-1), 1+i(n-1)\right) = 1$.

Thus G is a prime graph. So G is a prime graph.

Theorem 2.5: A cycle C_m and m copies of a cycle C_n are given, then the graph obtained by identifying each edge of C_m with an edge of corresponding copy of the cycle C_n is prime.

Proof: Let $v_1, v_2, v_3, \cdots, v_m$ be the vertices of C_m and $u_{1,i}, u_{2,i}, u_{3,i}, \cdots, u_{n,i}$ be the vertices of i^{th} copy of cycle C_n where $i = 1, 2, \cdots, m$. Let G be a graph obtained by identifying an edge $u_{1,i}u_{n,i}$ of i^{th} copy of cycle C_n with an edge $v_i v_{i+1}$ of cycle C_m, where $i = 1, 2, \cdots, m-1$ and an edge $u_{1,m}u_{n,m}$ of m^{th} copy of cycle C_n with an edge $v_m v_1$ of cycle C_m. Let V be the vertex set of G then $|V| = m(n-1)$. Define a function $f : V \to \{1, 2, \cdots, |V|\}$ as follows:

$$f(u) = \begin{cases} 1 + (i-1)(n-1) & \text{for } u = v_i,\ i = 1, 2, \cdots, m; \\ (i-1)(n-1) + j & \text{for } u = u_{j,i},\ j = 1, 2, \cdots, n-1,\ i = 2, 3, \cdots, m. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

Case 1: If $e = v_i v_{i+1}$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\left(1+(i-1)(n-1), 1+i(n-1)\right) = 1$, for $i = 1, 2, \cdots, m-1$.

Case 2: If $e = v_m v_1$ then $gcd\left(f(v_m), f(v_1)\right) = gcd\left(1+(m-1)(n-1), 1\right) = 1$.

Case 3: If $e = u_{j,i}u_{(j+1),i}$ then $gcd\left(f\left(u_{j,i}\right), f\left(u_{(j+1),i}\right)\right) = gcd\left((i-1)(n-1)+j, (i-1)(n-1)+j+1\right) = 1$, for $j = 2, 3, \cdots, n-2$ and $i = 1, 2, \cdots, m$.

Case 4: If $e = u_{(n-1),i}v_{i+1}$, $gcd\left(f\left(u_{(n-1),i}\right), f\left(v_{i+1}\right)\right) = gcd\left(i(n-1), 1+i(n-1)\right) = 1$, for $i = 1, 2, \cdots, m-1$.

Case 5: If $e = u_{(n-1),m}v_1$ then $gcd\left(f\left(u_{(n-1),m}\right), f\left(v_1\right)\right) = gcd\left(m(n-1), 1\right) = 1$.

Case 6: If $e = v_i u_{2,i}$ then $gcd\left(f(v_i), f\left(u_{2,i}\right)\right) = gcd\left(1+(i-1)(n-1), 2+(i-1)(n-1)\right) = 1$, for $i = 1, 2, \cdots, m$.

Thus G admits a prime labeling. So G is a prime graph.

S. K. Vaidya and U. M. Prajapati [9] have proved that switching the apex vertex in W_n is a prime graph and switching a rim vertex in W_n is a prime graph if $n+1$ is prime. But in the following theorem we have proved that W_n is switching invariant if n is even.

Theorem 2.6: W_{2n} is switching invariant.

Proof: Let $v_1, v_2, v_3, \cdots, v_{2n}$ be rim vertices and v_0 be an apex vertex of W_{2n}. According to the degree of vertices of W_{2n} we can take the following cases.

Case 1: Let G be a graph obtained by switching a rim vertex v_{2n}. Let p be a largest prime less than $2n$. Define a function $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ as follows:

$$f(v_i) = \begin{cases} 1 & \text{for } i = 0; \\ p+i & \text{for } i = 1, 2, \cdots, 2n-p+1; \\ i-2n+p & \text{for } i = 2n-p+2, 2n-p+3, \cdots, 2n-1; \\ p & \text{for } i = 2n. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

- If $e = v_0 v_i$, $i \neq 2n$ then $gcd\left(f(v_0), f(v_i)\right) = gcd\left(1, f(v_i)\right) = 1$.
- If $e = v_i v_{i+1}$, $i = 1, 2, \cdots, 2n-p$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\left(p+i, p+i+1\right) = 1$.
- If $e = v_{(2n-p+1)}v_{(2n-p+2)}$ then $gcd\left(f\left(v_{(2n-p+1)}\right), f\left(v_{(2n-p+2)}\right)\right) = gcd\left(2n+1, 2\right) = 1$.
- If $e = v_i v_{i+1}$, $i = 2n-p+2, 2n-p+3, \cdots, 2n-2$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\left(i-2n+p, i-2n+p+1\right) = 1$.
- If $e = v_i v_{2n}$, $i = 2, 3, \cdots, 2n-2$ then $gcd\left(f(v_i), f(v_{2n})\right) = gcd\left(p+i, p\right) = gcd\left(i, p\right) = 1$, as p is the largest prime less than $2n$.

Case 2: Let G be a graph obtained by switching an apex vertex v_0.

Define a function $f: V(G) \to \{1, 2, \cdots, |V(G)|\}$ as follows:

$$f(v_i) = \begin{cases} i & \text{for } i = 1, 2, \cdots, 2n; \\ 2n+1 & \text{for } i = 0. \end{cases}$$

Clearly f is an injective function. It can be easily verified that f is a prime labeling.

Thus from both the cases it follows that G is a prime graph.

Theorem 2.7: The complement of W_n is prime if and only if $3 \le n \le 6$.

Proof: We can easily see that $\overline{W_n}$ is prime for $n = 3, 4, 5$ and 6 from **Figure 1**.

Now if $n \ge 7$ then $(n-3) \ge 4$ and every rim vertex of $\overline{W_n}$ is adjacent to other $(n-3)$ rim vertices. We have total $\left\lfloor \frac{n+1}{2} \right\rfloor$ even numbers to assign $n+1$ vertices. If one of the rim vertices is labeled as even number then other $n-3$ vertices can not be labeled as even number. Also remaining two rim vertices are adjacent, so only one of them can be labeled as even number. The apex vertex can also be labeled as even number. Thus maximum three vertices can be labeled as even number. But if $n \ge 7$ then we have 4 or more even numbers to label. So it is not possible. Thus $\overline{W_n}$ is not prime for $n \ge 7$.

Theorem 2.8: Let $p \ge 3$ be a prime number and take $p-2$ copies of C_{p+1}, then the graph obtained by identifying one vertex of each copy of C_{p+1} with corresponding pendant vertex of $K_{1,p-2}$ is prime.

Proof: Let u_0 be an apex vertex and $u_1, u_2, u_3, \cdots, u_{p-2}$ be pendant vertices of $K_{1,p-2}$. Also let $v_{i,1}, v_{i,2}, v_{i,3}, \cdots, v_{i,p+1}$ be the vertices of i^{th} copy of C_{p+1}. Now let G be the graph obtained by identifying a pendant vertex u_i of $K_{1,p-2}$ with a vertex $v_{i,p+1}$ of i^{th} copy of C_{p+1}, where $i = 1, 2, \cdots, p-2$.

Define a function $f: V(G) \to \{1, 2, \cdots, |V|\}$, where $|V| = (p-2)(p+1)+1$ as follows:

$$f(u) = \begin{cases} 1 & \text{for } u = u_0; \\ i(p+1)+1 & \text{for } u = u_i = v_{i,p+1},\ i = 1, 2, \cdots, p-2; \\ (i-1)(p+1)+j+1 & \text{for } u = v_{i,j},\ i = 1, 2, \cdots, p-2,\ j = 1, 2, \cdots, p. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

Case 1: If $e = u_0 u_i = u_0 v_{i,p+1}$, $gcd(f(u_0), f(u_i)) = gcd(1, i(p+1)+1) = 1$, for $i = 1, 2, \cdots, p-2$.

Case 2: If $e = v_{i,j} v_{i,j+1}$, $gcd(f(v_{i,j}), f(v_{i,j+1})) = gcd((i-1)(p+1)+j+1, (i-1)(p+1)+j+2) = 1$ for $i = 1, 2, \cdots, p-2$ and $j = 1, 2, \cdots, p$.

Case 3: If $e = v_{i,1} v_{i,p+1}$ then for $i = 1, 2, \cdots, p-2$ and $j = 1, 2, \cdots, p$,

$$\begin{aligned} gcd(f(v_{i,1}), f(v_{i,p+1})) &= gcd((i-1)(p+1)+2, i(p+1)+1) \\ &= gcd((i-1)(p+1)+2, i(p+1)+1-(i-1)(p+1)-2) \\ &= gcd((i-1)(p+1)+2, p) \\ &= gcd((i-1)(p+1)+2-(i-1)p, p) \\ &= gcd(i+1, p) \\ &= 1 \qquad (\text{as } i < p-1 \text{ so } i+1 < p \text{ and } p \text{ is a prime.}) \end{aligned}$$

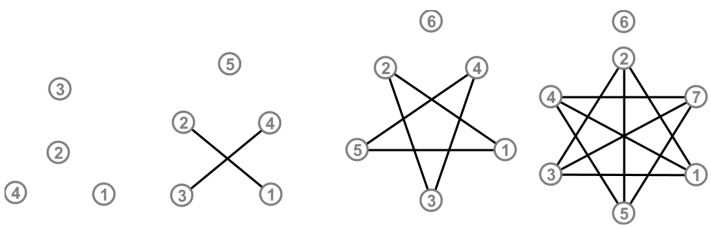

Figure 1. Prime labeling of $\overline{W_3}$, $\overline{W_4}$, $\overline{W_5}$ and $\overline{W_6}$.

Thus G admits a prime labeling. So G is a prime graph.

Theorem 2.9: If $n \geq 3$ is an odd integer then the prism graph $C_n \times P_2$ is not prime.

Proof: In the prism graph $C_n \times P_2$ there are two cycles C_n. So total number of vertices are $2n$. So we have to use 1 to $2n$ natural numbers to label these vertices, and from 1 to $2n$ there are n even integers. If n is odd then we can use at the most $\frac{n-1}{2}$ even integers to label the vertices of a cycle C_n. We have such two cycles, so we can use at the most $\frac{n-1}{2}+\frac{n-1}{2}=n-1$ even integers to label the vertices of $C_n \times P_2$. But from 1 to $2n$ there are n even integers. So such prime labeling is not possible.

Thus $C_n \times P_2$ is not prime if $n \geq 3$ is an odd integer.

Theorem 2.10: If $p \geq 3$ is a prime number then the prism graph $C_{p-1} \times P_2$ is prime.

Proof: In the prism graph $C_{p-1} \times P_2$, let $v_{1,1}, v_{1,2}, v_{1,3}, \cdots, v_{1,p-1}$ be the vertices of one cycle and $v_{2,1}, v_{2,2}$, $v_{2,3}, \cdots, v_{2,p-1}$ be the vertices of the other cycle and a vertex $v_{1,i}$ is joined with $v_{2,i}$ by an edge for $i = 1, 2, \cdots, p-1$. Define a function $f : V(G) \to \{1, 2, \cdots, 2p-2\}$ as follows:

$$f(v_{i,j}) = \begin{cases} j & \text{for } i = 1,\ j = 1, 2, \cdots, p-1; \\ p+j & \text{for } i = 2,\ j = 1, 2, \cdots, p-2; \\ p & \text{for } i = 2,\ j = p-1. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

Case 1: If $e = v_{1,j}v_{1,j+1}$ then $gcd\left(f\left(v_{1,j}\right), f\left(v_{1,j+1}\right)\right) = gcd(j, j+1) = 1$, for $j = 1, 2, \cdots, p-2$.

Case 2: If $e = v_{2,j}v_{2,j+1}$ then $gcd\left(f\left(v_{2,j}\right), f\left(v_{2,j+1}\right)\right) = gcd(p+j, p+j+1) = 1$, for $j = 1, 2, \cdots, p-3$.

Case 3: If $e = v_{1,1}v_{1,p-1}$ then $gcd\left(f\left(v_{1,1}\right), f\left(v_{1,p-1}\right)\right) = gcd(1, p-1) = 1$.

Case 4: If $e = v_{2,p-2}v_{2,p-1}$ then $gcd\left(f\left(v_{2,p-2}\right), f\left(v_{2,p-1}\right)\right) = gcd(2p-2, p) = 1$.

Case 5: If $e = v_{2,1}v_{2,p-1}$ then $gcd\left(f\left(v_{2,1}\right), f\left(v_{2,p-1}\right)\right) = gcd(p+1, p) = 1$.

Case 6: If $e = v_{1,j}v_{2,j}$ then $gcd\left(f\left(v_{1,j}\right), f\left(v_{2,j}\right)\right) = gcd(j, p+j) = gcd(j, p) = 1$, for $j = 1, 2, \cdots, p-2$.

Case 7: If $e = v_{1,p-1}v_{2,p-1}$ then $gcd\left(f\left(v_{1,p-1}\right), f\left(v_{2,p-1}\right)\right) = gcd(p-1, p) = 1$.

Thus G admits a prime labeling. So G is a prime graph.

Theorem 2.11: A graph obtained by joining every rim vertex of a wheel graph W_{p-1} with corresponding vertex of a cycle C_{p-1} is a prime graph, where p is a prime number not less than 3.

Proof: Let v_0 be an apex vertex and $v_1, v_2, v_3, \cdots, v_{p-1}$ be rim vertices of W_{p-1}. Also $u_1, u_2, u_3, \cdots, u_{p-1}$ are the vertices of C_{p-1}. Let G be the graph obtained by joining a vertex v_i of W_{p-1} with a vertex u_i of C_{p-1} by an edge, where $i = 1, 2, \cdots, p-1$. Define a function $f : V(G) \to \{1, 2, \cdots, 2p-1\}$ as follows:

$$f(u) = \begin{cases} i & \text{for } u = v_i, i = 1, 2, \cdots, p-1; \\ p & \text{for } u = v_0; \\ p+i & \text{for } u = u_i, i = 1, 2, \cdots, p-1. \end{cases}$$

Clearly f is an injective function. Let e be an arbitrary edge of G. To prove f is a prime labeling of G we have the following cases:

Case 1: If $e = v_iv_{i+1}$ then $gcd\left(f\left(v_i\right), f\left(v_{i+1}\right)\right) = gcd(i, i+1) = 1$, for $i = 1, 2, \cdots, p-2$.

Case 2: If $e = v_1v_{p-1}$ then $gcd\left(f\left(v_1\right), f\left(v_{p-1}\right)\right) = gcd(1, p-1) = 1$.

Case 3: If $e = v_0v_i$ then $gcd\left(f\left(v_0\right), f\left(v_i\right)\right) = gcd(p, i) = 1$, for $i = 1, 2, \cdots, p-1$.

Case 4: If $e = u_iu_{i+1}$ then $gcd\left(f\left(u_i\right), f\left(u_{i+1}\right)\right) = gcd(p+i, p+i+1) = 1$, for $i = 1, 2, \cdots, p-2$.

Case 5: If $e = u_1 u_{p-1}$ then $gcd\left(f\left(u_1\right), f\left(u_{p-1}\right)\right) = gcd\left(p+1, 2p-1\right) = 1$.

Case 6: If $e = v_i u_i$ then $gcd\left(f\left(v_i\right), f\left(u_i\right)\right) = gcd\left(i, p+i\right) = 1$, for $i = 1, 2, \cdots, p-1$.

Thus G admits a prime labeling. So G is a prime graph.

3. Concluding Remarks

Study of relatively prime numbers is very interesting in the theory of numbers and it is challenging to investigate prime labeling of some families of graphs. Here we investigate several results of some classes of graphs about prime labeling. Extending the study to other graph families is an open area of research.

Acknowledgements

The authors are highly thankful to the anonymous referee for valuable comments and constructive suggestions on the first draft of this paper.

References

[1] Gross, J. and Yellen, J. (2004) Handbook of Graph Theory, CRC Press, Boca Raton.

[2] Burton, D.M. (2007) Elementary Number Theory. 6th Edition, Tata McGraw-Hill Publishing Company Limited, New Delhi.

[3] Gallian, J.A. (2012) A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, **19**, # DS6.

[4] Tout, A., Dabboucy, A.N. and Howalla, K. (1982) Prime Labeling of Graphs. *National Academy Science Letters*, **11**, 365-368.

[5] Fu, H.L. and Huang, K.C. (1994) On Prime Labellings. *Discrete Mathematics*, **127**, 181-186. http://dx.doi.org/10.1016/0012-365X(92)00477-9

[6] Lee, S.M., Wui, I. and Yeh, J. (1988) On the Amalgamation of Prime Graphs. *Bull. of Malaysian Math. Soc.*, **11**, 59-67.

[7] Deretsky, T., Lee, S.M. and Mitchem, J. (1991) On Vertex Prime Labelings of Graphs. In: Alvi, J., Chartrand, G., Oellerman, O. and Schwenk, A., Eds., *Graph Theory, Combinatorics and Applications. Proceedings of the 6th International Conference Theory and Applications of Graphs*, Wiley, New York, 359-369.

[8] Vaidya, S.K. and Prajapati, U.M. (2011) Some Results on Prime and K-Prime Labeling. *Journal of Mathematics Research*, **3**, 66-75. http://dx.doi.org/10.5539/jmr.v3n1p66

[9] Vaidya, S.K. and Prajapati, U.M. (2012) Some Switching Invariant Prime Graphs. *Open Journal of Discrete Mathematics*, **2**, 17-20. http://dx.doi.org/10.4236/ojdm.2012.21004