Open Journal of Discrete Mathematics, 2014, 4, 55-59
Published Online July 2014 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2014.43008

Closure for Spanning Trees with *k*-Ended Stems

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Received 29 April 2014; revised 27 May 2014; accepted 25 June 2014

Abstract

Let T be a tree. The set of leaves of T is denoted by $\mathrm{Leaf}(T)$. The subtree $T-\mathrm{Leaf}(T)$ of T is called the stem of T. A stem is called a *k*-ended stem if it has at most *k*-leaves in it. In this paper, we prove the following theorem. Let G be a connected graph and $k \geq 2$ be an integer. Let u and v be a pair of nonadjacent vertices in G. Suppose that $|N_G(u) \cup N_G(v)| \geq |G|-k-1$. Then G has a spanning tree with *k*-ended stem if and only if $G+uv$ has a spanning tree with *k*-ended stem. Moreover, the condition on $|N_G(u) \cup N_G(v)|$ is sharp.

Keywords

Closure, Spanning Tree, Stem, *k*-End Stem

1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (*i.e.*, $|G| = |V(G)|$). For a vertex v of G, the degree of v in G is denoted by $\deg_G(v)$, and the set of vertices adjacent to v is called the neighborhood of v and denoted by $N_G(v)$. In particular, $\deg_G(v) = |N_G(v)|$. An edge joining two vertices x and y is denoted by xy or yx.

Let T be a tree. A vertex of T with degree one is often called a *leaf* of T, and the set of leaves of T is denoted by $\mathrm{Leaf}(T)$. The subtree $T-\mathrm{Leaf}(T)$ of T is called the *stem* of T and denoted by $\mathrm{Stem}(T)$. A spanning tree with specified stem was first considered in [1].

A tree having at most k leaves is called a *k-ended tree*. So a tree whose stem has at most k leaves in it is called a *tree with k-ended stem*. Notice that a tree with 2-ended stem is nothing but a *caterpillar*, whose stem is a path. We consider a spanning tree with *k*-ended stem.

How to cite this paper: Yan, Z. (2014) Closure for Spanning Trees with *k*-Ended Stems. *Open Journal of Discrete Mathematics*, **4**, 55-59. http://dx.doi.org/10.4236/ojdm.2014.43008

We make a remark about spanning trees with *k*-ended stem from the point of view of dominating set. A subgraph H of a graph G is said to *dominate* G if every vertex of G not contained in H has a neighbor in H. Namely, H dominates G if every vertex $v \in V(G) - V(H)$ satisfies $N_G(v) \cap V(H) \neq \emptyset$. So a graph G has a spanning tree with *k*-ended stem if and only if G has a *k*-ended tree that dominates G. There are many researches on dominating cycles and dominating paths (for example, see [2] and [3] with stronger definition of domination). Thus the concept of spanning trees with *k*-ended stem can be also considered as a generalization of dominating paths.

For an integer $k \geq 2$ and a graph G, $\sigma_k(G)$ denotes the minimum degree sum of k independent vertices of G. The following theorem gives a sufficient condition using $\sigma_k(G)$ for a graph to have a spanning tree with *k*-ended stem.

Theorem 1 (Tsugaki and Zhang [4]) *Let G be a connected graph and $k \geq 2$ be an integer. If*

$$\sigma_3(G) \geq |G| - 2k + 1,$$

then G has a spanning tree with *k*-ended stem.

Another result on spanning trees with *k*-ended stem is the following.

Theorem 2 (Kano and Yan [5]) *Let G be a connected graph and $k \geq 2$ be an integer. If G satisfies one of the following conditions, then G has a spanning tree with k-ended stem.*

(1) $\sigma_{k+1}(G) \geq |G| - k - 1$.

(2) G is claw-free and $\sigma_{k+1}(G) \geq |G| - 2k - 1$.

A closure operation is useful in the study of the existence of hamiltonian cycles, hamiltonian paths and other spanning subgraphs in graphs. It was first introduced by Bondy and Chvátal.

Theorem 3 (Bondy and Chval [6]) *Let G be a graph and let u and v be two nonadjacent vertices of G.*

(1) Suppose $\deg_G(u) + \deg_G(v) \geq |G|$. Then G has a hamiltonian cycle if and only if $G + uv$ has a hamiltonian cycle.

(2) Suppose $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then G has a hamiltonian path if and only if $G + uv$ has a hamiltonian path.

After [6], many researchers have defined other closure concepts for various graph properties. The following theorem gives a result on closure for spanning *k*-ended tree.

Theorem 4 (Broersma and Tuinstra [7]) *Let $k \geq 2$ be an integer, and let G be a graph. Let u and v be a pair of nonadjacent vertices of G with $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree.*

Another type of closure theorem on spanning *k*-ended tree can be found in Fujisawa, Saito and Schiermeyer [8]. The interested reader is referred to the survey [9] on closure concepts.

In this paper, we prove the following theorem.

Theorem 5 *Let G be a connected graph and $k \geq 2$ be an integer. Let u and v be a pair of nonadjacent vertices of G such that*

$$|N_G(u) \cup N_G(v)| \geq |G| - k - 1. \tag{1}$$

Then G has a spanning tree with *k*-ended stem if and only if $G + uv$ has a spanning tree with *k*-ended stem.

Before proving Theorem 5, we show that the condition (1) in Theorem 5 is sharp. We construct a graph G as follows. Let $k \geq 2$ and $m \geq 1$ be integers, and let K_m be a complete graph of order m, which is a subgraph of G. Let $u, u_1, \cdots, u_k, v, v_1, \cdots, v_k$ be $2k + 2$ vertices of G not contained in K_m. Join u and v to all the vertices of K_m by edges. Join $u_i, 1 \leq i \leq k$, to u and v_i by edges. Then the resulting graph is G (see **Figure 1**). It is immediate that $G + uv$ has a spanning tree with *k*-ended stem, where all the vertices of K_m and v are leaves of the spanning tree, and $|N_G(u) \cup N_G(v)| = |G| - k - 2$. However G has no spanning tree with *k*-ended stem. Therefore the condition (1) is sharp. Moreover, in **Figure 1**, $\deg_G(u) + \deg_G(v) = 2m + k = |G| + m - k - 2$. Since m be an arbitrary integer, the degree sum of u and v can be arbitrarily great. This implies that we can not find a condition similar to Theorem 4 for spanning tree with *k*-ended stem.

Some results on spanning *k*-ended trees and other spanning trees with given properties can be found in [10], and many current results on spanning trees can be found in [11].

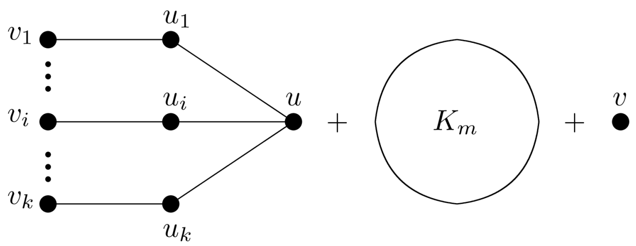

Figure 1. A graph G having no spanning tree with *k*-ended stem.

2. Proof of Theorem 5

In this section we prove Theorem 5. Without mentioning, we often use the fact that $V(T)$ is a disjoint union of $V(\text{Leaf}(T))$ and $V(\text{Stem}(T))$.

Proof of Theorem 5. Since the necessity of the theorem is trivial, we only prove the sufficiency. Assume, to the contrary, that $G+uv$ has a spanning tree with *k*-ended stem but G does not have a spanning tree with *k*-ended stem. Let us denote $G^*=G+uv$. Choose a spanning tree T with *k*-ended stem of G^* so that

(T1) $|\text{Leaf}(\text{Stem}(T))|$ is as small as possible, and

(T2) $|\text{Stem}(T)|$ is as small as possible subject to (T1).

It is obvious that the edge uv is contained in T since otherwise T is a spanning tree with *k*-ended stem of G. Let $\text{Leaf}(\text{Stem}(T))=X=\{x_1,x_2,\cdots,x_l\}$. Then obviously $2\le l\le k$.

Claim 1. (1) X is an independent set of G; (2) For every $x_i, 1\le i\le l$, there exists a vertex $y_i\in\text{Leaf}(T)$ that is adjacent to x_i in T and $N_{G^*}(y_i)\subseteq\text{Leaf}(T)\cup\{x_i\}$.

By the choice (T1), it is easy to see that if $l\ge 3$, then X is an independent set of G^*, and so is of G. Assume that $l=2$ and the two leaves x_1 and x_2 of $\text{Stem}(T)$ are adjacent in G^*. It is easy to see that x_1 and x_2 are not adjacent in $\text{Stem}(T)$ since otherwise we can obtain a spanning tree with 2-ended stem of G from T. If u and v are both contained in $\text{Stem}(T)$, then $T-uv+x_1x_2$ is a spanning tree with 2-ended stem of G, which contradicts the assumption. Hence we may assume that u is a vertex of $\text{Stem}(T)$ and v is a leaf of T by symmetry of u and v and by $uv\in E(T)$. Since G is connected, v is adjacent to a vertex w. If w is contained in $\text{Stem}(T)$, then $T-uv+wv$ is a spanning tree with 2-ended stem of G, a contradiction. If w is a leaf of T, let w^* be the vertex of $\text{Stem}(T)$ adjacent to w in T, and let z be a vertex of $\text{stem}(T)$ adjacent to w^*. Then $T-uv-w^*z+x_1x_2+wv$ is a spanning tree with 2-ended stem of G, a contradiction. Therefore, $X=\{x_1,x_2\}$ is an independent set of G^*. Hence (1) of Claim 1 follows.

Suppose that there exists a vertex x_s, $1\le s\le l$, such that every leaf y adjacent to x_s in T satisfies $N_{G^*}(y)\cap(\text{Stem}(T)-\{x_s\})\ne\varnothing$. Then for every leaf y adjacent to x_s in T, remove the edge yx_s from T and add an edge yz of G^*, where $z\in V(\text{Stem}(T))-\{x_s\}$. Denote the resulting tree of G^* by T^*. Then T^* is a spanning tree of G^* and satisfies $\text{Stem}(T^*)=\text{Stem}(T)-\{x_s\}$, which contradicts the condition (T2). Therefore, (2) of Claim 1 holds.

Hereafter, we take the vertices y_i $(1\le i\le l)$ as in Claim 1(2). Let $Y=\{y_1,y_2,\cdots,y_l\}$. Since the edge uv is contained in T, let $T-uv=T_1\cup T_2$, where $u\in V(T_1)$ and $v\in V(T_2)$. Since G is a connected graph, there exist a vertex $a\in V(T_1)$ and a vertex $b\in V(T_2)$ which are adjacent in G.

Claim 2. $l=k$.

The claim holds when $k=2$, and so we assume that $k\ge 3$. If $l\le k-2$, then $T-uv+ab$ is a spanning tree with *k*-ended stem of G, which contradicts the assumption. Next we consider the case where $l=k-1$. If either v is a leaf of T or the degree of v in $\text{Stem}(T)$ is 1 or greater than 2, then $T-uv+ab$ is a spanning tree with *k*-ended stem of G, which contradicts the assumption. Thus the degree of v in $\mathit{Stem}(T)$ is 2. By symmetry of u and v, we may assume that the degree of u in $\text{Stem}(T)$ is also 2, and it is clear that uv is an edge of $\text{Stem}(T)$.

First we consider $l=k-1$ and $k\ge 4$. If a vertex $x\in X\cap V(T_1)$ is adjacent to v in G, then $T-uv+xv$ is a spanning tree with *k*-ended stem of G, a contradiction. Thus no vertex of $X\cap V(T_1)$ is adjacent to v in G. By symmetry of u and v, no vertex of $X\cap V(T_2)$ is adjacent to u in G. Assume that a vertex $x\in X$ is adjacent to u in G and T_1 contains at least two vertices of X. Then the path in $\text{Stem}(T)$ connecting

x and u contains a vertex of degree at least 3 in $\mathrm{Stem}(T)$. Let e be an edge of the path which is incident with a vertex with degree at least 3 in $\mathrm{Stem}(T)$. Then $T' = T + xu - e$ is a spanning tree with k-ended stem of G^*, and u has degree at least 3 in $\mathrm{Stem}(T')$. Then we can derive a contradiction by the same argument as in the above first paragraph. Therefore if a vertex x of X is adjacent to u in G, then T_1 contains exactly one vertex of X. Note that x may be adjacent to u but not to v. Since $|X| = l \geq 3$ by $k \geq 4$, $|T_2|$ contains at least two vertices of X, which means no vertex of X is adjacent to v in G by the same argument on T_1 and x given above.

By the above fact and Claim (1), we obtain

$$N_G(u) \cup N_G(v) \subseteq \left(V(G) - Y - X - \{u,v\}\right) \cup \{x\},$$

which implies $|N_G(u) \cup N_G(v)| \leq |G| - 2l - 1 < |G| - k - 2$ by $l = k - 1$ and $k \geq 4$, which contradicts (1).

Next we consider the case where $k = 3$ and $l = k - 1 = 2$. In this case, $\mathrm{Stem}(T)$ is a path, and let x_1 and x_2 be the leaves of $\mathrm{Stem}(T)$ where $x_1 \in V(T_1)$ and $x_2 \in V(T_1)$. By the same argument as in the above paragraph, we have that neither u and x_2 nor v and x_1 are adjacent in G. We shall show that u and x_1 are not adjacent in G. Assume that u and x_1 are adjacent in G. Let a^* be the vertex adjacent to a in T if $a \in \mathrm{Leaf}(T)$ or $a^* = a$ if $a \in \mathrm{Stem}(T)$, and let c be a vertex adjacent to a^* in $\mathrm{Stem}(T)$. Then $T - uv - a^*c + ux_1 + ab$ is a spanning tree with 3-ended stem of G, a contradiction. Similarly, v and x_2 are not adjacent in G.

By the above fact and Claim 1, we obtain

$$N_G(u) \cup N_G(v) \subseteq V(G) - \{y_1, y_2\} - \{x_1, x_2\} - \{u,v\},$$

which implies $|N_G(u) \cup N_G(v)| \leq |G| - 6 < |G| - k - 2$ by $k = 3$, which contradicts (1). Hence Claim 2 holds.

We consider the following two cases:

Case 1. u and v are both contained in $\mathrm{Stem}(T)$.

Subcase 1.1 Both u are v are vertices of $\mathrm{Stem}(\mathrm{Stem}(T))$.

In this case, by Claim 1 (2), we have

$$N_G(u) \cup N_G(v) \subseteq V(G) - Y - \{u,v\},$$

then $|N_G(u) \cup N_G(v)| \leq |G| - k - 2$ by Claim 2, which contradicts (1).

Subcase 1.2 u is a leaf of $\mathrm{Stem}(T)$ and v is a vertex of $\mathrm{Stem}(\mathrm{Stem}(T))$.

In this case, without loss of generality, let $u = x_1$. If either v has degree greater than 2 in $\mathrm{Stem}(T)$ or $b = v$, then $T - vu + ab$ is a spanning tree with k-ended stem of G, which is a contradiction. Hence v has degree 2 in $\mathrm{Stem}(T)$ and $b \neq v$.

Let b^* be the vertex adjacent to b in T if $b \in \mathrm{Leaf}(T)$ or $b^* = b$ if $b \in \mathrm{Stem}(T)$. Then $b^* \notin \{x_1, x_2, \cdots, x_k\}$, since otherwise, $T - vx_1 + ab$ is a spanning tree with k-ended stem of G. Let $P_T(v, x_i)$ be the path connecting v and x_i in $\mathrm{Stem}(T)$. Then there exists at least one vertex $x \in X$ such that $P_T(v,x)$ pass through b^*. We assign an orientation in $P_T(v,x)$ from v to x, and b^{*+} be the successor of b^*. If v and x are adjacent in G, then $T - vx_1 - b^*b^{*+} + ab + vx$ is a spanning tree with k-ended stem of G, which contradicts the assumption. Therefore, by Claim 1(2), we have

$$N_G(v) \cup N_G(x_1) \subseteq V(G) - \{v, x_1, x, y_2, \cdots, y_k\},$$

Then $|N_G(v) \cup N_G(x_1)| \leq |G| - k - 2$, which contradicts (1).

Case 2. u is a vertex of $\mathrm{Stem}(T)$ and v is a leaf of T.

Subcase 2.1 u is a leaf of $\mathrm{Stem}(T)$.

In this case, if v is adjacent to a vertex w in G, where w is a vertex of $\mathrm{Stem}(T)$, then $T - uv + vw$ is a spanning tree with k-ended stem of G, which contradicts the assumption. So the neighborhood of v is contained in $\mathrm{Leaf}(T)$. Without loss of generality, let $u = x_1$ and $v = y_1$.

By Claim 1 (2), we have

$$N_G(u) = N_G(x_1) \subseteq V(G) - X - Y, \text{ and } N_G(v) = N_G(y_1) \subseteq \mathrm{Leaf}(T) - Y.$$

Hence

$$N_G(u) \cup N_G(v) \subseteq V(G) - X - Y.$$

That means $\left|N_G(u)\bigcup N_G(v)\right| \le |G|-2k \le |G|-k-2$, which contradicts (1).

Subcase 2.2 u is a vertex of $\text{Stem}(\text{Stem}(T))$.

In this case, by Claim 1 (2), $N_G(u)\bigcap Y=\varnothing$. If there exists some $y\in Y$ such that $yv\in E(G)$, then $T-uv+yv$ is a spanning tree with k-ended stem of G, which contradicts the assumption. Hence $N_G(v)\bigcap Y=\varnothing$. We have

$$N_G(u)\bigcup N_G(v)\subseteq V(G)-Y-\{u,v\},$$

then $\left|N_G(u)\bigcup N_G(v)\right| \le |G|-k-2$, which contradicts (1).

Consequently, the proof is complete. □

Acknowledgments

The author would like to thank Prof. Mikio Kano for introducing me to problems on spanning trees with specified stems and for his valuable suggestions.

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