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Identities of Symmetry for *q*-Euler Polynomials

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Abstract

In this paper, we derive eight basic identities of symmetry in three variables related to q-Euler polynomials and the q -analogue of alternating power sums. These and most of their corollaries are new, since there have been results only about identities of symmetry in two variables. These abundance of symmetries shed new light even on the existing identities so as to yield some further interesting ones. The derivations of identities are based on the p-adic integral expression of the generating function for the q -Euler polynomials and the quotient of integrals that can be expressed as the exponential generating function for the q -analogue of alternating power sums.

Keywords: q -Euler Polynomial, q -Analogue of Alternating Power sum, Fermionic Integral, Identities of Symmetry

1. Introduction and Preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. For a continuous function $f:\mathbb{Z}_p \to \mathbb{C}_p$, the p-adic fermionic integral of f is defined by

$$\int_{\mathbb{Z}_p} f(z)\,\mathrm{d}\mu_{-1}(z) = \lim_{N\to\infty}\sum_{j=0}^{p^N-1} f(j)(-1)^j .$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1)\,\mathrm{d}\mu_{-1}(z) + \int_{\mathbb{Z}_p} f(z)\,\mathrm{d}\mu_{-1}(z) = 2f(0). \quad (1)$$

Let $|\ |_p$ be the normalized absolute value of $\mathbb{C}_p$, such that $|p|_p = 1/p$, and let

$$E = \left\{ t \in \mathbb{C}_p \,\middle|\, |t|_p < p^{\frac{-1}{p-1}} \right\}. \quad (2)$$

Assume that $q, t \in \mathbb{C}_p$, with $q-1, t \in E$, so that $q^z = \exp(z \log q)$ and e^{zt} are, as functions of z, analytic functions on $\mathbb{Z}_p$. By applying (1) to f, with $f(z) = q^z e^{tz}$, we get the p-adic integral expression of the generating function for q-Euler numbers $E_{n,q}$:

$$\int_{\mathbb{Z}_p} q^z \mathrm{e}^{zt}\,\mathrm{d}\mu_{-1}(z) = \frac{2}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} \left(t \in E\right). \quad (3)$$

So we have the following p-adic integral expression of the generating function for the q-Euler polynomials $E_{n,q}(x)$:

$$\begin{aligned}\int_{\mathbb{Z}_p} q^z \mathrm{e}^{(x+z)t}\,\mathrm{d}\mu_{-1}(z) &= \frac{2}{q\mathrm{e}^t+1}\mathrm{e}^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\left(t\in E, x\in\mathbb{Z}_p\right).\end{aligned} \quad (4)$$

Note here that in [7] ζ was used in place of q, and that q -Euler numbers and polynomials were coined respectively as ζ -Euler numbers and polynomials.

Let $T_{k,q}(n)$ denote the q-analogue of alternating kth power sum of the first $n+1$ nonnegative integers, namely

$$\begin{aligned}T_{k,q}(n) &= \sum_{i=0}^{n}(-1)^i i^k q^i \\ &= (-1)^0 0^k q^0 + (-1)^1 1^k q^1 + \cdots + (-1)^n n^k q^n .\end{aligned} \quad (5)$$

In particular,

$$\begin{aligned} T_{0,q}(n) &= \frac{(-q)^{n+1}-1}{(-q)-1} = [n+1]_{-q}, \\ T_{k,q}(0) &= \begin{cases} 1, & \text{for } k=0, \\ 0, & \text{for } k>0. \end{cases}\end{aligned} \quad (6)$$

From (3) and (5), one easily derives the following identities: for any odd positive integer w,

$$\frac{\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{wy} e^{wyt} d\mu_{-1}(y)} = \sum_{i=0}^{w-1} (-1)^i q^i e^{it} = \sum_{k=0}^{\infty} T_{k,q}(w-1) \frac{t^k}{k!} \, (t \in E) \quad . \tag{7}$$

In what follows, we will always assume that the p-adic fermionic integrals of the various exponential functions on $\mathbb{Z}_p$ are defined for $t \in E$ (cf. (2)), and therefore it will not be mentioned.

[1,2,5,8,9] are some of the previous works on identities of symmetry involving Bernoulli polynomials and power sums. These results were generalized in [4] to obtain identities of symmetry involving three variables in contrast to the previous works involving just two variables.

In this paper, we will produce 8 basic identities of symmetry in three variables w_1, w_2, w_3 related to q-Euler polynomials and the q-analogue of alternating power sums (cf. (44), (45), (48), (51), (55), (57), (59), (60)). These and most of their corollaries seem to be new, since there have been results only about identities of symmetry in two variables in the literature. These abundance of symmetries shed new light even on the existing identities. For instance, it has been known that (8) and (9) are equal and (10) and (11) are so (cf. [7,(2.11),(2,16)]). In fact, (8)-(11) are all equal, as they can be derived from one and the same p-adic integral. Perhaps, this was neglected to mention in [7]. In addition, we have a bunch of new identities in (12)-(15). All of these were obtained as corollaries(cf. Cor. 4.9, 4.12, 4.15) to some of the basic identities by specializing the variable w_3 as 1. Those would not be unearthed if more symmetries had not been available. Related to q-Bernoulli polynomials and the q-analogue of power sums, identities of symmetry in three variables were also obtained in [3] as an extension of identities of symmetry in two variables in [6].

Let w_1, w_2, be any odd positive integers. Then we have:

$$\sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2}}(w_1 y_1) T_{n-k,q^{w_1}}(w_2-1) w_1^{n-k} w_2^k \tag{8}$$

$$= \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(w_2 y_1) T_{n-k,q^{w_2}}(w_1-1) w_2^{n-k} w_1^k \tag{9}$$

$$= w_1^n \sum_{i=0}^{w_1-1} (-1)^i q^{w_2 i} E_{n,q^{w_1}}\left(w_2 y_1 + \frac{w_2}{w_1} i\right) \tag{10}$$

$$= w_2^n \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{n,q^{w_2}}\left(w_1 y_1 + \frac{w_1}{w_2} i\right) \tag{11}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1 w_2}}(y_1) T_{l,q^{w_2}}(w_1-1) T_{m,q^{w_1}}(w_2-1) w_1^{k+m} w_2^{k+l} \tag{12}$$

$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_1}}(w_2-1) w_2^k \sum_{i=0}^{w_1-1} (-1)^i q^{w_2 i} E_{k,q^{w_1 w_2}}\left(y_1 + \frac{i}{w_1}\right) \tag{13}$$

$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_2}}(w_1-1) w_1^k \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{k,q^{w_1 w_2}}\left(y_1 + \frac{i}{w_2}\right) \tag{14}$$

$$= (w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} q^{w_2 i + w_1 j} E_{n,q^{w_1 w_2}}\left(y_1 + \frac{i}{w_1} + \frac{j}{w_2}\right). \tag{15}$$

The derivations of identities are based on the p-adic integral expression of the generating function for the q-Euler polynomials in (4) and the quotient of integrals in (7) that can be expressed as the exponential generating function for the q -analogue of alternating power sums. We indebted this idea to the papers [5,6].

2. Several Types of Quotients of Fermionic Integrals

Here we will introduce several types of quotients of p-adic fermionic integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^3$ from which some interesting identities follow owing to the built-in symmetries in w_1, w_2, w_3. In the following, w_1, w_2, w_3 are all positive integers and all of the explicit expressions of integrals in (17), (19), (21), and (23) are obtained from the identity in (3).

(a) Type Λ_{23}^i (for $i = 0,1,2,3$)

$$I(\Lambda_{23}^i) = \frac{\int_{\mathbb{Z}_p^3} q^{w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3} e^{\left(w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 \left(\sum_{j=1}^{3-i} y_j\right)\right) t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3)}{\left(\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)\right)^i} \tag{16}$$

$$= \frac{2^{3-i} e^{w_1 w_2 w_3 \left(\sum_{j=1}^{3-i} y_j\right) t} \left(q^{w_1 w_2 w_3} e^{w_1 w_2 w_3 t} + 1\right)^i}{\left(q^{w_2 w_3} e^{w_2 w_3 t} + 1\right)\left(q^{w_1 w_3} e^{w_1 w_3 t} + 1\right)\left(q^{w_1 w_2} e^{w_1 w_2 t} + 1\right)}. \tag{17}$$

(b) Type Λ_{13}^i (for $i = 0,1,2,3$)

$$I(\Lambda_{13}^i) = \frac{\int_{\mathbb{Z}_p^3} q^{w_1 x_1 + w_2 x_2 + w_3 x_3} e^{\left(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 \left(\sum_{j=1}^{3-i} y_j\right)\right) t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3)}{\left(\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)\right)^i} \tag{18}$$

$$= \frac{2^{3-i} e^{w_1 w_2 w_3 \left(\sum_{j=1}^{3-i} y_j\right) t} \left(q^{w_1 w_2 w_3} e^{w_1 w_2 w_3 t} + 1\right)^i}{\left(q^{w_1} e^{w_1 t} + 1\right)\left(q^{w_2} e^{w_2 t} + 1\right)\left(q^{w_3} e^{w_3 t} + 1\right)}. \tag{19}$$

(c-0) Type Λ_{12}^0

$$I\left(\Lambda_{12}^0\right) = \int_{\mathbb{Z}_p^3} q^{w_1 x_1 + w_2 x_2 + w_3 x_3} e^{\left(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_2 w_3 y + w_1 w_3 y + w_1 w_2 y\right) t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3) \tag{20}$$

$$= \frac{8 e^{\left(w_2 w_3 + w_1 w_3 + w_1 w_2\right) y t}}{\left(q^{w_1} e^{w_1 t} + 1\right)\left(q^{w_2} e^{w_2 t} + 1\right)\left(q^{w_3} e^{w_3 t} + 1\right)}. \tag{21}$$

(c-1) Type Λ_{12}^1

$$I\left(\Lambda_{12}^1\right) = \frac{\int_{\mathbb{Z}_p^3} q^{w_1 x_1 + w_2 x_2 + w_3 x_3} e^{\left(w_1 x_1 + w_2 x_2 + w_3 x_3\right) t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p^3} q^{w_2 w_3 z_1 + w_1 w_3 z_2 + w_1 w_2 z_3} e^{\left(w_2 w_3 z_1 + w_1 w_3 z_2 + w_1 w_2 z_3\right) t} d\mu_{-1}(z_1) d\mu_{-1}(z_2) d\mu_{-1}(z_3)} \tag{22}$$

$$= \frac{\left(q^{w_2 w_3} e^{w_2 w_3 t} + 1\right)\left(q^{w_1 w_3} e^{w_1 w_3 t} + 1\right)\left(q^{w_1 w_2} e^{w_1 w_2 t} + 1\right)}{\left(q^{w_1} e^{w_1 t} + 1\right)\left(q^{w_2} e^{w_2 t} + 1\right)\left(q^{w_3} e^{w_3 t} + 1\right)}. \tag{23}$$

All of the above p-adic integrals of various types are invariant under all permutations of w_1, w_2, w_3, as one can see either from p-adic integral representations in (16), (18), (20), and (22) or from their explicit evaluations in (17), (19), (21), and (23).

3. Identities for *q*-Euler Polynomials

In the following w_1, w_2, w_3, are all odd positive integers except for (a – 0) and (c – 0), where they are any positive integers.

(a – 0) First, let's consider Type Λ_{23}^i, for each $i = 0,1,2,3$. The following results can be easily obtained from (4) and (7).

$$I\left(\Lambda_{23}^0\right) = \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} e^{w_2 w_3 \left(x_1 + w_1 y_1\right) t} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} e^{w_1 w_3 \left(x_2 + w_2 y_2\right) t} d\mu_{-1}(x_2) \int_{\mathbb{Z}_p} q^{w_1 w_2 x_3} e^{w_1 w_2 \left(x_3 + w_3 y_3\right) t} d\mu_{-1}(x_3) \tag{24}$$

$$= \left(\sum_{k=0}^{\infty} \frac{E_{k,q^{w_2 w_3}}(w_1 y_1)}{k!} (w_2 w_3 t)^k\right) \left(\sum_{l=0}^{\infty} \frac{E_{l,q^{w_1 w_3}}(w_2 y_2)}{l!} (w_1 w_3 t)^l\right) \left(\sum_{m=0}^{\infty} \frac{E_{m,q^{w_1 w_2}}(w_3 y_3)}{m!} (w_1 w_2 t)^m\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_2 w_3}}(w_1 y_1) E_{l,q^{w_1 w_3}}(w_2 y_2) E_{m,q^{w_1 w_2}}(w_3 y_3) w_1^{l+m} w_2^{k+m} w_3^{k+l} \right) \frac{t^n}{n!}, \tag{24}$$

where the inner sum is over all nonnegative integers k, l, m, with $k + l + m = n$, and

$$\binom{n}{k,l,m} = \frac{n!}{k! l! m!}. \tag{25}$$

(a-1) Here we write $I\left(\Lambda_{23}^1\right)$ in two different ways:

$$I(\Lambda_{23}^{1}) = \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} \mathrm{e}^{w_2 w_3 (x_1 + w_1 y_1) t} \mathrm{d}\mu_{-1}(x_1)$$

1) $$\int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} \mathrm{e}^{w_1 w_3 (x_2 + w_2 y_2) t} \mathrm{d}\mu_{-1}(x_2) \tag{26}$$

$$\frac{\int_{\mathbb{Z}_p} q^{w_1 w_2 x_3} \mathrm{e}^{w_1 w_2 x_3 t} \mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} \mathrm{e}^{w_1 w_2 w_3 x_4 t} \mathrm{d}\mu_{-1}(x_4)}$$

$$= \left(\sum_{k=0}^{\infty} E_{k,q^{w_2 w_3}}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!}\right) \left(\sum_{l=0}^{\infty} E_{l,q^{w_1 w_3}}(w_2 y_2) \frac{(w_1 w_3 t)^l}{l!}\right) \left(\sum_{m=0}^{\infty} T_{m,q^{w_1 w_2}}(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_2 w_3}}(w_1 y_1) E_{l,q^{w_1 w_3}}(w_2 y_2) T_{m,q^{w_1 w_2}}(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \right) \frac{t^n}{n!}. \tag{27}$$

2) Invoking (7), (26) can also be written as

$$I(\Lambda_{23}^{1}) = \sum_{i=0}^{w_3 - 1} (-1)^i q^{w_1 w_2 i} \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} e^{w_2 w_3 (x_1 + w_1 y_1) t} \mathrm{d}\mu_{-1}(x_1) \int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} e^{w_1 w_3 \left(x_2 + w_2 y_2 + \frac{w_2}{w_3} i\right) t} \mathrm{d}\mu_{-1}(x_2)$$

$$= \sum_{i=0}^{w_3 - 1} (-1)^i q^{w_1 w_2 i} \left(\sum_{k=0}^{\infty} E_{k,q^{w_2 w_3}}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!}\right) \left(\sum_{l=0}^{\infty} E_{l,q^{w_1 w_3}}\left(w_2 y_2 + \frac{w_2}{w_3} i\right) \frac{(w_1 w_3 t)^l}{l!}\right) \tag{28}$$

$$= \sum_{n=0}^{\infty} \left(w_3^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2 w_3}}(w_1 y_1) \sum_{i=0}^{w_3 - 1} (-1)^i q^{w_1 w_2 i} E_{n-k,q^{w_1 w_3}}\left(w_2 y_2 + \frac{w_2}{w_3} i\right) w_1^{n-k} w_2^k \right) \frac{t^n}{n!}.$$

(a-2) Here we write $I(\Lambda_{23}^{2})$ in three different ways:

$$I(\Lambda_{23}^{2}) = \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} \mathrm{e}^{w_2 w_3 (x_1 + w_1 y_1) t} \mathrm{d}\mu_{-1}(x_1)$$

1) $$\frac{\int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} \mathrm{e}^{w_1 w_3 x_2 t} \mathrm{d}\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} \mathrm{e}^{w_1 w_2 w_3 x_4 t} \mathrm{d}\mu_{-1}(x_4)}$$

$$\frac{\int_{\mathbb{Z}_p} q^{w_1 w_2 x_3} \mathrm{e}^{w_1 w_2 x_3 t} \mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} \mathrm{e}^{w_1 w_2 w_3 x_4 t} \mathrm{d}\mu_{-1}(x_4)} \tag{29}$$

$$= \left(\sum_{k=0}^{\infty} E_{k,q^{w_2 w_3}}(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!}\right) \left(\sum_{l=0}^{\infty} T_{l,q^{w_1 w_3}}(w_2 - 1) \frac{(w_1 w_3 t)^l}{l!}\right) \left(\sum_{m=0}^{\infty} T_{m,q^{w_1 w_2}}(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_2 w_3}}(w_1 y_1) T_{l,q^{w_1 w_3}}(w_2 - 1) T_{m,q^{w_1 w_2}}(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \right) \frac{t^n}{n!}. \tag{30}$$

2) Invoking (7), (29) can also be written as

$$I(\Lambda_{23}^{2}) = \sum_{i=0}^{w_2 - 1} (-1)^i q^{w_1 w_3 i} \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} \mathrm{e}^{w_2 w_3 \left(x_1 + w_1 y_1 + \frac{w_1}{w_2} i\right) t} \mathrm{d}\mu_{-1}(x_1) \tag{31}$$

$$\frac{\int_{\mathbb{Z}_p} q^{w_1 w_2 x_3} \mathrm{e}^{w_1 w_2 x_3 t} \mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 w_3 x_4} e^{w_1 w_2 w_3 x_4 t} \mathrm{d}\mu_{-1}(x_4)}$$

$$= \sum_{i=0}^{w_2 - 1} (-1)^i q^{w_1 w_3 i} \left(\sum_{k=0}^{\infty} E_{k,q^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_2} i\right) \frac{(w_2 w_3 t)^k}{k!}\right) \left(\sum_{l=0}^{\infty} T_{l,q^{w_1 w_2}}(w_3 - 1) \frac{(w_1 w_2 t)^l}{l!}\right)$$

$$= \sum_{n=0}^{\infty} \left(w_2^n \sum_{k=0}^{n} \binom{n}{k} \times \sum_{i=0}^{w_2 - 1} (-1)^i q^{w_1 w_3 i} E_{k,q^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_2} i\right) T_{n-k,q^{w_1 w_2}}(w_3 - 1) w_1^{n-k} w_3^k \right) \frac{t^n}{n!}. \tag{32}$$

3) Invoking (7) once again, (31) can be written as

$$I(\Lambda_{23}^{2}) = \sum_{i=0}^{w_2 - 1} \sum_{j=0}^{w_3 - 1} (-1)^{i+j} q^{w_1 (w_3 i + w_2 j)} \int_{\mathbb{Z}_p} q^{w_2 w_3 x_1} \mathrm{e}^{w_2 w_3 \left(x_1 + w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right) t} \mathrm{d}\mu_{-1}(x_1)$$

$$= \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j} q^{w_1(w_3 i+w_2 j)}$$
$$\left(\sum_{n=0}^{\infty}E_{n,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\frac{(w_2w_3t)^n}{n!}\right) \quad (33)$$
$$= \sum_{n=0}^{\infty}\left((w_2w_3)^n\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j} q^{w_1(w_3 i+w_2 j)} E_{n,q^{w_2w_3}}(w_3 i+w_2 j)\right)\frac{t^n}{n!}.$$

(a-3)

$$I\left(\Lambda_{23}^{3}\right)=\frac{\int_{\mathbb{Z}_p}q^{w_2w_3x_1}e^{w_2w_3x_1t}\mathrm{d}\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}\mathrm{d}\mu_{-1}(x_4)}\times\frac{\int_{\mathbb{Z}_p}q^{w_1w_3x_2}e^{w_1w_3x_2t}\mathrm{d}\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}\mathrm{d}\mu_{-1}(x_4)}\times\frac{\int_{\mathbb{Z}_p}q^{w_1w_2x_3}e^{w_1w_2x_3t}\mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}q^{w_1w_2w_3x_4}e^{w_1w_2w_3x_4t}\mathrm{d}\mu_{-1}(x_4)}$$
$$=\left(\sum_{k=0}^{\infty}T_{k,q^{w_2w_3}}(w_1-1)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}T_{l,q^{w_1w_3}}(w_2-1)\frac{(w_1w_3t)^l}{l!}\right)\left(\sum_{m=0}^{\infty}T_{m,q^{w_1w_2}}(w_3-1)\frac{(w_1w_2t)^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}T_{k,q^{w_2w_3}}(w_1-1)T_{l,q^{w_1w_3}}(w_2-1)T_{m,q^{w_1w_2}}(w_3-1)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}. \quad (34)$$

b) For Type $\Lambda_{13}^{i}\ (i=0,1,2,3)$, we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute w_2w_3, w_1w_3, w_1w_2 respectively for w_1, w_2, w_3 in (16), this amounts to replacing t by $w_1w_2w_3t$ and q by $q^{w_1w_2w_3}$ in (18). So, upon replacing w_1, w_2, w_3 respectively by w_2w_3, w_1w_3, w_1w_2, and then dividing by $(w_1w_2w_3)^n$ and replacing $q^{w_1w_2w_3}$ by q, in each of the expressions of Theorem 4.1 through Corollary 4.15, we will get the corresponding symmetric identities for Type $\Lambda_{13}^{i}\ (i=0,1,2,3)$.

(c-0)

$$I\left(\Lambda_{12}^{0}\right)=\int_{\mathbb{Z}_p}q^{w_1x_1}\mathrm{e}^{w_1(x_1+w_2y)t}\mathrm{d}\mu_{-1}(x_1)\int_{\mathbb{Z}_p}q^{w_2x_2}\mathrm{e}^{w_2(x_2+w_3y)t}\mathrm{d}\mu_{-1}(x_2)\int_{\mathbb{Z}_p}q^{w_3x_3}\mathrm{e}^{w_3(x_3+w_1y)t}\mathrm{d}\mu_{-1}(x_3)$$
$$=\left(\sum_{k=0}^{\infty}\frac{E_{k,q^{w_1}}(w_2y)}{k!}(w_1t)^k\right)\left(\sum_{l=0}^{\infty}\frac{E_{l,q^{w_2}}(w_3y)}{l!}(w_2t)^l\right)\left(\sum_{m=0}^{\infty}\frac{E_{m,q^{w_3}}(w_1y)}{m!}(w_3t)^m\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1}}(w_2y)E_{l,q^{w_2}}(w_3y)E_{m,q^{w_3}}(w_1y)w_1^kw_2^lw_3^m\right)\frac{t^n}{n!}. \quad (35)$$

(c-1)

$$I\left(\Lambda_{12}^{1}\right)=\frac{\int_{\mathbb{Z}_p}q^{w_1x_1}\mathrm{e}^{w_1x_1t}\mathrm{d}\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}q^{w_1w_2z_3}\mathrm{e}^{w_1w_2z_3t}\mathrm{d}\mu_{-1}(z_3)}\times\frac{\int_{\mathbb{Z}_p}q^{w_2x_2}\mathrm{e}^{w_2x_2t}\mathrm{d}\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}q^{w_2w_3z_1}\mathrm{e}^{w_2w_3z_1t}\mathrm{d}\mu_{-1}(z_1)}\times\frac{\int_{\mathbb{Z}_p}q^{w_3x_3}\mathrm{e}^{w_3x_3t}\mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}q^{w_3w_1z_2}\mathrm{e}^{w_3w_1z_2t}\mathrm{d}\mu_{-1}(z_2)}$$
$$=\left(\sum_{k=0}^{\infty}T_{k,q^{w_1}}(w_2-1)\frac{(w_1t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}T_{l,q^{w_2}}(w_3-1)\frac{(w_2t)^l}{l!}\right)\left(\sum_{m=0}^{\infty}T_{m,q^{w_3}}(w_1-1)\frac{(w_3t)^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}T_{k,q^{w_1}}(w_2-1)T_{l,q^{w_2}}(w_3-1)T_{m,q^{w_3}}(w_1-1)w_1^kw_2^lw_3^m\right)\frac{t^n}{n!}. \quad (36)$$

4. Main Theorems

As we noted earlier in the last paragraph of Section 2, the various types of quotients of p-adic fermionic integrals are invariant under any permutation of w_1, w_2, w_3. So the corresponding expressions in Section 3 are also invariant under any permutation of w_1, w_2, w_3. Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single one labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of S_3. In particular, the number of possible distinct expressions are 1, 2, 3, or 6. (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4.8 and 4.17, leaving the others as easy exercises for the reader. As for the case of Theorem 4.8, in addition to (50)-(52), we get the following three ones:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_2w_3}}(w_1y_1)T_{l,q^{w_1w_2}}(w_3-1)T_{m,q^{w_1w_3}}(w_2-1)w_1^{l+m}w_3^{k+m}w_2^{k+l}, \tag{37}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_3}}(w_2y_1)T_{l,q^{w_2w_3}}(w_1-1)T_{m,q^{w_1w_2}}(w_3-1)w_2^{l+m}w_1^{k+m}w_3^{k+l}, \tag{38}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_2}}(w_3y_1)T_{l,q^{w_1w_3}}(w_2-1)T_{m,q^{w_2w_3}}(w_1-1)w_3^{l+m}w_2^{k+m}w_1^{k+l}. \tag{39}$$

But, by interchanging l and m, we see that (37), (38), and (39) are respectively equal to (50), (51), and (52).

As to Theorem 4.17, in addition to (60) and (61), we have:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_2-1)T_l(w_3-1)T_m(w_1-1)w_1^kw_2^lw_3^m, \tag{40}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_3-1)T_l(w_1-1)T_m(w_2-1)w_2^kw_3^lw_1^m, \tag{41}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_3-1)T_l(w_2-1)T_m(w_1-1)w_1^kw_3^lw_2^m, \tag{42}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_2-1)T_l(w_1-1)T_m\left|w_3-1\right|w_3^kw_2^lw_1^m. \tag{43}$$

However, (40) and (41) are equal to (60), as we can see by applying the permutations $k\to l, l\to m, m\to k$ for (40) and $k\to m, l\to k, m\to l$ for (41). Similarly, we see that (42) and (43) are equal to (61), by applying permutations $k\to l, l\to m, m\to k$ for (42) and $k\to m, l\to k, m\to l$ for (43).

Theorem 4.1 *Let* w_1, w_2, w_3 *be any positive integers. Then we have :*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_2w_3}}(w_1y_1)E_{l,q^{w_1w_3}}(w_2y_2)E_{m,q^{w_1w_2}}(w_3y_3)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_2w_3}}(w_1y_1)E_{l,q^{w_1w_2}}(w_3y_2)E_{m,q^{w_1w_3}}(w_2y_3)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_3}}(w_2y_1)E_{l,q^{w_2w_3}}(w_1y_2)E_{m,q^{w_1w_2}}(w_3y_3)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_3}}(w_2y_1)E_{l,q^{w_1w_2}}(w_3y_2)E_{m,q^{w_2w_3}}(w_1y_3)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_2}}(w_3y_1)E_{l,q^{w_2w_3}}(w_1y_2)E_{m,q^{w_1w_3}}(w_2y_3)w_3^{l+m}w_1^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_2}}(w_3y_1)E_{l,q^{w_1w_3}}(w_2y_2)E_{m,q^{w_2w_3}}(w_1y_3)w_3^{l+m}w_2^{k+m}w_1^{k+l}.
\end{aligned} \tag{44}$$

Theorem 4.2 *Let* w_1, w_2, w_3 *be any odd positive integers. Then we have:*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_2w_3}}(w_1y_1)E_{l,q^{w_1w_3}}(w_2y_2)T_{m,q^{w_1w_2}}(w_3-1)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_2w_3}}(w_1y_1)E_{l,q^{w_1w_2}}(w_3y_2)T_{m,q^{w_1w_3}}(w_2-1)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_1w_3}}(w_2y_1)E_{l,q^{w_2w_3}}(w_1y_2)T_{m,q^{w_1w_2}}(w_3-1)w_2^{l+m}w_1^{k+m}w_3^{k+l}
\end{aligned} \tag{45}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1w_3}}(w_2y_1) E_{l,q^{w_1w_2}}(w_3y_2)$$
$$T_{m,q^{w_2w_3}}(w_1-1) w_2^{l+m} w_3^{k+m} w_1^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1w_2}}(w_3y_1) E_{l,q^{w_1w_3}}(w_2y_2)$$
$$T_{m,q^{w_2w_3}}(w_1-1) w_3^{l+m} w_2^{k+m} w_1^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1w_2}}(w_3y_1) E_{l,q^{w_2w_3}}(w_1y_2)$$
$$T_{m,q^{w_1w_3}}(w_2-1) w_3^{l+m} w_1^{k+m} w_2^{k+l}.$$

Putting $w_3 = 1$ in (45), we get the following corollary.

Corollary 4.3 *Let w_1, w_2 be any odd positive integers. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2}}(w_1y_1) E_{n-k,q^{w_1}}(w_2y_2) w_1^{n-k} w_2^{k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(w_2y_1) E_{n-k,q^{w_2}}(w_1y_2) w_2^{n-k} w_1^{k}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1w_2}}(y_1) E_{l,q^{w_1}}(w_2y_2)$$
$$T_{m,q^{w_2}}(w_1-1) w_2^{k+m} w_1^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1}}(w_2y_1) E_{l,q^{w_1w_2}}(y_2)$$
$$T_{m,q^{w_2}}(w_1-1) w_2^{l+m} w_1^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1w_2}}(y_1) E_{l,q^{w_2}}(w_1y_2)$$
$$T_{m,q^{w_1}}(w_2-1) w_1^{k+m} w_2^{k+l}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_2}}(w_1y_1) E_{l,q^{w_1w_2}}(y_2)$$
$$T_{m,q^{w_1}}(w_2-1) w_1^{l+m} w_2^{k+l}. \tag{46}$$

Letting further $w_2 = 1$ in (46), we have the following corollary.

Corollary 4.4 *Let w_1 be any odd positive integer. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} E_{k,q}(w_1y_1) E_{n-k,q^{w_1}}(y_2) w_1^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(y_1) E_{n-k,q}(w_1y_2) w_1^{k}$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1}}(y_1) E_{l,q^{w_1}}(y_2) T_{m,q}(w_1-1) w_1^{k+l}. \tag{47}$$

Theorem 4.5 *Let w_1, w_2, w_3 be any odd positive integers. Then we have:*

$$w_1^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1w_2}}(w_3y_1) w_3^{n-k} w_2^{k}$$
$$\sum_{i=0}^{w_1-1} (-1)^i q^{w_2w_3i} E_{n-k,q^{w_1w_3}}\left(w_2y_2 + \frac{w_2}{w_1} i\right)$$
$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1w_3}}(w_2y_1) w_2^{n-k} w_3^{k}$$
$$\sum_{i=0}^{w_1-1} (-1)^i q^{w_2w_3i} E_{n-k,q^{w_1w_2}}\left(w_3y_2 + \frac{w_3}{w_1} i\right) \tag{48}$$
$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1w_2}}(w_3y_1) w_3^{n-k} w_1^{k}$$
$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1w_3i} E_{n-k,q^{w_2w_3}}\left(w_1y_2 + \frac{w_1}{w_2} i\right)$$
$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2w_3}}(w_1y_1) w_1^{n-k} w_3^{k}$$
$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1w_3i} E_{n-k,q^{w_1w_2}}\left(w_3y_2 + \frac{w_3}{w_2} i\right)$$
$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1w_3}}(w_2y_1) w_2^{n-k} w_1^{k}$$
$$\sum_{i=0}^{w_3-1} (-1)^i q^{w_1w_2i} E_{n-k,q^{w_2w_3}}\left(w_1y_2 + \frac{w_1}{w_3} i\right)$$
$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2w_3}}(w_1y_1) w_1^{n-k} w_2^{k}$$
$$\sum_{i=0}^{w_3-1} (-1)^i q^{w_1w_2i} E_{n-k,q^{w_1w_3}}\left(w_2y_2 + \frac{w_2}{w_3} i\right).$$

Letting $w_3 = 1$ in (48), we obtain alternative expressions for the identities in (46).

Corollary 4.6 *Let w_1, w_2 be any odd positive integers. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2}}(w_1y_1) E_{n-k,q^{w_1}}(w_2y_2) w_1^{n-k} w_2^{k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(w_2y_1) E_{n-k,q^{w_2}}(w_1y_2) w_2^{n-k} w_1^{k} \tag{49}$$
$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1w_2}}(y_1) w_2^{k}$$
$$\sum_{i=0}^{w_1-1} (-1)^i q^{w_2i} E_{n-k,q^{w_1}}\left(w_2y_2 + \frac{w_2}{w_1} i\right)$$

$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(w_2 y_1) w_2^{n-k}$$
$$\sum_{i=0}^{w_1-1} (-1)^i q^{w_2 i} E_{n-k,q^{w_1 w_2}}\left(y_2 + \frac{i}{w_1}\right)$$
$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1 w_2}}(y_1) w_1^k$$
$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{n-k,q^{w_2}}\left(w_1 y_2 + \frac{w_1}{w_2} i\right)$$
$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2}}(w_1 y_1) w_1^{n-k}$$
$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} E_{n-k,q^{w_1 w_2}}\left(y_2 + \frac{i}{w_2}\right).$$

Putting further $w_2 = 1$ in (49), we have the alternative expressions for the identities for (47).

Corollary 4.7 *Let* w_1 *be any odd positive integer. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(y_1) E_{n-k,q}(w_1 y_2) w_1^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(y_2) E_{n-k,q}(w_1 y_1) w_1^k$$
$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(y_1) \sum_{i=0}^{w_1-1} (-1)^i q^i E_{n-k,q^{w_1}}\left(y_2 + \frac{i}{w_1}\right).$$

Theorem 4.8 *Let* w_1, w_2, w_3 *be any odd positive integers. Then we have:*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_2 w_3}}(w_1 y_1) T_{l,q^{w_1 w_3}}(w_2 - 1) T_{m,q^{w_1 w_2}}(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \tag{50}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1 w_3}}(w_2 y_1) T_{l,q^{w_1 w_2}}(w_3 - 1) T_{m,q^{w_2 w_3}}(w_1 - 1) w_2^{l+m} w_3^{k+m} w_1^{k+l} \tag{51}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1 w_2}}(w_3 y_1) T_{l,q^{w_2 w_3}}(w_1 - 1) T_{m,q^{w_1 w_3}}(w_2 - 1) w_3^{l+m} w_1^{k+m} w_2^{k+l} \tag{52}$$

Putting $w_3 = 1$ in (50)-(52), we get the following corollary.

Corollary 4.9 *Let* w_1, w_2 *be any odd positive integers. Then we have:*

$$\sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_2}}(w_1 y_1) T_{n-k,q^{w_1}}(w_2 - 1) w_1^{n-k} w_2^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(w_2 y_1) T_{n-k,q^{w_2}}(w_1 - 1) w_2^{n-k} w_1^k$$
$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,q^{w_1 w_2}}(y_1) T_{l,q^{w_2}}(w_1 - 1) T_{m,q^{w_1}}(w_2 - 1) w_1^{k+m} w_2^{k+l}. \tag{53}$$

Letting further $w_2 = 1$ in (53), we get the following corollary. This was also obtained in [7,(2.12)].

Corollary 4.10 *Let* w_1 *be any odd positive integer. Then we have:*

$$E_{n,q}(w_1 y_1) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q^{w_1}}(y_1) T_{n-k,q}(w_1 - 1) w_1^k. \tag{54}$$

Theorem 4.11 *Let* w_1, w_2, w_3 *be any odd positive integers. Then we have:*

$$w_1^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_1 w_2}}(w_3 - 1) w_2^{n-k} w_3^k$$
$$\sum_{i=0}^{w_1-1} (-1)^i q^{w_2 w_3 i} E_{k,q^{w_1 w_3}}\left(w_2 y_1 + \frac{w_2}{w_1} i\right)$$
$$= w_1^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_1 w_3}}(w_2 - 1) w_3^{n-k} w_2^k$$
$$\sum_{i=0}^{w_1-1} (-1)^i q^{w_2 w_3 i} E_{k,q^{w_1 w_2}}\left(w_3 y_1 + \frac{w_3}{w_1} i\right)$$
$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_1 w_2}}(w_3 - 1) w_1^{n-k} w_3^k$$
$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1 w_3 i} E_{k,q^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_2} i\right)$$
$$= w_2^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_2 w_3}}(w_1 - 1) w_3^{n-k} w_1^k$$
$$\sum_{i=0}^{w_2-1} (-1)^i q^{w_1 w_3 i} E_{k,q^{w_1 w_2}}\left(w_3 y_1 + \frac{w_3}{w_2} i\right)$$
$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_1 w_3}}(w_2 - 1) w_1^{n-k} w_2^k$$
$$\sum_{i=0}^{w_3-1} (-1)^i q^{w_1 w_2 i} E_{k,q^{w_2 w_3}}\left(w_1 y_1 + \frac{w_1}{w_3} i\right)$$
$$= w_3^n \sum_{k=0}^{n} \binom{n}{k} T_{n-k,q^{w_2 w_3}}(w_1 - 1) w_2^{n-k} w_1^k$$
$$\sum_{i=0}^{w_3-1} (-1)^i q^{w_1 w_2 i} E_{k,q^{w_1 w_3}}\left(w_2 y_1 + \frac{w_2}{w_3} i\right). \tag{55}$$

Putting $w_3 = 1$ in (55), we obtain the following corollary. In Section 1, the identities in (53), (56), and (58) are combined to give those in (8)-(15).

Corollary 4.12 *Let* w_1, w_2 *be any odd positive inte-*

gers. Then we have:

$$\begin{aligned}
&w_1^n\sum_{i=0}^{w_1-1}(-1)^i q^{w_2 i}E_{n,q^{w_1}}\left(w_2y_1+\frac{w_2}{w_1}i\right)\\
&=w_2^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,q^{w_2}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}E_{k,q^{w_1}}\left(w_2y_1\right)T_{n-k,q^{w_2}}\left(w_1-1\right)w_2^{n-k}w_1^k\\
&=\sum_{k=0}^{n}\binom{n}{k}E_{k,q^{w_2}}\left(w_1y_1\right)T_{n-k,q^{w_1}}\left(w_2-1\right)w_1^{n-k}w_2^k\\
&=w_1^n\sum_{k=0}^{n}\binom{n}{k}T_{n-k,q^{w_1}}\left(w_2-1\right)w_2^k\\
&\quad\sum_{i=0}^{w_1-1}(-1)^i q^{w_2 i}E_{k,q^{w_1w_2}}\left(y_1+\frac{i}{w_1}\right)\\
&=w_2^n\sum_{k=0}^{n}\binom{n}{k}T_{n-k,q^{w_2}}\left(w_1-1\right)w_1^k\\
&\quad\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{k,q^{w_1w_2}}\left(y_1+\frac{i}{w_2}\right).
\end{aligned}\tag{56}$$

Letting further $w_2=1$ in (56), we get the following corollary. This is the identity obtained in (54) together with the multiplication formula for q-Euler polynomials [cf.7,(2.17)].

Corollary 4.13 *Let w_1 be any odd positive integer. Then we have:*

$$\begin{aligned}
E_{n,q}\left(w_1y_1\right)&=w_1^n\sum_{i=0}^{w_1-1}(-1)^i q^i E_{n,q^{w_1}}\left(y_1+\frac{i}{w_1}\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}E_{k,q^{w_1}}\left(y_1\right)T_{n-k,q}\left(w_1-1\right)w_1^k.
\end{aligned}$$

Theorem 4.14 *Let w_1,w_2,w_3 be any odd positive integers. Then we have:*

$$\begin{aligned}
&\left(w_1w_2\right)^n\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}(-1)^{i+j}q^{w_3\left(w_2i+w_1j\right)}\\
&E_{n,q^{w_1w_2}}\left(w_3y_1+\frac{w_3}{w_1}i+\frac{w_3}{w_2}j\right)\\
&=\left(w_2w_3\right)^n\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j}q^{w_1\left(w_3i+w_2j\right)}\\
&E_{n,q^{w_2w_3}}\left(w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\\
&=\left(w_3w_1\right)^n\sum_{i=0}^{w_3-1}\sum_{j=0}^{w_1-1}(-1)^{i+j}q^{w_2\left(w_1i+w_3j\right)}\\
&E_{n,q^{w_1w_3}}\left(w_2y_1+\frac{w_2}{w_3}i+\frac{w_2}{w_1}j\right).
\end{aligned}\tag{57}$$

Letting $w_3=1$ in (57), we have the following corollary.

Corollary 4.15 *Let w_1,w_2 be any odd positive integers. Then we have:*

$$\begin{aligned}
&w_1^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,q^{w_1}}\left(w_2y_1+\frac{w_2}{w_1}j\right)\\
&=w_2^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,q^{w_2}}\left(w_1y_1+\frac{w_1}{w_2}i\right)\\
&=\left(w_1w_2\right)^n\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}(-1)^i q^{w_2i+w_1j}\\
&\quad E_{n,q^{w_1w_2}}\left(y_1+\frac{i}{w_1}+\frac{j}{w_2}\right).
\end{aligned}\tag{58}$$

Theorem 4.16 *Let w_1,w_2,w_3 be any positive integers. Then we have:*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_3}}\left(w_1y\right)E_{l,q^{w_1}}\left(w_2y\right)\\
&E_{m,q^{w_2}}\left(w_3y\right)w_3^k w_1^l w_2^m\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,q^{w_2}}\left(w_1y\right)E_{l,q^{w_1}}\left(w_3y\right)\\
&E_{m,q^{w_3}}\left(w_2y\right)w_2^k w_1^l w_3^m.
\end{aligned}\tag{59}$$

Theorem 4.17 *Let w_1,w_2,w_3 be any odd positive integers. Then we have:*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}T_{k,q^{w_3}}\left(w_1-1\right)T_{l,q^{w_1}}\left(w_2-1\right)\\
&T_{m,q^{w_2}}\left(w_3-1\right)w_3^k w_1^l w_2^m
\end{aligned}\tag{60}$$

$$\begin{aligned}
&=\sum_{k+l+m=n}\binom{n}{k,l,m}T_{k,q^{w_2}}\left(w_1-1\right)T_{l,q^{w_1}}\left(w_3-1\right)\\
&T_{m,q^{w_3}}\left(w_2-1\right)w_2^k w_1^l w_3^m.
\end{aligned}\tag{61}$$

Putting $w_3=1$ in (60) and (61), we get the following corollary.

Corollary 4.18 *Let w_1,w_2 be any odd positive integers. Then we have:*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}T_{k,q^{w_1}}\left(w_2-1\right)T_{n-k,q}\left(w_1-1\right)w_1^k\\
&=\sum_{k=0}^{n}\binom{n}{k}T_{k,q^{w_2}}\left(w_1-1\right)T_{n-k,q}\left(w_2-1\right)w_2^k.
\end{aligned}$$

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