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Coefficient Estimates for a Certain General Subclass of Analytic and Bi-Univalent Functions

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Abstract

Motivated and stimulated especially by the work of Xu *et al.* [1], in this paper, we introduce and discuss an interesting subclass $\mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ of analytic and bi-univalent functions defined in the open unit disc $\mathbb{U}$. Further, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass. Many relevant connections with known or new results are pointed out.

Keywords

Analytic Functions, Univalent Functions, Bi-Univalent Functions, Bi-Starlike Functions

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\beta)$ of starlike functions of order β $(0 \le \beta < 1)$ in $\mathbb{U}$ and the class $\mathcal{SS}^*(\alpha)$ of strongly starlike functions of order α $(0 < \alpha \le 1)$ in $\mathbb{U}$.
It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

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$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$. For a brief history and interesting examples of functions in the class Σ see [2] and the references therein.

In fact, the study of the coefficient problems involving bi-univalent functions was revived recently by Srivastava *et al*. [2]. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in these subclasses were found in several recent investigations (see, for example, [3]-[13]). The aforecited all these papers on the subject were motivated by the pioneering work of Srivastava *et al*. [2]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} := \{1,2,3,\cdots\})$ is still an open problem.

Motivated by the aforecited works (especially [1]), we introduce the following subclass $\mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ of the analytic function class $\mathcal{A}$.

Definition 1 *Let* $f \in \mathcal{A}$ *and the functions* $\varphi, \psi : \mathbb{U} \to \mathbb{C}$ *be so constrained that* $\min\{\Re(\varphi(z)), \Re(\psi(z))\} > 0,\ z \in \mathbb{U}$ *and* $\varphi(0) = \psi(0) = 1$. *We say that* $f \in \mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ *if the following conditions are satisfied:* $f \in \Sigma$,

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \in \varphi(\mathbb{U}) \quad (0 \leq \lambda < 1; z \in \mathbb{U}) \tag{1.3}$$

and

$$\frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \in \psi(\mathbb{U}) \quad (0 \leq \lambda < 1;\ w \in \mathbb{U}), \tag{1.4}$$

where the function g is the extension of f^{-1} to $\mathbb{U}$.

We note that, for the different choices of the functions φ and ψ, we get interesting known and new subclasses of the analytic function class $\mathcal{A}$. For example, if we set

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \text{ and } \psi(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \quad (0 < \alpha \leq 1; z \in \mathbb{U}),$$

in the class $\mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ then we have $\mathcal{SS}_{\Sigma}^{*}(\alpha,\lambda)$. Also, $f \in \mathcal{SS}_{\Sigma}^{*}(\alpha,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1;\ 0 \leq \lambda < 1;\ z \in \mathbb{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1;\ 0 \leq \lambda < 1;\ w \in \mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Similarly, if we let

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} \text{ and } \psi(z) = \frac{1-(1-2\beta)z}{1+z} \quad (0 \leq \beta < 1;\ z \in \mathbb{U}),$$

in the class $\mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ then we get $\mathcal{S}_{\Sigma}^{*}(\beta,\lambda)$. Further, we say that $f \in \mathcal{S}_{\Sigma}^{*}(\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \Re\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right) > \beta \quad (0 \le \beta < 1;\ 0 \le \lambda < 1;\ z \in \mathbb{U})$$

and

$$\Re\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right) > \beta \quad (0 \le \beta < 1;\ 0 \le \lambda < 1;\ w \in \mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

The classes $\mathcal{SS}_{\Sigma}^{*}(\alpha,\lambda)$ and $\mathcal{S}_{\Sigma}^{*}(\beta,\lambda)$ were introduced and studied by Murugusundaramoorthy *et al*. [12], Definition 1.1 and Definition 1.2]. The classes $\mathcal{SS}_{\Sigma}^{*}(\alpha,0) := \mathcal{SS}_{\Sigma}^{*}(\alpha)$ and $\mathcal{S}_{\Sigma}^{*}(\beta,0) := \mathcal{S}_{\Sigma}^{*}(\beta)$ are strongly bi-starlike functions of order α and bi-starlike functions of order β respectively. The classes $\mathcal{SS}_{\Sigma}^{*}(\alpha)$ and $\mathcal{S}_{\Sigma}^{*}(\beta)$ were introduced and studied by Brannan and Taha [14], Definition 1.1 and Definition 1.2]. In addition, we note that, $\mathcal{G}_{\Sigma}^{\varphi,\psi}(0) := \mathcal{B}_{\Sigma}^{\varphi,\psi}$ was introduced and studied by Bulut [4], Definition 3].

Motivated and stimulated by Bulut [4] and Xu *et al*. [1] (also [10]), in this paper, we introduce a new subclass $\mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ and obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in aforementioned class, employing the techniques used earlier by Xu *et al*. [1].

2. A Set of General Coefficient Estimates

In this section we state and prove our general results involving the bi-univalent function class $\mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$ given by Definition 1.

Theorem 1 *Let* $f(z)$ *be of the form* (1.1). *If* $f \in \mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$, *then*

$$|a_2| \le \min\left\{\sqrt{\frac{|\varphi'(0)|^2+|\psi'(0)|^2}{2(1-\lambda)^2}}, \frac{\sqrt{|\varphi''(0)|+|\psi''(0)|}}{2(1-\lambda)}\right\} \tag{1.5}$$

and

$$|a_3| \le \min\left\{\frac{|\varphi'(0)|^2+|\psi'(0)|^2}{2(1-\lambda)^2}+\frac{|\varphi''(0)|+|\psi''(0)|}{8(1-\lambda)}, \frac{(3-\lambda)|\varphi''(0)|+(1+\lambda)|\psi''(0)|}{8(1-\lambda)^2}\right\}. \tag{1.6}$$

Proof 1 *Since* $f \in \mathcal{G}_{\Sigma}^{\varphi,\psi}(\lambda)$. *From* (1.3) *and* (1.4), *we have*,

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = \varphi(z) \quad (z \in \mathbb{U})$$

and

$$\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)} = \psi(w) \quad (w \in \mathbb{U}),$$

where

$$\varphi(z) = 1+\varphi_1 z+\varphi_2 z^2+\cdots$$

and

$$\psi(z) = 1+\psi_1 z+\psi_2 z^2+\cdots$$

satisfy the conditions of Definition 1. Now, upon equating the coefficients of $\dfrac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}$ with those of $\varphi(z)$ and the coefficients of $\dfrac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}$ with those of $\psi(w)$, we get

$$(1-\lambda)a_2 = \varphi_1 \tag{1.7}$$

$$(\lambda^2-1)a_2^2 + 2(1-\lambda)a_3 = \varphi_2 \tag{1.8}$$

$$-(1-\lambda)a_2 = \psi_1 \tag{1.9}$$

and

$$(\lambda^2-4\lambda+3)a_2^2 - 2(1-\lambda)a_3 = \psi_2. \tag{1.10}$$

From (1.7) and (1.9), we get

$$\varphi_1 = -\psi_1 \tag{1.11}$$

and

$$2(1-\lambda)^2 a_2^2 = \varphi_1^2 + \psi_1^2. \tag{1.12}$$

From (1.8) and (1.10), we obtain

$$2(1-\lambda)^2 a_2^2 = \varphi_2 + \psi_2. \tag{1.13}$$

Therefore, we find from (1.12) and (1.13) that

$$a_2^2 = \frac{\varphi_1^2+\psi_1^2}{2(1-\lambda)^2}. \tag{1.14}$$

and

$$a_2^2 = \frac{\varphi_2+\psi_2}{2(1-\lambda)^2}. \tag{1.15}$$

Since $\varphi(z)\in\varphi(\mathbb{U})$ and $\psi(z)\in\psi(\mathbb{U})$, we immediately have

$$|a_2|^2 \le \frac{|\varphi'(0)|^2+|\psi'(0)|^2}{2(1-\lambda)^2}$$

and

$$|a_2|^2 \le \frac{|\varphi''(0)|^2+|\psi''(0)|^2}{4(1-\lambda)^2}$$

respectively. So we get the desired estimate on $|a_2|$ as asserted in (1.5).

Next, in order to find the bound on $|a_3|$, by subtracting (1.10) from (1.8), we get

$$4(1-\lambda)a_3 - 4(1-\lambda)a_2^2 = \varphi_2 - \psi_2. \tag{1.16}$$

Upon substituting the values of a_2^2 from (1.14) and (1.15) into (1.16), we have

$$a_3 = \frac{\varphi_1^2+\psi_1^2}{2(1-\lambda)^2} + \frac{\varphi_2-\psi_2}{4(1-\lambda)}$$

and

$$a_3 = \frac{(3-\lambda)\varphi_2+(1+\lambda)\psi_2}{4(1-\lambda)^2}$$

respectively. Since $\varphi(z)\in\varphi(\mathbb{U})$ and $\psi(z)\in\psi(\mathbb{U})$, we readily get

$$|a_3| \le \frac{|\varphi'(0)|^2+|\psi'(0)|^2}{2(1-\lambda)^2} + \frac{|\varphi''(0)|+|\psi''(0)|}{8(1-\lambda)},$$

and

$$|a_3| \le \frac{(3-\lambda)|\varphi''(0)| + (1+\lambda)|\psi''(0)|}{8(1-\lambda)^2}.$$

This completes the proof of Theorem 1.

If we choose

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad \text{and} \quad \psi(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \quad (0 < \alpha \le 1,\ z \in \mathbb{U})$$

in Theorem 1, we have the following corollary.

Corollary 1 *Let* $f(z)$ *be of the form* (1.1) *and in the class* $\mathcal{SS}_{\Sigma}^{*}(\alpha,\lambda)$. *Then*

$$|a_2| \le \min\left\{\frac{2\alpha}{1-\lambda}, \frac{\sqrt{2}\alpha}{1-\lambda}\right\}$$

and

$$|a_3| \le \min\left\{\frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha^2}{1-\lambda}, \frac{2\alpha^2}{(1-\lambda)^2}\right\}.$$

If we set

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad \psi(z) = \frac{1-(1-2\beta)z}{1+z} \quad (0 \le \beta < 1,\ z \in \mathbb{U})$$

in Theorem 1, we readily have the following corollary.

Corollary 2 *Let* $f(z)$ *be of the form* (1.1) *and in the class* $\mathcal{S}_{\Sigma}^{*}(\beta,\lambda)$. *Then*

$$|a_2| \le \min\left\{\frac{2(1-\beta)}{1-\lambda}, \frac{\sqrt{2(1-\beta)}}{1-\lambda}\right\}$$

and

$$|a_3| \le \min\left\{\frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{1-\beta}{1-\lambda}, \frac{2(1-\beta)}{(1-\lambda)^2}\right\}.$$

Remark 1 The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollaries 1 and 2 are improvement of the estimates obtained in [10], Theorems 4 and 5]. Taking $\lambda = 0$ in Corollaries 1 and 2, the estimates on the coefficients $|a_2|$ and $|a_3|$ are improvement of the estimates in [14], Theorems 2.1 and 4.1]. When $\lambda = 0$ the results discussed in this article reduce to results in [4]. Similarly, various other interesting corollaries and consequences of our main result can be derived by choosing different φ and ψ.

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References

[1] Xu, Q.-H., Gui, Y.-C. and Srivastava, H.M. (2012) Coefficient Estimates for a Certain Subclass of Analytic and Bi-Univalent Functions. *Applied Mathematics Letters*, **25**, 990-994.

[2] Srivastava, H.M., Mishra, A.K. and Gochhayat, P. (2010) Certain Subclasses of Analytic and Bi-Univalent Functions. *Applied Mathematics Letters*, **23**, 1188-1192.

[3] Ali, R.M., Lee, S.K., Ravichandran, V. and Supramanian, S. (2012) Coefficient Estimates for Bi-Univalent Ma-Minda Starlike and Convex Functions. *Applied Mathematics Letters*, **25**, 344-351.

[4] Bulut, S. (2013) Coefficient Estimates for a Class of Analytic and Bi-Univalent Functions. *Novi Sad Journal of Mathematics*, **43**, 59-65.

[5] Çağlar, M., Orhan, H. and Yağmur, N. (2012) Coefficient Bounds for New Subclasses of Bi-Univalent Functions. *FILOMAT*, **27**, 1165-1171.

[6] Frasin, B.A. and Aouf, M.K. (2011) New Subclasses of Bi-Univalent Functions. *Applied Mathematics Letters*, **24**, 1569-1573.

[7] Hayami, T. and Owa, S. (2012) Coefficient Bounds for Bi-Univalent Functions. *Pan American Mathematical Journal*, **22**, 15-26.

[8] Li, X.-F. and Wang, A.-P. (2012) Two New Subclasses of Bi-Univalent Functions. *International Mathematical Forum*, **7**, 1495-1504.

[9] Magesh, N., Rosy, T. and Varma, S. (2013) Coefficient Estimate Problem for a New Subclass of Biunivalent Functions. *Journal of Complex Analysis*, 2013, Article ID: 474231, 3 Pages.

[10] Murugusundaramoorthy, G., Magesh, N. and Prameela, V. (2013) Coefficient Bounds for Certain Subclasses of Bi-Univalent Functions. *Abstract and Applied Analysis*, 2013, Article ID: 573017, 3 Pages.

[11] Srivastava, H.M., Bulut, S., Çağlar, M. and Yağmur, N. (2013) Coefficient Estimates for a General Subclass of Analytic and Bi-Univalent Functions. *FILOMAT*, **27**, 831-842.

[12] Srivastava, H.M., Murugusundaramoorthy, G. and Magesh, N. (2013) On Certain Subclasses of Bi-Univalent Functions Associated with Hohlov Operator. *Global J. Math. Anal*, **1**, 67-73.

[13] Xu, Q.-H., Xiao, H.-G. and Srivastava, H. M. (2012) A Certain General Subclass of Analytic and Bi-Univalent Functions and Associated Coefficient Estimate Problems. *Applied Mathematics and Computation*, **218**, 11461-11465.

[14] Brannan, D.A. and Taha, T.S. (1986) On Some Classes of Bi-Univalent Functions. *Studia Universitatis Babeş-Bolyai. Mathematica*, **31**, 70-77.