**Advances in Pure Mathematics, 2014, 4, 82-88**
Published Online March 2014 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2014.43013

Scientific Research

# Fixed Point Theorem for Maps Satisfying a General Contractive Condition of Integral Type

**Sumitra[1], Saurabh Manro[2*]**

[1]Department of Mathematics, Faculty of Science, Jazan University, Jazan, KSA
[2]B. No. 33, H. No. 223, Peer Khana Road, Khanna, Ludhiana, India
Email: mathsqueen_d@yahoo.com, *sauravmanro@yahoo.com, *sauravmanro@hotmail.com

Received 26 December 2013; revised 26 January 2014; accepted 1 February 2014



## Abstract

**The aim of this paper is to prove common fixed point theorems for variants of weak compatible maps in a complex valued-metric space. In this paper, we generalize various known results in the literature using (*CLRg*) property. The concept of (*CLRg*) does not require a more natural condition of closeness of range.**

## Keywords

**Weakly Compatible Maps; (*CLRg*) Property; Common Fixed Point**

## 1. Introduction

Recently, Azam *et al.* [1] introduced complex-valued metric space which is more general than classical metric space. Sastry *et al.* [2] proved that every complex-valued metric space is metrizable and hence is not real generalizations of metric spaces. But indeed it is a metric space and it is well known that complex numbers have many applications in Control theory, Fluid dynamics, Dynamic equations, Electromagnetism, Signal analysis, Quantum mechanics, Relativity, Geometry, Fractals, Analytic number theory, Algebraic number theory etc. For more detail, one can refer to [3]-[5]. The aim of this paper is to prove a common fixed point theorem for variants of weak compatible maps in a complex valued-metric space. As a consequence, we extend and generalize various known results in the literature using (*CLRg*) property in complex valued metric space.

*Corresponding author.

**How to cite this paper:** Sumitra and Manro, S. (2014) Fixed Point Theorem for Maps Satisfying a General Contractive Condition of Integral Type. *Advances in Pure Mathematics*, **4**, 82-88. http://dx.doi.org/10.4236/apm.2014.43013

## 2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$, recall a natural partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\operatorname{Re}(z_1) \le \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) \le \operatorname{Im}(z_2)$,

$z_1 \prec z_2$ if and only if $\operatorname{Re}(z_1) \le \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

**Definition 2.1. [1].** Let $X$ be a nonempty set such that the map $d: X \times X \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a complex-valued metric on $X$, and $(X, d)$ is called a complex-valued metric space.

**Example 2.1.** [1] Define complex-valued metric $d: X \times X \to \mathbb{C}$ by $d(z_1, z_2) = \mathrm{e}^{3\mathrm{i}} |z_1 - z_2|$. Then $(X, d)$ is a complex-valued metric space.

**Definition 2.2.** [1]. Let $(X, d)$ complex -valued metric space and $x \in X$. Then sequence $\{x_n\}$ sequence is

i) **convergent** if for every $0 \prec c \in \mathbb{C}$, there is a natural number $N$ such that $d(x_n, x) \prec c$, for all $n \ge N$. We write it as $\lim_{n\to\infty} x_n = x$.

ii) a **Cauchy sequence**, if for every $0 \prec c \in \mathbb{C}$, there is a natural number $N$ such that $d(x_n, x_m) \prec c$, for all $n, m \ge N$.

**Definition 2.3.** [5] [6]. A pair of self-maps $f$ and $g$ of a complex-valued metric space $(X, d)$ are weakly compatible if $fgz = gfz$ for all $z \in X$ at which $fz = gz$.

**Example 2.2.** [6]. Define complex-valued metric $d: X \times X \to \mathbb{C}$ defined by

$$d(z_1, z_2) = \mathrm{e}^{\mathrm{i}a} |z_1 - z_2|,$$

where $a$ is any real constant. Then $(X, d)$ is a complex-valued metric space. Suppose self maps $f$ and $g$ be defined as:

$$fz = \begin{cases} 2\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} & \text{if} \quad \operatorname{Re}(z) \ne 0, \\ 3\mathrm{e}^{\frac{\mathrm{i}\pi}{3}} & \text{if} \quad \operatorname{Re}(z) = 0 \end{cases}$$

and

$$gz = \begin{cases} 2\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} & \text{if} \quad \operatorname{Re}(z) \ne 0, \\ 4\mathrm{e}^{\frac{\mathrm{i}\pi}{6}} & \text{if} \quad \operatorname{Re}(z) = 0. \end{cases}$$

Clearly, $f$ and $g$ are weakly compatible self maps.

In 2011, Sintunavarat and Kumam [7] introduced a new property called as "common limit in the range of $g$ property" *i.e.*, $(CLR_g)$ property, defined as:

**Definition 2.4.** A pair $(f, g)$ of self-mappings is said to be satisfy the common limit in the range of $g$ property if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz$ for some $z \in X$.

## 3. Main Results

**Definition 3.1.** Let $(X, d)$ be a complex valued metric space and $(f, g)$ be a pair of self mappings on $X$ and $x, y \in X$. Let us consider the following sets:

$$M_1 = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$$

$$M_2 = \left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2}\right\}$$

$$M_3 = \left\{d(gx, gy), \frac{d(gx, fx) + d(gy, fy)}{2}, \frac{d(gx, fy) + d(gy, fx)}{2}\right\}$$

and define the following conditions:

A) For arbitrary $x, y \in X$, there exists

$$u(x,y) \in M_1$$

such that

$$d(fx, fy) \prec u(x,y);$$

B) For arbitrary $x, y \in X$, there exists

$$u(x,y) \in M_2$$

such that

$$d(fx, fy) \prec u(x,y);$$

C) For arbitrary $x, y \in X$, there exists

$$u(x,y) \in M_3$$

such that

$$d(fx, fy) \prec u(x,y).$$

Conditions A), B) and C) are called strict contractive conditions.

**Theorem 3.1.** Let *f* and *g* be two weakly compatible self mappings of a complex valued metric space $(X, d)$ such that

(3.1) *f*, *g* satisfy (*CLRg*) property;

(3.2) for all $x, y \in X$, there exists

$$u(x,y) \in M_3$$

such that

$$d(fx, fy) \prec u(x,y).$$

Then *f* and *g* have a unique common fixed point in *X*.

**Proof.** Since *f* and *g* satisfy the (*CLRg*) property, there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

We first show that $fx = gx$. Suppose not, *i.e.*, $fx \neq gx$.

From (3.2),

$$d(fx_n, fx) \prec u(x_n, x) \tag{3.3}$$

where

$$u(x_n, x) \in M_3$$

$$M_3 = \left\{ d(gx_n, gx), \frac{d(gx_n, fx_n) + d(gx, fx)}{2}, \frac{d(gx_n, fx) + d(gx, fx_n)}{2} \right\}$$

Three cases arises:

i) If

$$M_3 = d(gx_n, gx) = u(x_n, x)$$

then (3.3) implies

$$d(fx_n, fx) \prec d(gx_n, gx).$$

Taking limit as $n \to \infty$,

$$d(gx, fx) \prec d(gx, gx) = 0,$$

which gives, $|d(fx, gx)| < 0$, contradiction.

ii) If

$$M_3 = \frac{d(gx_n, fx_n) + d(gx, fx)}{2} = u(x_n, x)$$

then (3.3) implies,

$$d(fx_n, fx) \prec \frac{1}{2}\left[d(gx_n, fx_n) + d(gx, fx)\right].$$

Taking limit as $n \to \infty$,

$$d(gx, fx) \prec \frac{1}{2}\left[d(gx, gx) + d(gx, fx)\right] = \frac{1}{2}\left[d(gx, fx)\right]$$

*i.e.*, $d(gx, fx) \prec 0$ which gives, $|d(fx, gx)| < 0$, a contradiction.

iii) If

$$M_3 = \frac{d(gx_n, fx) + d(gx, fx_n)}{2} = u(x_n, x),$$

then (3.3) gives,

$$d(fx_n, fx) \prec \frac{1}{2}\left[d(gx_n, fx) + d(gx, fx_n)\right].$$

Making limit as $n \to \infty$,

$$d(gx, fx) \prec \frac{1}{2}\left[d(gx, fx) + d(gx, gx)\right] = \frac{1}{2} d(gx, fx)$$

*i.e.*, $d(gx, fx) \prec 0$ which gives, $|d(fx, gx)| < 0$, a contradiction.

Hence, from all three cases, $gx = fx$.

Now let $z = fx = gx$. Since $f$ and $g$ are weakly compatible mappings $fgx = gfx$ which implies that $fz = fgx = gfx = gz$.

We claim that $fz = z$. Let, if possible, $fz \neq z$.

Now

$$d(fz, z) = d(fz, fx) \prec u(z, x) \tag{3.4}$$

where

$$\begin{aligned} u(z, x) \in M_3 &= \left\{ d(gz, gx), \frac{d(gz, fz) + d(gx, fx)}{2}, \frac{d(gz, fx) + d(gx, fz)}{2} \right\} \\ &= \{ d(fz, z), 0, d(fz, z) \} \\ &= \{ d(fz, z), 0 \}. \end{aligned}$$

Two cases arises:

i) If $u(z, x) = M_3 = d(fz, z)$, then (3.4) gives,

$$d(fz, z) \prec d(fz, z),$$

which gives, $|d(fz, z)| < |d(fz, z)|$, a contradiction.

ii) If $u(z, x) = M_3 = 0$, then (3.4) gives,

$$d(fz, z) \prec 0,$$

which gives, $|d(fz, z)| < 0$ a contradiction.

Hence, from two cases, it is clear that

$$fz = z = gz.$$

Hence $z$ is a common fixed point of $f$ and $g$.

For uniqueness, suppose that $w$ is another common fixed point of $f$ and $g$.

We shall prove that $z = w$. Let, if possible, $z \neq w$.

Then

$$d(z,w) = d(fz, fw) \prec u(z,w). \tag{3.5}$$

where

$$\begin{aligned} u(z,w) \in M_3 &= \left\{ d(gz, gw), \frac{d(gz, fz) + d(gw, fw)}{2}, \frac{d(gz, fw) + d(gw, fz)}{2} \right\} \\ &= \left\{ d(z,w), \frac{d(z,z) + d(w,w)}{2}, \frac{d(z,w) + d(w,z)}{2} \right\} \\ &= \{ d(z,w), 0, d(z,w) \} \\ &= \{ d(z,w), 0 \}. \end{aligned}$$

Again, two possible cases

i) If $u(z,w) = M_3 = d(z,w)$, then by (3.5), we have

$$d(z,w) \prec d(z,w)$$

which gives, $|d(z,w)| < |d(z,w)|$, a contradiction.

ii) If $u(z,w) = M_3 = 0$, then by (3.5), we have

$$d(z,w) \prec 0$$

which gives, $|d(z,w)| < 0$, a contradiction.

Hence, $z = w$.

So, we can say that $f$ and $g$ have a unique common fixed point.

**Remark 3.1.** Theorem 3.1 also holds true if $M_3$ is replaced by $M_2$.

**Definition 3.2.** Let $(X, d)$ be a complex valued metric space, and let $f, g$: $X \to X$. Then $f$ is called a $g$-quasi-contraction, if for some constant $\alpha \in (0,1)$ and for every $x, y \in X$, there exists

$$u(x,y) \in M_1$$

such that

$$d(fx, fy) \precsim \alpha \cdot u(x,y).$$

**Theorem 3.2.** Let $f$ and $g$ be two weakly compatible self mappings of a complex valued metric space $(X, d)$ such that

(3.6) $f$ is a $g$-quasi-contraction;

(3.7) $f$ and $g$ satisfy $(CLRg)$ property.

Then $f$ and $g$ have a unique common fixed point.

**Proof.** Since $f$ and $g$ satisfy the $(CLRg)$ property, there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \quad \text{for some} \quad x \in X$$

We first claim that $fx = gx$. Suppose not. Since, $f$ is a $g$-quasi-contraction, therefore

$$d(fx_n, fx) \precsim \alpha \cdot u(x_n, x) \tag{3.8}$$

for some $u(x_n, x) \in M_1 = \{ d(gx_n, gx), d(gx_n, fx_n), d(gx, fx), d(gx_n, fx), d(gx, fx_n) \}$

Following five cases arises:

i) If $u(x_n, x) \in M_1 = d(gx_n, gx)$, then by (3.8), we have

$$d(fx_n, fx) \precsim \alpha \cdot d(gx_n, fx)$$

taking limit as $n \to \infty$,we have

$$d(gx, fx) \precsim \alpha \cdot d(gx, gx) = 0,$$

which gives, $\left|d\left(gx, fx\right)\right| < 0$, a contradiction.

ii) If $u\left(x_n, x\right) \in M_1 = d\left(gx_n, fx_n\right)$, then by (3.8), we have

$$d\left(fx_n, fx\right) \precsim \alpha \cdot d\left(gx_n, fx_n\right)$$

taking limit as $n \to \infty$,we have

$$d\left(gx, fx\right) \precsim \alpha \cdot d\left(gx, gx\right) = 0,$$

which gives, $\left|d\left(gx, fx\right)\right| < 0$, a contradiction.

iii) If $u\left(x_n, x\right) \in M_1 = d\left(gx_n, fx\right)$, then by (3.8), we have

$$d\left(fx_n, fx\right) \precsim \alpha \cdot d\left(gx_n, fx\right)$$

taking limit as $n \to \infty$, we have

$$d\left(gx, fx\right) \precsim \alpha \cdot d\left(gx, fx\right)$$
$$\left(1-\alpha\right) d\left(gx, fx\right) \precsim 0$$
$$d\left(gx, fx\right) \precsim 0$$

which gives, $\left|d\left(gx, fx\right)\right| < 0$, a contradiction.

iv) If $u\left(x_n, x\right) \in M_1 = d\left(gx, fx_n\right)$, then by (3.8), we have

$$d\left(fx_n, fx\right) \precsim \alpha \cdot d\left(gx, fx_n\right)$$

taking limit as $n \to \infty$,we have

$$d\left(gx, fx\right) \precsim \alpha \cdot d\left(gx, gx\right) = 0,$$

which gives, $\left|d\left(gx, fx\right)\right| < 0$, a contradiction.

v) If $u\left(x_n, x\right) \in M_1 = d\left(gx, fx\right)$, then by (3.8), we have

$$d\left(fx_n, fx\right) \precsim \alpha \cdot d\left(gx, fx\right)$$

taking limit as $n \to \infty$,we have

$$d\left(gx, fx\right) \precsim \alpha \cdot d\left(gx, fx\right),$$
$$\left(1-\alpha\right) d\left(gx, fx\right) \precsim 0,$$
$$d\left(gx, fx\right) \precsim 0,$$

which gives, $\left|d\left(gx, fx\right)\right| < 0$, a contradiction.

Thus from all fives possible cases, $gx = fx$.

Now, let $z = fx = gx$. Since $f$ and $g$ are weakly compatible mappings $fgx = gfx$ which implies that $fz = fgx = gfx = gz$.

We claim that $fz = z$. Suppose not, then by (3.6), we have

$$d\left(fz, z\right) = d\left(fz, fx\right) \precsim \alpha \cdot u\left(z, x\right) \tag{3.9}$$

where

$$\begin{aligned} u\left(z, x\right) \in M_1 &= \left\{d\left(gz, gx\right), d\left(gz, fz\right), d\left(gx, fx\right), d\left(gz, fx\right), d\left(gx, fz\right)\right\} \\ &= \left\{d\left(fz, z\right), 0, 0, d\left(fz, z\right), d\left(z, fz\right)\right\} \\ &= \left\{d\left(fz, z\right), 0\right\}. \end{aligned}$$

Two cases arises:

i) If $u\left(z, x\right) \in M_1 = d\left(fz, z\right)$, then by (3.9), we have

$$d\left(fz, z\right) \precsim \alpha \cdot d\left(fz, z\right),$$
$$\left(1-\alpha\right) d\left(fz, z\right) \precsim 0,$$
$$d\left(fz, z\right) \precsim 0$$

which gives, $\left|d\left(fz, z\right)\right| < 0$ a contradiction.

ii) If $u(z,x) \in M_1 = 0$, then by (3.9), we have

$$d(fz,z) \precsim \alpha \cdot 0 = 0$$
$$d(fz,z) \precsim 0$$

which gives, $|d(fz,z)| < 0$ a contradiction.

Thus, $fz = z = gz$.

Hence, $z$ is a common fixed point of $f$ and $g$.

For uniqueness, suppose that $w$ is another common fixed point of $f$ and $g$ in $X$.

By (3.6), we have

$$d(z,w) = d(gz,gw) \precsim \alpha \cdot u(z,w), \tag{3.10}$$

where

$$\begin{aligned} u(z,w) \in M_1 &= \{d(gz,gw), d(gz,fz), d(gw,fw), d(gz,fw), d(gw,fz)\} \\ &= \{d(z,w), 0, 0, d(z,w), d(w,z)\} \\ &= \{d(z,w), 0\}. \end{aligned}$$

Two possible cases arises:

i) If $u(z,w) \in M_1 = 0$, then by (3.9), we have

$$d(z,w) \precsim \alpha \cdot 0 = 0$$

which gives $|d(z,w)| < 0$, a contradiction.

ii) If $u(z,w) \in M_1 = d(z,w)$, then by (3.9), we have

$$d(z,w) \precsim \alpha \cdot d(z,w)$$
$$(1-\alpha)d(z,w) \precsim 0$$
$$d(z,w) \precsim 0$$

which gives $|d(z,w)| < 0$, a contradiction.

Hence, $z = w$ *i.e.*, $f$ and $g$ have a unique common fixed point.

## References


[1] Azam, A., Fisher, B. and Khan, M. (2011) Common Fixed Point Theorems in Complex Valued Metric Spaces. *Numerical Functional Analysis and Optimization*, **32**, 243-253. http://dx.doi.org/10.1080/01630563.2011.533046

[2] Sastry, K.P.R., Naidu, G.A. and Bekeshie, T. (2012) Metrizability of Complex Valued Metric Spaces and Remarks on Fixed Point Theorems in Complex Valued Metric Spaces. *International Journal of Mathematical Archive*, **3**, 2686-2690.

[3] Manro, S. (2013) Some Common Fixed Point Theorems in Complex Valued Metric Spaces Using Implicit Relation. *International Journal of Analysis and Applications*, **2**, 62-70.

[4] Sintunavarat, W. and Kumam, P. (2012) Generalized Common Fixed Point Theorems in Complex Valued Metric Spaces with Applications. *Journal of Inequalities and Applications*, **1**, 1-12.

[5] Verma, R.K. and Pathak, H.K. (2012) Common Fixed Point Theorems Using Property (*E.A*) in Complex Valued Metric Spaces. *Thai Journal of Mathematics*, **11**, 347-355.

[6] Jungck, G. and Rhoades, B.E. (1998) Fixed Point for Set Valued Functions without Continuous. *Indian Journal of Pure and Applied Mathematics*, **29**, 227-238.

[7] Sintunavarat, W. and Kumam, P. (2011) Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces. *Journal of Applied Mathematics*, **2011**, Article ID: 637958.