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Approximative Method of Fixed Point for Φ-Pseudocontractive Operator and an Application to Equation with Accretive Operator

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ABSTRACT

In this paper, Φ-pseudo-contractive operators and Φ-accretive operators, more general than the strongly pseudo-contractive operators and strongly accretive operators, are introduced. By setting up a new inequality, authors proved that if $T : X \to X$ is a uniformly continuous Φ-pseudo-contractive operator then T has unique fixed point q and the Mann iterative sequence with random errors approximates to *q*. As an application, the iterative solution of nonlinear equation with Φ-accretive operator is obtained. The results presented in this paper improve and generalize some corresponding results in recent literature.

KEYWORDS

1. Introduction and Preliminaries

In 1994, Chidume [1] solved a problem dealt with the fixed point for the class of Lipschitz strictly (strongly) pseudo-contractive operators in uniformly smooth Banach space X. That is, he proved that the Ishikawa iterative sequence converges strongly to the unique fixed point of T in K where $K \subset X$ and $T : K \to K$ is Lipschitz strictly (strongly) pseudo-contractive. Chang [2], in 1998, improved the result, *i.e.*, he proved that the conclusion of Chidume holds if T is uniformly continuous and the fixed point set of T is nonempty (*i.e.*, $F(T) \neq \varnothing$). Recently, Liu [3] proved, if the strongly pseudocontractive operators are replaced by the more general ϕ-strongly pseudo-contractive operators then the conclusion of Chidume still holds.

The objective of this paper is to introduce Φ-pseudo-contractive operators—a class of operators which are more general than the ϕ-strongly pseudo-contractive operators and to study the problems of existence, uniqueness and the iterative approximate method of fixed point by setting up a new inequality in arbitrary Banach space. As an application, the iterative solution of nonlinear equation with Φ-accretive operator is obtained. The results presented in this paper improve and generalize the conclusions of Chidame, Chang and Liu.

To set the framework, we recall some basic notations as follows.

Throughout this paper, we assume that X is a real Banach space with dual X^*, $(\cdot, \cdot)$ denotes the generalized duality pairing. The mapping $J : X \to 2^{X^*}$ defined by

$$Jx = \left\{ j \in X^* : (x, j) = \|x\|\|j\|, \|j\| = \|x\| \right\} \ \forall x \in X \tag{1}$$

is called the normalized duality mapping [4].

Now, we introduce Φ-pseudo-contractive operators as follows.

Definition 1. Let K be nonempty subset of X. An operator $T : K \to K$ is said to be Φ-pseudo-contractive, if there exists a strictly increasing function

$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and

$j(x - y) \in J(x - y)$ such that

$$\left(Tx - Ty, j\left(x - y\right)\right) \le \|x - y\|^2 - \Phi\left(\|x - y\|\right) \,\forall x, y \in K . \tag{2}$$

An operator $A : K \to K$ is said to be Φ-accretive, if

$$\left(Ax - Ay, j\left(x - y\right)\right) \ge \Phi\left(\|x - y\|\right) \,\forall x, y \in K . \tag{3}$$

It is easy to verify that the operator T is Φ-pseudo-accretive if and only if $I - T$ is Φ-accretive where I is an identity mapping on X. Hence, the mapping theory for accretive operators is intimately connected with the fixed point theory for pseudo-contractive operators.

We like to point out: every strongly pseudo-contractive operator is ϕ-strongly pseudo-contractive with $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(s) = ks$ where $k \in (0,1)$, and every ϕ-strongly pseudo-contractive operator must be the Φ-pseudo-contractive operator with $\Phi : [0, \infty) \to [0, \infty)$ defined by $\Phi(s) = \phi(s)s$.

Obviously, if a Φ-pseudo-contractive operator has a fixed point then it is unique.

Definition 2. Let $T : K \to K$ be an operator. For any given $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = \left(1 - \alpha_n\right) x_n + \alpha_n T x_n + \gamma_n u_n \left(n \ge 0\right) \tag{4}$$

is called Mann iteration sequence with random errors. Here $\{u_n\}$ is a bounded sequence in X and is said random error terms of iterative process, and the parameters $\{\alpha_n\}$ and $\{\gamma_n\}$ both are sequences in $[0,1]$. By the way, Xu introduced another definition of Mann iterative sequence with random errors in [5].

In particular, the parameters $\gamma_n = 0$ for all $n \ge 0$ in Equation (4) then $\{x_n\}$ is called Mann [6] iteration sequence.

2. Main Results

First, we have an existence theorem of fixed point as follow.

Theorem 1. If $T : X \to X$ is a continuous Φ-pseudo-contractive operator with bounded range then T has an unique fixed point in X.

Proof. Define $T_n : X \to X$ by

$$T_n x = x - c_n T x \,\forall x \in X \text{ and } n \ge 1$$

where $c_n \in (0,1)$ and $\lim\limits_{n \to \infty} c_n = 1$. Note that T is Φ-pseudo-contractive, thus

$$\begin{aligned}
\left(T_n x - T_n y, j\left(x - y\right)\right) &= \|x - y\|^2 - c_n \left(Tx - Ty, j\left(x - y\right)\right) \\
&\ge \left(1 - c_n\right)\|x - y\|^2 + c_n \Phi\left(\|x - y\|\right) \\
&\ge c_n \Phi\left(\|x - y\|\right)
\end{aligned} \tag{5}$$

for all $x, y \ge X, n \ge 1$ and some $j(x - y) \in J(x + y)$.

Clearly, T_n is a continuous strongly accretive operator. It follows from the Theorem 13.1 of Deimling [7] that there exists an $x_n \in X$ such that $T_n x_n = 0$ for any $n \ge 1$. Next, the sequence $\{x_n\}_{n-1}$ is bounded. In fact, if $\|Tx\| \le M$ then $\|x_n\| \le M$ for all $x \in X$ and

$$\lim_{n \to \infty} \|x_n - Tx_n\| = \lim_{n \to \infty} \left(1 - c_n\right)\|Tx_n\| = 0 \tag{6}$$

Since T is Φ-pseudo-contractive, that is, $(I - T)$ is Φ-accretive, so

$$\begin{aligned}
\Phi\left(\|x_m - x_k\|\right) &\le \left(\left(I - T\right)x_m - \left(I - T\right)x_k, j\left(x_m - x_k\right)\right) \\
&\le \left(\|x_m - Tx_k\| + \|x_m - Tx_k\|\right)\|x_m - x_k\| \\
&\le 2M \left(\|x_m - Tx_m\| + \|x_k - Tx_k\|\right)
\end{aligned} \tag{7}$$

for any $x_m - x_n \in \{x_n\}$. Equations of (6) and (7) ensure that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Consequently, $\{x_n\}_{n=1}^{\infty}$ converges to some $q \in X$. By the continuity of T we have

$$Tq = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} \frac{1}{c_n} x_n = q .$$

Suppose that there exists a $q^* \in X$ such that $Tq^* \in q^*$ then

$$\Phi\left(\left\|q-q^*\right\|\right) \le \left\|q-q^*\right\|^2 -\left(Tq-Tq^*, j\left(q-q^*\right)\right)=0$$

which means that $q = q^*$. *i.e.*, q is unique fixed point of T. The proof is completed.

The following two Lemmas will play crucial roles in the proof of Theorem 2.

Lemma 1. [2] If X be a real Banach space then there exists $j(x+y) \in J(x+y)$ such that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \ \forall x, y \in X . \tag{8}$$

Second, to set up a new inequality as follows.

Lemma 2. Let $\Phi:[0,\infty) \to [0,\infty)$ be a strictly increasing function with $\Phi(0)=0$ and let $\{b_n\}$ and $\{c_n\}$ be two nonnegative real sequences satisfying

$$\sum c_n < \infty, \sum b_n = \infty \text{ and } \lim_{n \to \infty} b_n . \tag{9}$$

Suppose $\{a_n\}$ is a nonnegative real sequences. If there exists a integer $N_0 > 0$ satisfying

$$a_{n+1}^2 \le a_n^2 + o(b_n) + c_n - b_n \Phi(a_{n+1}) \forall n \ge N_0 \tag{10}$$

then $\lim_{n \to \infty} a_n = 0$.

Proof. Let $\inf\{a_n\} = 2\sigma$. If $\sigma > 0$, then $\Phi(a_{n+1}) > \Phi(\sigma)$ for all $n \ge 0$. From the conditions of Equation (9) there exists an integer $N > 0$ 0 such that

$$2o(b_n) \le \Phi(\sigma)b_n \ \forall n \ge N . \tag{11}$$

So, we have

$$a_{n+1}^2 \le a_n^2 + c_n - \frac{1}{2}\Phi(\sigma)b_n \ \forall n \ge N .$$

By induction, we obtain

$$\frac{1}{2}\Phi(\sigma)\sum_{j=N}^{+\infty} b_j \le a_N^2 + \sum_{j=N}^{+\infty} c_j \le +\infty . \tag{12}$$

Equation (12) is in contradiction with $\sum_{N=0}^{+\infty} b_n = +\infty$. It implies that $\sigma = 0$. Therefore, there exists a subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $\lim_{j \to \infty} a_{n_j} = 0$. So, for any given $\varepsilon > 0$ there exists an integer $j_0 \ge 0$ such that

$$\begin{aligned} &a_{n_j} < \varepsilon \forall j \ge j_0 \ and \\ &o(b_n)+c_n \le \Phi(\varepsilon)b_n \forall n \ge n_{j_0} > 0 \end{aligned} . \tag{13}$$

If J_0 is fixed, we will prove that $a_{n_{j0}+k} < \varepsilon$ for all integers $k \ge 1$. The proof is by induction. For $k = 1$, suppose $a_{n_{j0}+k} < \varepsilon$. It follows from Equations (10) and (13) that

$$\varepsilon^2 \le a_{n_{j_0}+1}^2 \le a_{n_{j_0}}^2 + o\left(b_{n_{j_0}}\right) + c_{n_{j_0}} - \Phi(\varepsilon)b_{n_{j_0}} \le a_{n_{j_0}}^2 < \varepsilon^2 .$$

It is a contradiction. Hence, $a_{n_{j_0}+1} \le \varepsilon$ holds for $k = 1$. Assume now that $a_{n_{j_0}+p} \le \varepsilon$ for some integer $p > 1$. We prove that $a_{n_{j_0}+p+1} \le \varepsilon$. Again, assuming the contrary,

Using Equations (10) and (13), as above, it leads to a contradiction as follows

$$\begin{aligned} \varepsilon^2 &\le a_{n_{j_0}+p+1}^2 \\ &\le a_{n_{j_0}}^2 + o\left(b_{n_{j_0}+p}\right) + c_{n_{j_0}+p} - \Phi(\varepsilon)b_{n_{j_0}+p} \\ &\le a_{n_{j_0}+p}^2 < \varepsilon^2 \end{aligned}$$

Where $n_{j_o} + p \ge n_{j_o} > j_o$. Therefore, $a_{n_{j_0}+p} < \varepsilon$ holds for all integers $k \ge 1$, *i.e.*, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n_{j_0}+k} = 0$

The Proof is completed.

Theorem 2. Let $T : X \to X$ be an $\to$ a uniformly continuous Φ-pseudo-contractive operator with bounded range. Suppose that the iterative sequence $\{x_n\}$ is defined by Equation (4) satisfying

$$\sum \gamma_n < \infty, \lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum \alpha_n = \infty. \tag{14}$$

then $\{x_n\}$ converges strongly to unique fixed point of T.

Proof. From Theorem 1, we know that there exists a unique fixed point q of T.

Putting $M_0 = \sup\{\|Tx\| : x \in X\} + \sup\{\|u_n\| : n \geq 0\} + \|x_0\|$ for any given $x_0 \in X$, then

$$\|x_1\| \leq (1 - \alpha_0)\|x_0\| + \alpha_0\|Tx_0\| \leq M_0.$$

Using induction, we have $\|x_0\| \leq M_0$ for all $n \geq 0$. Let $M = M_0 + \|q\|$. Since $\|x_n - x_{n+1}\| = a_n\|x_n - Tx_n\| \leq 2M\alpha_n \to 0 (as\ n \to \infty)$, therefore,
$e_n : \|Tx_n - Tx_{n+1}\| \to 0 (as\ n \to \infty)$
by the uniformly continuity of T.

From Equation (14) there exists an integer $N_0 > 0$ such that

$$0 \leq \alpha_n \leq \frac{1}{6} \ \forall n \geq N_0. \tag{15}$$

By Equations (4), (8) and (15) we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1 - \alpha_n)(x_n - q) + a_n(Tx_n - q) + \gamma_n u_n\|^2 \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\gamma_n(u_n, j(x_{n+1} - q)) \\
&\quad + 2\alpha_n(Tx_n - q, j(x_{n+1} - q)) \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 \\
&\quad + 2\alpha_n(Tx_n - Tx_{n+1}, j(x_{n+1} - q)) \\
&\quad + 2\alpha_n(Tx_n - q, j(x_{n+1} - q)) + 2M^2\gamma_n \\
&\leq (1 - 2\alpha_n + \alpha_n^2)^2 \|x_n - q\|^2 + 2\alpha_n\|x_{n+1} - q\|^2 \\
&\quad + 2M\alpha_n e_n + 2M^2\gamma_n - 2\alpha_n\Phi(\|x_{n+1} - q\|) \\
&\leq \|x_n - q\|^2 + 2M^2\alpha_n^2 + 3M\alpha_n e_n + 2M^2\gamma_n - \alpha_n\Phi(\|x_{n+1} - q\|) \\
&= \|x_n - q\|^2 + o(\alpha_n) + c_n - \alpha_n\Phi(\|x_{n+1} - q\|)
\end{aligned} \tag{16}$$

for all $n \geq N$ where $o(\alpha_n) = 2M^2\alpha_n^2 + 3M\alpha_n e_n$ and $c_n = 2M^2\gamma_n$. It follows from the Lemma 2 that $\lim_{n \to \infty}\|x - q\|$. The Proof is completed.

Last, as an application, the existence, the uniqueness and the approximate method of solution of nonlinear equation with Φ-accretive operator is obtained. That is

Theorem 3. Suppose that $A : X \to X$ is an $\to$ a uniformly continuous Φ-accretive operator and the range of either A or $I - A$ be bounded. For any given $f \in X$, the equation $Ax = f$ has unique solution in X, and if $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha)x + \alpha[f + (I - A)x] + \gamma_n u_n (n \geq 0)$$

satisfying the conditions of equation (14) then it converges strongly to the solution.

Proof. We define $S : X \to X$ by $Sx = f + x - Ax$ for all $x \in X$. Clearly, S is Φ-pseudo-contractive and continuous if A is Φ-accretive and continuous, and the range of S is bounded if $(I - A)$ is. It is easy to see that x^* is a solution of the equation $Ax = f$ if and only if that x^* is a fixed point of S. It follows from the Theorem 2 that $Ax = f$ has an unique solution $x^* \in X^*$ and the iterative sequence $\{x_n\}$ is defined by Equation (4) converges strongly to x^*. *i.e.*, the sequence $\{x_n\}$ is an iterative solution of the equation $Ax = f$.

The proof is completed.

Remark. Theorem 2 improves a number of results (for example, Theorem 4.2 of [2], Theorem 2 of [1] and Corollary 3.3 and 3.4 of [3]) in the following senses.

1) The existence and the convergence of the fixed point for Φ-pseudo-contractive operator are studied simultaneously.

2) The operators may not be strongly pseudo-contractive or ϕ-strongly pseudo-contractive.

3) The continuity of operator may not be Lipschitzian.

4) The errors come from the iterative process have been considered appropriately.

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