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Common Fixed Point Theorems for Totally Quasi-G-Asymptotically Nonexpansive Semigroups with the Generalized f-Projection

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ABSTRACT

In this paper, we introduce some new classes of the totally quasi-G-asymptotically nonexpansive mappings and the totally quasi-G-asymptotically nonexpansive semigroups. Then, with the generalized f-projection operator, we prove some strong convergence theorems of a new modified Halpern type hybrid iterative algorithm for the totally quasi-G-asymptotically nonexpansive semigroups in Banach space. The results presented in this paper extend and improve some corresponding ones by many others.

KEYWORDS

Totally Quasi-G-Asymptotically Nonexpansive Semigroup; Generalized f-Projection Operator; Modified Halpern Type Hybrid Iterative Algorithm; Strong Convergence Theorem

1. Introduction

In this paper, we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real number and the set of nature number respectively. Let E be a real Banach space with its dual E^* and C be a nonempty, closed and convex subset of E. The mapping $J: E \to 2^{E^*}$ is the normalized duality mapping, defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x\| = \|x^*\|\right\}, x \in E$$

Recall that a mapping $T: C \to C$ is said to be *nonexpansive* [1,2], if for each $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|.$$

A mapping $T: C \to C$ is said to be *totally asymptotically nonexpansive*, if there exists nonnegative real sequences $\{\mu_n\}$ and $\{v_n\}$ with $\mu_n \to 0, v_n \to 0$ as $(n \to \infty)$ and a strictly increasing continuous function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, such that for each $x, y \in C$,

$$\left\|T^n x - T^n y\right\| \le \|x - y\| + v_n \varphi(\|x - y\|) + \mu_n, \forall n \ge 0.$$

We use $\phi: E \times E \to \mathbb{R}^+$ to denote the Lyapunov function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \forall x, y \in E.$$

Obviously, we have

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$$(\|x\|-\|y\|)^2 \le \phi(x,y) \le (\|y\|+\|x\|)^2.$$

Recently, Chang *et al.* [3-5] and Li [6] introduced the uniformly totally quasi-ϕ-asymptotically nonexpansive mappings and studied the strong convergence of some iterative methods for the mappings in Banach space.

Definition 1.1 [1] *A countable family of mapping* $\{T_i\}$ *is said to be uniformly totally quasi-*ϕ*-asymptotically nonexpansive, if* $\bigcap_{n=1}^{\infty} F(T_i) \neq \varnothing$, *and there exist nonnegative sequences* $\{\mu_n\}$, $\{\nu_n\}$ *with* $\mu_n \to 0, \nu_n \to 0$ (*as* $n \to \infty$) *and a strictly increasing continuous function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(0)=0$, *such that for each* $i \ge 0$, *and each* $x \in C, p \in \bigcap_{n=1}^{\infty} F(T_i)$,,

$$\phi(p, T_i x) \le \phi(p,x) + \mu_n \psi(\phi(p,x)) + \nu_n. \tag{1}$$

More recently, Wang *et al.* [7] studied the strong convergence for a countable family of total quasi-ϕ-asymptotically nonexpansive mappings by using the hybrid algorithm in 2-uniformly convex and uniformly smooth real Banach spaces. Quan *et al.* [8] introduced total quasi-ϕ-asymptotically nonexpansive semigroup containing many kinds of generalized nonexpansive mappings as its special cases and used the modified Halpern-Mann iteration algorithm to prove strong convergence theorems in Banach spaces.

We use $F(\mathscr{T})$ to denote the common fixed point set of the semigroup $\mathscr{T}$, *i.e.* $F(\mathscr{T}) = \bigcap_{t \ge 0} F(T(t))$.

Definition 1.2 [8] *One-parameter family* $\mathscr{T} := \{T(t): C \to C, t \ge 0\}$ *is said to be a quasi-*ϕ*-asymptotically nonexpansive semigroup, if* $F(\mathscr{T}) \neq \varnothing$ *and the following conditions are satisfied*:

(a) $T(0)x = x$ for each $x \in C$;

(b) For each $x \in C$, $T(s+t)x = T(s)T(t)x$, $\forall t,s \in \mathbb{R}^+$;

(c) For each $x \in C$, the mapping $t \to T(t)x$ is continuous;

(d) For each $x \in C$, $p \in F(\mathscr{T})$, there exists a sequences $\{k_n\} \subset [1,+\infty)$ with $k_n \to 1$ as $n \to \infty$, such that

$$\phi(p, T^n(t)x) \le k_n \phi(p,x), \forall n \in \mathbb{N}. \tag{2}$$

One-parameter family $\mathscr{T} := \{T(t): C \to C, t \ge 0\}$ is said to be a totally quasi-ϕ-asymptotically nonexpansive semigroup, if $F(\mathscr{T}) \neq \varnothing$, the conditions (a)-(c) and the following condition are satisfied:

(e) If $F(\mathscr{T}) \neq \varnothing$, there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$, such that

$$\phi(p, T^n(t)x) \le \phi(p,x) + \mu_n \psi(\phi(p,x)) + \nu_n, \forall n \in \mathbb{N}, \tag{3}$$

for all $x \in C$, $p \in F(\mathscr{T})$.

On the other hand, Wu *et al.* [9] introduced the generalized f-projection which extends the generalized projection and always exists in a real reflexive Banach space. Li *et al.* [10] proved some properties of the generalized f-projection operator and studied the strong convergence theorems for the relatively nonexpansive mappings.

In 2013, by using the generalized f-projection operator, Seawan *et al.* [11] introduced the modified Mann type hybrid projection algorithm for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with Kadec-Klee property.

Motivated by the above researches, in this paper, we introduce a new class of the totally quasi-G-asymptotically nonexpansive mappings which contains the class of the totally quasi-ϕ-asymptotically nonexpansive mappings and we extend from a countable family of mappings to the totally quasi-G-asymptotically nonexpansive semigroup. Then we modify the Halpern type hybrid projection algorithm by using the generalized f-projection operator for uniformly total quasi-G-asymptotically nonexpansive semigroup and prove some strong convergence theorems under some suitable conditions. The results presented in this paper extend and improve some corresponding ones by many others, such as [1,2,7,8,10,11].

2. Preliminaries

This section contains some definitions and lemmas which will be used in the proofs of our main results in the next section.

Throughout this paper, we assume that E be a real Banach space with its dual space E^*. A Banach space E is said to be strictly convex, if $\frac{\|x+y\|}{2}<1$ for all $x,y\in E$ with $\|x\|=\|y\|=1$ and $x\neq y$. E is said to be uniformly convex, if $\lim_{n\to\infty}\|x_n-y_n\|=0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E with $\|x_n\|=\|y_n\|=1$ and $\lim_{n\to\infty}\frac{\|x_n+y_n\|}{2}=1$. A Banach space E is said to be smooth, if $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for each $x,y\in E$ with $\|x\|=\|y\|=1$. E is said to be uniformly smooth, if the limit is attainted uniformly for each $\|x\|=\|y\|=1$.

It is well known that the normalized dual mapping $J:E\to E^*$ holds the properties:

(1) If E is a smooth Banach space, then J is single-valued and semi-continuous;

(2) If E is uniformly smooth Banach space, then J is uniformly norm-to-norm continuous operator on each bounded subset of E.

A Banach space E is said to have Kadec-Klee property, if for any sequence $\{x_n\}\in E$ satisfies $x_n\to x\in E$ and $\|x_n\|\to\|x\|$, then $x_n\to x$. As we all know, if E is uniformly convex, then E has the Kadec-Klee property.

Now, we give a functional $G:C\times E^*\to\mathbb{R}\cup\{+\infty\}$, defined by

$$G(\xi,\eta^*)=\|\xi\|^2-2\langle\xi,\eta^*\rangle+\|\eta^*\|^2+2\rho f(\xi),\tag{4}$$

where $\xi\in C$, $\eta^*\in E^*$, ρ is a positive real number and $f:C\to\mathbb{R}\cup\{+\infty\}$ is proper, convex and lower semi-continuous. From the definition of G and f, it is easy to see the following properties:

(1) $G(\xi,\eta^*)$ is convex and continuous with respect to η^* when ξ is fixed;

(2) $G(\xi,\eta^*)$ is convex and lower semi-continuous with respect to ξ when η^* is fixed.

Definition 2.1 [9] $\prod_C^f:E^*\to 2^C$ *is said to be a generalized f-projection operator, if for any* $\eta^*\in E^*$,

$$\prod_C^f\eta^*=\left\{u\in C:G(u,\eta^*)=\inf_{x\in C}G(x,\eta^*)\right\}.\tag{5}$$

Lemma 2.2 [9] *Let* E *be a real reflexive Banach space with its dual* E^*, C *be a nonempty closed and convex subset of* E. *Then* $\prod_C^f y^*$ *is a nonempty closed and convex subset of* C *for all* $y^*\in E^*$. *Moreover, if* E *is strictly convex, then* $\prod_C^f$ *is a single-valued mapping.*

Recall that if E is a smooth Banach space, then the normalized dual mapping J is single-valued, *i.e.* there exists unique $\eta^*\in E^*$ such that $\eta^*=Jx$ for each $x\in E$. Then (4) is equivalent to

$$G(\xi,Jx)=\|x\|^2-2\langle\xi,Jx\rangle+\|Jx\|^2+2\rho f(\xi).\tag{6}$$

And in a smooth Banach space, the definition of the generalized f-projection operator transforms into:

Definition 2.3 [10] *Let* E *be a real smooth Banach space and* C *be a nonempty, closed and convex subset of* E. *The mapping* $\prod_C^f:E^*\to 2^C$ *is called generalized f-projection operator, if for all* $x\in E$,

$$\prod_C^f x=\left\{u\in C:G(u,Jx)=\inf_{\xi\in C}G(\xi,Jx)\right\}.\tag{7}$$

Now, we give the definition of the totally quasi-G-asymptotically nonexpansive mapping and the totally quasi-G-asymptotically nonexpansive semigroup.

Definition 2.4 *A mapping* $T:C\to C$ *is said to be a quasi-G-asymptotically nonexpansive, if* $F(T)\neq\varnothing$ *and there exists a sequence* $\{k_n\}\subset[1,+\infty]$ *with* $k_n\to 1$ *(as* $n\to\infty$*), such that*

$$G(p,JT^nx)\le k_nG(p,Jx),\forall n\ge 0,\tag{8}$$

for any $x\in C$ and $p\in F(T)$.

A mapping $T:C\to C$ is said to be a totally quasi-G-asymptotically nonexpansive, if $F(T)\neq\varnothing$ and there exist sequences $\{\mu_n\}$, $\{\delta_n\}$ with $\mu_n,\delta_n\to 0$ as $n\to\infty$ and a strictly increasing continuous function

$\tau:\mathbb{R}\to\mathbb{R}$ with $\tau(0)=0$, such that

$$G\left(p,JT^n x\right)\le G(p,x)+\mu_n\tau\left(G(p,Jx)\right)+\delta_n,\forall n\in\mathbb{N},\tag{9}$$

for all $x\in C$ and $p\in F(T)$.

Remark 2.5 It is easy to see that a quasi-ϕ-asymptotically nonexpansive mapping is a quasi-G-asymptotically nonexpansive mapping with $f(p)=0$ for all $p\in F(\mathscr{T})$. A totally quasi-ϕ-asymptotically nonexpansive mapping is a totally quasi-G-asymptotically nonexpansive mapping with $\delta_n=\mu_n\psi\left(f(p)\right)$. Therefore, our totally quasi-G-asymptotically nonexpansive mappings here are more widely than the totally quasi-ϕ-asymptotically nonexpansive mappings which contain many kinds of generalized nonexpansive mappings as their special cases.

Definition 2.6 *One-parameter family* $\mathscr{T}:=\{T(t):C\to C,t\ge 0\}$ *is said to be a quasi-G-asymptotically nonexpansive semigroup on* C, *if the conditions* (*a*)-(*c*) *in Definition* 1.2 *and the following condition are satisfied*:

(f) There exists a sequence $\{k_n\}\subset[1,+\infty]$ with $k_n\to 1$ as $n\to\infty$ such that

$$G\left(p,T^n(t)x\right)\le k_n G(p,Jx)\tag{10}$$

holds for all $x,y\in C$, $n\in\mathbb{N}$.

One-parameter family $\mathscr{T}:=\{T(t):C\to C,t\ge 0\}$ is said to be a totally quasi-G-asymptotically nonexpansive semigroup on C, if the above conditions (a)-(c) in Definition 1.2 and the following condition are satisfied:

(g) if $F(\mathscr{T})\ne\varnothing$ and there exist sequences $\{\mu_n\},\{\delta_n\}$ with $\mu_n,\nu_n\to 0$ as $n\to\infty$ and a strictly increasing continuous function $\tau:\mathbb{R}\to\mathbb{R}$ with $\tau(0)=0$ such that for all $x\in C$ and $p\in F(\mathscr{T})$,

$$G\left(p,JT^n(t)x\right)\le G(p,Jx)+\mu_n\tau\left(G(p,Jx)\right)+\delta_n\tag{11}$$

holds for each $n\in\mathbb{N}$.

Remark 2.7 It is easy to see that a quasi-ϕ-asymptotically nonexpansive semigroup is a quasi-G-asymptotically nonexpansive semigroup with $f(p)=0$ for all $p\in F(\mathscr{T})$. A totally quasi-ϕ-asymptotically nonexpansive semigroup is a totally quasi-G-asymptotically nonexpansive semigroup with $\delta_n=\mu_n\psi\left(f(p)\right)$. When we use $t_m\left(m\in\mathbb{N}^+\right)$ instead of t in Definition 2.6 and denote $T(t_m)$ by T_m, $\mathscr{T}:=\{T_m:C\to C\}_{m=1}^{\infty}$, then a quasi-G-asymptotically nonexpansive semigroup becomes a countable family of total quasi-G-asymptotically nonexpansive mappings which contains a countable family of total quasi-ϕ-asymptotically nonexpansive mappings (see [3,4,7]) as it's special case. So our totally quasi-G-asymptotically nonexpansive semigroup here is the most widely family of the nonexpansive mappings so far.

The following Lemmas are necessary for proving the main results in this paper.

Lemma 2.8 [12] *Let* E *be a uniformly convex and smooth Banach space*, *and* $\{x_n\}$, $\{y_n\}$ *be two sequences of* E. *If* $\phi(x_n,y_n)\to 0$ *and either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded*, *then* $\|x_n-y_n\|\to 0$.

Lemma 2.9 [13] *If* E *is a strictly convex*, *reflexive and smooth Banach space*, *then for* $x,y\in E$, $\phi(x,y)=0$ *if and only if* $x=y$.

Lemma 2.10 [14] *Let* E *be a real Banach space and* $f:E\to\mathbb{R}\cup\{+\infty\}$ *be a lower semicontinuous convex functional. Then there exists* $x^*\in E^*$ *and* $\alpha\in\mathbb{R}$ *such that*

$$f(x)\ge\left\langle x,x^*\right\rangle+\alpha,\tag{12}$$

for each $x\in E$.

Lemma 2.11 [10] *Let* E *be a real reflexive and smooth Banach space and* C *be a nonempty*, *closed and convex subset of* E. *Let* $x\in E$, $z\in\prod_C^f x$. *Then*

$$\phi(y,z)+G(z,Jx)\le G(y,Jx),\forall y\in C.\tag{13}$$

Lemma 2.12 *Let* E *be a uniformly smooth and strictly convex Banach space*, C *be a nonempty closed and convex subset of* E. *Let* $T:C\to C$ *be a totally quasi-G-asymptotically nonexpansive mapping defined by* (9). *If* $\mu_1=\delta_1=0$, *then the fixed point set* $F(T)$ *of* T *is closed and convex subset of* C.

Proof Let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$, we prove that $p \in F(T)$. In fact, since T is a quasi-G-asymptotically nonexpansive mapping, we have

$$G(p_n, JTp) \le G(p_n, Jp) + \mu_1 \tau(G(p_n, Jp)) + \delta_1.$$

Since $\mu_1 = \delta_1 = 0$, it is equivalent to that

$$\|p_n\|^2 - 2\langle p_n, JTp\rangle + \|JTp\|^2 + 2\rho f(p_n) \le \|p_n\|^2 - 2\langle p_n, Jp\rangle + \|Jp\|^2 + 2\rho f(p_n).$$

So,

$$\phi(p_n, Tp) \le \phi(p_n, p) \to 0.$$

By lemma 2.8, we have that $p \in F(T)$ which implies that $F(T)$ is closed. Next we prove that $F(T)$ is convex, *i.e.* for any $x, y \in F(T)$, $\lambda \in (0,1)$, we prove that $z = \lambda x + (1-\lambda) y \in F(T)$. In fact,

$$\begin{aligned} G(z, JT^n z) &= \|z\|^2 - 2\langle z, JT^n z\rangle + \|JT^n z\|^2 + 2\rho f(z) \\ &= \|z\|^2 - 2\lambda\langle x, JT^n z\rangle - 2(1-\lambda)\langle y, JT^n z\rangle + \|JT^n z\|^2 + 2\rho f(z) \\ &= \|z\|^2 + \lambda G(x, Jz) + (1-\lambda) G(y, JT^n z) - \lambda\|x\|^2 - (1-\lambda)\|y\|^2. \end{aligned} \tag{14}$$

$$\begin{aligned} &\lambda G(x, Jz) + (1-\lambda) G(y, JT^n z) \\ &\le \lambda\left[G(x, Jz) + \mu_n \tau(G(x, Jz)) + \delta_n\right] + (1-\lambda)\left[G(y, Jz) + \mu_n \tau(G(y, Jz)) + \delta_n\right] \\ &= \lambda\left[\|x\|^2 - 2\langle x, Jz\rangle + \|Jz\|^2 + 2\rho f(x) + \mu_n \tau(G(x, Jz)) + \delta_n\right] \\ &\quad + (1-\lambda)\left[\|y\|^2 - 2\langle y, Jz\rangle + \|Jz\|^2 + 2\rho f(y) + \mu_n \tau(G(y, Jz)) + \delta_n\right] \\ &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \|z\|^2 + 2\rho f(z) + \delta_n \\ &\quad + \lambda \mu_n \tau(G(x, Jz)) + (1-\lambda)\mu_n \tau(G(y, Jz)). \end{aligned} \tag{15}$$

Submitting (15) into (14), we have

$$\phi(z, T^n z) = \|z\|^2 - 2\langle z, JT^n z\rangle + \|JT^n z\|^2 \le \lambda \mu_n \tau(G(x, Jz)) + (1-\lambda)\mu_n \tau(G(y, Jz)) + \delta_n.$$

This implies that $T^n z \to z$ and $T^{n+1} z = TT^n z \to z$. Hence we have $z = Tz$, *i.e.* $z \in F(T)$. This completes the proof of Lemma 2.12.

3. Main Results

Theorem 3.1 *Let E be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed and convex subset of E. Let* $f: E \to \mathbb{R}$ *be a convex and lower semicontinuous function with* $C \subset \operatorname{int}(D(f))$ *such that* $f(x) > 0$ *for all* $x \in C$ *and* $f(0) = 0$. *Let* $\mathscr{T} = \{T(t): C \to C, t \ge 0\}$ *be a closed and totally quasi-G-asymptotically nonexpansive semigroup defined by Definition 2.6. Assume that* $T(t)$ *is uniformly asymptotically regular for all* $t \ge 0$ *and* $F(\mathscr{T}) = \bigcap_{t \ge 0} F(T(t)) \ne \varnothing$. *Let the sequence* $\{x_n\}$ *be defined by*

$$\begin{cases} x_1 \in E, \text{chosen arbitrarily; } C_1 = C, \\ y_{n,t} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n) J T^n(t) x_n\right], \\ C_{n+1} = \left\{z \in C_n : \sup\limits_{t \ge 0} G(z, J y_{n,t}) \le \alpha_n G(z, J x_1) + (1-\alpha_n) G(z, J x_n) + \xi_n\right\}, \\ x_{n+1} = \prod\limits_{C_{n+1}}^{f} x_1, \end{cases} \tag{16}$$

where $\xi_n = \mu_n \sup_{p\in F(\mathscr{T})} \tau\left(G\left(p, Jx_n\right)\right) + \delta_n$ and the sequence $\{\alpha_n\} \subset (0,1)$. If $\lim_{n\to\infty}\alpha_n = 0$ and $\mu_1 = \delta_1 = 0$, then $\{x_n\}$ converges strongly to $\prod_{F(\mathscr{T})}^{f} x_1$.

Proof We divide the proof into five steps.

Step 1. Firstly, we prove that $F(\mathscr{T})$ and C_n are closed and convex subsets in C.

Since $T(t)$ is a totally quasi-G-asymptotically nonexpansive mapping, it follows the Lemma 2.12 that $F(T(t))$ is a closed and convex subset of C. So $F(\mathscr{T}) = \bigcap_{t\ge 0} F(T(t))$ is closed and convex subset of C.

Again, by the assumption, $C_1 = C$ is closed and convex. Suppose that C_n is the closed and convex subset of C for $n \ge 2$. In view of the definition of G, we have that

$$
\begin{aligned}
C_{n+1} &= \left\{ z \in C_n : \sup_{t\ge 0} G\left(z, Jy_{n,t}\right) \le \alpha_n G\left(z, Jx_1\right) + \left(1-\alpha_n\right) G\left(z, Jx_n\right) + \xi_n \right\} \\
&= \bigcap_{t\ge 0} \left\{ z \in C_n : G\left(z, Jy_{n,t}\right) \le \alpha_n G\left(z, Jx_1\right) + \left(1-\alpha_n\right) G\left(z, Jx_n\right) + \xi_n \right\} \bigcap C_n \\
&= \bigcap_{t\ge 0} \left\{ z \in C_n : 2\left\langle z, Jx_n\right\rangle - Jy_{n,t} \le \left\|x_n\right\|^2 - \left\|y_{n,t}\right\|^2 + \xi_n \right\} \bigcap C_n.
\end{aligned}
$$

This shows that C_{n+1} is closed and convex for all $n \ge 1$.

Step 2. Next, we prove that $F(\mathscr{T}) \subset C_n$.

In fact, $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_n$, for some $n \ge 2$. Since $\mathscr{T} = \left\{T(t) : C \to C, t \ge 0\right\}$ is a totally quasi-G-asymptotically nonexpansive semigroup, for each $p \in F(T) \subset C_n$, we have

$$
\begin{aligned}
G\left(p, y_{n,t}\right) &= G\left(p, \alpha_n Jx_1 + \left(1-\alpha_n\right) JT^n(t) x_n\right) \\
&= \|p\|^2 - 2\left\langle p, \alpha_n Jx_1 + \left(1-\alpha_n\right) JT^n(t) x_n \right\rangle + \left\|\alpha_n Jx_1 + \left(1-\alpha_n\right) JT^n(t) x_n\right\|^2 + 2\rho f(p) \\
&\le \|p\|^2 - 2\alpha_n \left\langle p, Jx_1\right\rangle - 2\left(1-\alpha_n\right)\left\langle p, JT^n(t) x_n\right\rangle + \alpha_n \left\|Jx_1\right\|^2 \\
&\quad + \left(1-\alpha_n\right)\left\|JT^n(t) x_n\right\|^2 + 2\rho f(p) \\
&= \alpha_n G\left(p, Jx_1\right) + \left(1-\alpha_n\right) G\left(p, JT^n(t) x_n\right) \\
&\le \alpha_n G\left(p, Jx_1\right) + \left(1-\alpha_n\right) G\left(p, Jx_n\right) + \left(1-\alpha_n\right)\left(\mu_n \tau\left(G\left(p, Jx_n\right)\right) + \delta_n\right) \\
&\le G\left(p, Jx_1\right) + \left(1-\alpha_n\right) G\left(p, Jx_n\right) + \xi_n,
\end{aligned}
$$

where $\xi_n = \mu_n \sup_{t\ge 0} \tau\left(G\left(p, Jx_n\right)\right) + \delta_n$. This shows that $p \in C_{n+1}$, which implies that $F(\mathscr{T}) \subset C_n$ for all $n \ge 1$.

Step 3. We prove that $\{x_n\}$ is bounded and $\left\{G\left(x_n, x_1\right)\right\}$ is convergent.

Since $f : E \to \mathbb{R}$ is a convex and lower semicontinuous function, by virtue of Lemma 2.10, we have that there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that $f(x) \ge \left\langle x, x^*\right\rangle + \alpha$ for each $x \in E$. Then for each $x_n \in E$, we have that

$$
\begin{aligned}
G\left(x_n, Jx_1\right) &= \left\|x_n\right\|^2 - 2\left\langle x_n, Jx_1\right\rangle + \left\|x_1\right\|^2 + 2\rho f\left(x_n\right) \\
&\ge \left\|x_n\right\|^2 - 2\left\langle x_n, Jx_1\right\rangle + \left\|x_1\right\|^2 + 2\rho\left\langle x_n, x^*\right\rangle + 2\rho\alpha \\
&= \left\|x_n\right\|^2 - 2\left\langle x_n, Jx_1 - \rho x^*\right\rangle + \left\|x_1\right\|^2 + 2\rho\alpha \\
&\ge \left\|x_n\right\|^2 - 2\left\|x_n\right\|\left\|Jx_1 - \rho x^*\right\| + \left\|x_1\right\|^2 + 2\rho\alpha \\
&= \left(\left\|x_n\right\| - \left\|Jx_1 - \rho x^*\right\|\right)^2 + \left\|x_1\right\|^2 - \left\|Jx_1 - \rho x^*\right\|^2 + 2\rho\alpha.
\end{aligned}
\tag{17}
$$

Again since $x_n = \prod_{C_n}^{f} x_1$ and $F(\mathscr{T}) \subset C_n$, from Lemma 2.11, we have $G\left(x_n, Jx_1\right) \le G\left(p, Jx_1\right)$ for any $p \in F(\mathscr{T})$. Hence, from (17), we have

$$G(p,Jx_1)\geq G(x_n,Jx_1)\geq\left(\|x_n\|-\|Jx_1-\rho x^*\|\right)^2+\|x_1\|^2-\|Jx_1-\rho x^*\|^2+2\rho\alpha.$$

Therefore $\{x_n\}$ and $\{G(x_n,Jx_1)\}$ are bounded. As $x_{n+1}=\prod_{C_{n+1}}^f x_1\in C_{n+1}\subset C_n$ and $x_n=\prod_{C_n}^f x_1$, by using Lemma 2.11, we have that

$$G(x_{n+1},Jx_1)-G(x_n,Jx_1)\geq\phi(x_{n+1},x_n)\geq 0.$$

This implies that $\{G(x_n,Jx_1)\}$ is bounded and nondecreasing. Hence the limit $\lim_{n\to\infty}G(x_n,x_1)$ exists.

Step 4. Next, we prove that $x_n\to \overline{x}\in F(\mathscr{T})$.

By the definition of C_n, for any positive integer $m\geq n$, we have $x_m=\prod_{C_m}^f x_1\in C_m\subset C_n$. Again from Lemma 2.11, we have that

$$\phi(x_m,x_n)<=G(x_m,Jx_1)-G(x_n,Jx_1)\to 0$$

as $m,n\to\infty$. It follows from Lemma 2.8 that $\lim_{n,m\to\infty}\|x_m-x_n\|=0$. Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty closed and convex subset of Banach space E, we can assume that $x_n\to\overline{x}\in C$. Therefore, we have

$$\lim_{n\to\infty}\xi_n=\lim_{n\to\infty}\left[\mu_n\sup_{p\in F(T)}\tau G(p,Jx_n)+\mu_n\right]=0. \tag{18}$$

Since $x_{n+1}\in C_{n+1}$ and $\alpha_n\to 0$, it follows from the definition of C_n that we have

$$\sup_{t\geq 0}G(x_{n+1},Jy_{n,t})\leq\alpha_n G(x_{n+1},Jx_1)+(1-\alpha_n)G(x_{n+1},Jx_n)+\xi_n.$$

$$\begin{aligned}&\|x_{n+1}\|^2-2\langle x_{n+1},Jy_{n,t}\rangle+\|y_{n,t}\|^2+2\rho f(x_{n+1})\\&\leq\alpha_n\left(\|x_{n+1}^2-2\langle x_{n+1},Jy_{n,t}\rangle+\|x_1\|^2+2\rho f(x_{n+1})\right)\\&\quad+(1-\alpha_n)\left(\|x_{n+1}\|^2-2\langle x_{n+1},Jx_n\rangle+\|x_n\|^2+2\rho f(x_{n+1})+\xi_n\right).\end{aligned}$$

$$\begin{aligned}\|x_{n+1}\|^2-2\langle x_{n+1},Jy_{n,t}\rangle+\|y_{n,t}\|^2&\leq(1-\alpha_n)\left(\|x_{n+1}\|^2-2\langle x_{n+1},Jx_n\rangle+\|x_n\|^2\right)\\&\quad+\alpha_n\left(\|x_{n+1}^2-2\langle x_{n+1},Jy_{n,t}\rangle+\|x_1\|^2\right)+\xi_n.\end{aligned}$$

That is

$$\phi(x_{n+1},y_{n,t})\leq\alpha_n\phi(x_{n+1},x_1)+(1-\alpha_n)\phi(x_{n+1},x_1)+\xi_n. \tag{19}$$

Since $x_n\to u$ and $\alpha_n\to 0$, from (18), (19), we can get

$$\lim_{n\to\infty}\phi(x_{n+1},y_{n,t})=0.$$

Then, by Lemma 2.8, we have

$$\lim_{n\to\infty}y_{n,t}=\overline{x}. \tag{20}$$

As J is uniformly continuous on each bounded subset of E, we have $Jx_n\to J\overline{x}$. Then from (20), for any $t\geq 0$, we have

$$\begin{aligned}0=\lim_{n\to\infty}\|Jy_{n,t}-J\overline{x}\|&=\lim_{n\to\infty}\|\alpha_n Jx_1+(1-\alpha_n)JT^n(t)x_n-J\overline{x}\|\\&\geq\lim_{n\to\infty}\left[(1-\alpha_n)\|JT^n(t)x_n-Ju\|-\alpha_n\|Jx_1-J\overline{x}\|\right]\\&=\lim_{n\to\infty}(1-\alpha_n)\|JT^n(t)x_n-J\overline{x}\|.\end{aligned}$$

Since $\lim_{n\to\infty}(1-\alpha_n)=1$, we have that

$$\lim_{n\to\infty}\left\|JT^n(t)x_n - J\bar{x}\right\|=0,$$

uniformly for all $t\ge 0$.

Since J is uniformly continuous, we obtain that

$$\lim_{n\to\infty}\left\|T^n(t)x_n-\bar{x}\right\|=0, \tag{21}$$

uniformly for all $t\ge 0$.

Since $T(t)$ is asymptotically regular for all $t\ge 0$, from (21), we have

$$\lim_{n\to\infty}\left\|T^{n+1}(t)x_n-\bar{x}\right\|=\lim_{n\to\infty}\left(\left\|T^{n+1}(t)x_n-T^n(t)x_n\right\|+\left\|T^n(t)x_n-\bar{x}\right\|\right)=0$$

Then $T^{n+1}(t)x_n=T(t)T^n(t)x_n\to\bar{x}$ as $n\to\infty$. By virtue of the closedness of $T(t)$ and $T^n(t)x_n\to\bar{x}$ as $n\to\infty$, we can obtain that $T(t)\bar{x}=\bar{x}$, which implies $\bar{x}\in F(T(t))$ for all $t\ge 0$.

Hence, $x_n\to\bar{x}\in F(\mathscr{T})=\bigcap_{t\ge 0}F(T(t))$.

Step 5. Finally, we prove that $x_n\to\bar{x}=\prod_{F(\mathscr{T})}^f x_1$.

Since $F(\mathscr{T})\subset C_n\subset E$ is closed and convex, by Lemma 2.2, we know that $\prod_{F(\mathscr{T})}^f x_1$ is single-valued. Assume that $\varpi=\prod_{F(\mathscr{T})}^f x_1$. Since $\varpi\in F(\mathscr{T})\subset C_n$ and $x_n=\prod_{C_n}^f x_1$, we have $G(x_n,Jx_1)\le G(\varpi,Jx_1)$ for all $n\ge 1$. As we know, $G(y,Jx)$ is convex and lower semicontinuous with respect to y when x is fixed. So we have

$$G(\bar{x},Jx_1)\le \liminf_{n\to\infty}G(x_n,Jx_1)\le \limsup_{n\to\infty}G(x_n,Jx_1)\le G(\varpi,x_1).$$

As $\bar{x}\in F(\mathscr{T})$, from the definition of $\prod_{F(\mathscr{T})}^f x_1$, we can obtain that $\bar{x}=\varpi=\prod_{F(\mathscr{T})}^f x_1$ and $x_n\to\bar{x}$ as $n\to\infty$. This completes the proof of Theorem 3.1.

Just as in Remark 2.7, we use $t_m\left(m\in\mathbb{N}^+\right)$ instead of t in Definition 2.6 and denote $T(t_m)$ by T_m, $\mathscr{T}:=\{T_m:C\to C\}_{m=1}^{\infty}$ becomes a countable family of total quasi-G-asymptotically nonexpansive mappings. Then we get the following corollary.

Corollary 3.2 *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed and convex subset of* E. *Let* $\mathscr{T}:=\{T_m:C\to C\}_{m=1}^{\infty}$ *be a countable family of closed and totally quasi-G-asymptotically nonexpansive mappings. Let* $f:E\to\mathbb{R}$ *be a convex and lower semicontinuous function with* $C\subset\operatorname{int}(D(f))$ *such that* $f(x)>0$ *for all* $x\in C$ *and* $f(0)=0$. *Assume that* T_m *is uniformly asymptotically regular for all* $m\in\mathbb{N}^+$ *and* $F(\mathscr{T})=\bigcap_{m\in\mathbb{N}^+}F(T_m)\ne\varnothing$. *Let the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_1\in E,\text{chosen arbitrarily; } C_1=C \\ y_{n,m}=J^{-1}\left[\alpha_n Jx_1+(1-\alpha_n)JT_m^n x_n\right], \\ C_{n+1}=\left\{z\in C_n:\sup\limits_{m\in\mathbb{N}^+}G(z,Jy_{n,m})\le\alpha_n G(z,Jx_1)+(1-\alpha_n)G(z,Jx_n)+\xi_n\right\}, \\ x_{n+1}=\prod\limits_{c_{n+1}}^{f}x_1, \end{cases} \tag{22}$$

where, $\xi_n=\mu_n\sup_{p\in F(\mathscr{T})}\tau(G(p,Jx_n))+\delta_n$ and $\{\alpha_n\}\subset(0,1)$. If $\lim_{n\to\infty}\alpha_n=0$ and $\mu_1=\delta_1=0$, then $\{x_n\}$ converges strongly to $\prod_{F(\mathscr{T})}^f x_1$.

In Corollary 3.2, when $f(x)\equiv 0$ for all $x\in C$, $\mathscr{T}=\{T_m:C\to C\}_{m=1}^{\infty}$ be a countable family of closed and totally quasi-ϕ-asymptotically nonexpansive mappings. Then we can get the following theorem.

Corollary 3.3 *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E. Let $\mathscr{T} = \{T_m : C \to C\}_{m=1}^{\infty}$ be a countable family of closed and totally quasi-ϕ-asymptotically nonexpansive mappings. Assume that T_m is uniformly asymptotically regular for all $m \in \mathbb{N}^+$ and $F(\mathscr{T}) = \bigcap_{m\in\mathbb{N}^+} F(T_m) \neq \varnothing$. Let the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_1 \in E, \text{chosen arbitrarily; } C_1 = C, \\ y_{n,m} = J^{-1}\left[\alpha_n Jx_1 + (1-\alpha_n) JT_m^n x_n\right], \\ C_{n+1} = \left\{ z \in C_n : \sup\limits_{m\in\mathbb{N}^+} \phi(z, Jy_{n,m}) \le \alpha_n \phi(z, Jx_1) + (1-\alpha_n)\phi(z, Jx_n) + \xi_n \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \tag{23}$$

where, $\xi_n = \mu_n \sup_{p\in F(\mathscr{T})} \tau(\phi(p, Jx_n)) + \delta_n$ and $\{\alpha_n\} \subset (0,1)$. If $\lim_{n\to\infty}\alpha_n = 0$ and $\delta_1 = 0$, then $\{x_n\}$ converges strongly to $\prod_{F(\mathscr{T})}^{f} x_1$.

Remark 3.4 The results in this paper improve and extend many recent corresponding main results of other authors (see, for example, [3,4,7,8,10,11,15-19]) in the following ways: (a) we introduce a new class of totally quasi-G-asymptotically nonexpansive mappings which contains the classes of the totally quasi-ϕ-asymptotically nonexpansive mappings and many non-expansive mappings; (b) we extend from a countable family of mappings to the totally quasi-G-asymptotically nonexpansive semigroup; (c) we modify the Halpern type hybrid projection algorithm by using the generalized f-projection operator for uniformly total quasi-G-asymptotically nonexpansive semigroup. For example, Corollary 3.2 extends the main result of Seawan *et al.* [11] from the modified Mann type iterative algorithm to modified Halpern iterative by the generalized f-projection method. Corollary 3.3 is the main result of Chang *et al.*[3].

Contributions

All authors contributed equally and significantly in this research work. All authors read and approved the final manuscript.

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