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Optimal Risk-Sensitive Filtering for System Stochastic of Second and Third Degree

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Abstract

The risk-sensitive filtering design problem with respect to the exponential mean-square cost criterion is considered for stochastic Gaussian systems with polynomial of second and third degree drift terms and intensity parameters multiplying diffusion terms in the state and observations equations. The closed-form optimal filtering equations are obtained using quadratic value functions as solutions to the corresponding Focker-Plank-Kolmogorov equation. The performance of the obtained risk-sensitive filtering equations for stochastic polynomial systems of second and third degree is verified in a numerical example against the optimal polynomial filtering equations (and extended Kalman-Bucy for system polynomial of second degree), through comparing the exponential mean-square cost criterion values. The simulation results reveal strong advantages in favor of the designed risk-sensitive equations for some values of the intensity parameters.

Keywords: Optimal Nonlinear Filtering, Risk-Sensitive Filtering, Extended Kalman-Bucy Filtering

1. Introduction

Since the linear optimal filter was obtained by Kalman and Bucy (60's), numerous works are based on it, see for example [1-5], of the variety of all those. The deterministic filter model introduced by Mortensen [6] provides an alternative to stochastic filtering theory. In this model, errors in the state dynamics and the observations are modeled as deterministic "disturbance functions", and an exponential mean-square cost criterion disturbance error is to be minimized. Special conditions are given for the existence, continuity and boundedness of $f\left(X\left(t\right)\right)$ in the state equation, which is considered nonlinear, and the linear function $h\left(X\left(t\right)\right)$ in the observation equation. A concept of stochastic risk-sensitive estimator, introduced more recently by McEneaney [7], regard a dynamic system where $f\left(X\left(t\right)\right)$ is a nonlinear function, linear observations and existence of parameter $\tilde{\varepsilon}$ multiplying the diffusion term in both equations (state and observations). In [8] were obtained the suboptimal risk-sensitive filtering equations for polynomial systems of third degree and applied to the pendulum equations [9], in which the original system was linearized applying Taylor series around the equilibrium point. In [10,11] it is regarded $f\left(X\left(t\right)\right)$ as nonlinear function. This paper presents an application of the equations obtained in [10,11] for singular form of $f\left(X\left(t\right)\right)$ (polynomial of second and third degree).

The goal of this work is to obtain the optimal risk-sensitive filtering equations when the form of $f\left(X\left(t\right)\right)$ is polynomial of second and third degree and parameter $\tilde{\varepsilon}$ multiplying the diffusion term in the state and observations equations. There filtering equations are obtained taking a value function as solution of the nonlinear parabolic partial differential equation and exponential mean-square exponential cost criterion to be minimized.

Undefined parameters in the value function are calculated through ordinary differential equations composed by collecting terms corresponding to each power of the state-dependent polynomial in the nonlinear parabolic PDE equations. This procedure leads to the obtention of the optimal risk-sensitive filtering equations.

The closed-form for risk-sensitive filtering equations is explicitly obtained in this work. Although the difficulty presented by systems of second and third degree, in this work is shown an advantage for risk-sensitive filtering equations versus extended Kalman-Bucy and polynomial filtering equations under certain values of the parameter ε. This performance is shown verified in a numerical example against the mean-square optimal for polynomial filtering equations (and extended Kalman-Bucy for systems of second degree), through comparing the exponential mean-square cost criterion values in finite horizon time. The simulation results reveal strong advantages in favor of the designed risk-sensitive filtering equations for all values of the intensity parameters (in **Table 1**) multiplying diffusion terms in state and observation equations. Tables of the criterion values and graphs of the simulations are included. This exponential mean-square cost criterion function contains the parameter ε which appear in the dynamic system, which leads to a more robust solution. This work is organized as follows: filtering problem statement, optimal risk-sensitive filtering for stochastic system of second degree, optimal risk-sensitive filtering for stochastic system of third degree, application for systems of second degree, application for systems of third degree, conclusions and references.

2. Filtering Problem Statement

Consider the following stochastic model (1), where $X(t)$ denotes the state process, $Y(t)$ denotes a continuous accumulated observations process, $X(t)$ satisfies the diffusion model given by

$$dX(t) = f(X(t))dt + \sqrt{\frac{\varepsilon}{2\gamma^2}}dW(t) \quad (1)$$

where $f(X(t))$ represents the nominal dynamics, and W is a Brownian motion, and the observation process $Y(t)$ satisfies the equation:

$$dY(t) = h(X(t))dt + \sqrt{\frac{\varepsilon}{2\gamma^2}}d\tilde{W}(t), \quad Y(0) = 0, \quad (2)$$

where ε is a parameter and W and $\tilde{W}$ are independent Brownian motions, which are also independent of the initial state $X(0)$. $X(0)$ has probability density $k_\varepsilon \exp(-\varepsilon^{-1}\varphi(X(0)))$ for some constant k_ε.

Let us consider

$$J = \varepsilon \log E\left[\exp\frac{1}{\varepsilon}\int_0^T L(X(t), m(t), t)dt \Big/ Y(t)\right] \quad (3)$$

the exponential mean-square cost criterion to be minimize. In the rest of the paper the assumptions (A1)-(A4) (from [10]) are hold:

- (A1) $f, g, h \in \mathbb{R}^n$ with f_x, h_x bounded.
- (A2) $D_1(|x|^2 - 1) \le \phi(x) \le D_2(|x|^2 + 1)$. Here f_x is the matrix of partial derivatives of f with h_x defined similarly. $\phi(x)$ is a continuous, real-valued function satisfying (A2) for some positive D_1, D_2.
- (A3) $f, h \in \mathbb{R}^n$ with f, h bounded and f_{xx}, h_{xx} bounded and globally Hölder continuous. (A function u is globally Hölder continuous if there exists $\alpha \in (0,1], K < \infty$ such that $|u(x) - u(y)| \le K|x - y|^\alpha$ for all x, y).
- (A4) Given $R < \infty$, there exists $K_R < \infty$ such that $|\phi(x) - \phi(y)| \le K_R|x - y|$ for all $|x|, |y| < \infty$.

Let $q(T, X(t))$ denotes the unnormalized conditional density of $X(T)$, given accumulated observations $Y(t)$ for $0 \le t \le T$. It satisfies the Zakai stochastic PDE, in a sense made precise, for instance in [12]. It is assumed that

$$\begin{aligned} q(0, X(t)) &= \exp(-\varepsilon^{-1}\varphi(X(t))), \\ q(T, X(t)) &= p(T, X(t))\exp\left[-\varepsilon^{-1}Y(T)\cdot h(x(t))\right], \end{aligned} \quad (4)$$

where $p(T, X(t))$ is called pathwise unnormalized filter density. p satisfies the following linear second-order parabolic PDE with coefficients depending on $Y(t)$:

$$\frac{\partial p}{\partial T} = \left(\breve{L}(T)\right)^* p + \frac{K}{\varepsilon}p. \quad (5)$$

where, for every $g \in \mathbb{R}^n$, let

$$\begin{aligned} L_g &= \frac{\varepsilon}{2}tr(ag_{XX}) + f\cdot g_X, \\ \breve{L}(T)g &= Lg - a(Y(T)\cdot h)_X \cdot g_X \\ K(T, X(t)) &= \frac{1}{2}a(X(t))(Y(T)\cdot h)_X \cdot (Y(T)\cdot h)_X \\ &\quad - L(Y(T)\cdot h) - \frac{1}{2}|h|^2 \end{aligned} \quad (6)$$

L denotes the differential generator of the Markov diffusion $X(t)$ in (1). By assumptions (A1) and (A3) in [10], K is bounded and continuous. $(\breve{L}(T))^*$ is the formal adjoint of $\breve{L}(T)$. Since $Y(0) = 0$, $p(0, X(t)) = q(0, X(t))$. The initial condition for (5) is (4). For some given $Y \in C_0(0,T]$, (where C_0 denote the space of continuous $Y(t)$ such that $Y(0) = 0$, with the sup norm $\|\ \|$). The pathwise filter density p is the unique "strong" solution to (5) and (4) in a sense made precise in [12]. Further, p is a classical solution to (5) and (4) with p continuous on $[0,T_1]\times\mathbb{R}^n$ and partial derivatives p_T, p_{X_i}, $p_{X_iX_j}$, $i, j = 1, \cdots, n$ continuous for $0 < T \le T_1$ [13,14].

Moreover, $p(T, X(t); Y) > 0$. We rewrite (5) as follows:

$$\frac{\partial p}{\partial T} = \frac{1}{2}tr(a(X(t))p_{XX}) + A\cdot p_X + \frac{B}{\varepsilon}p \quad (7)$$

where

$$\begin{aligned} A &= -f(X(t)) + a(X(t))(Y(t)\cdot h(x(t)))_X \\ &\quad + \varepsilon div(a_X(t)) \end{aligned} \quad (8)$$

$$B = (T, X(t))$$

$$
\begin{aligned}
&= \frac{\varepsilon^2}{2} tr\left(a_{XX}\left(X(t)\right)\right) \\
&\quad - \varepsilon div\left[f(X(t)) - a\left(X(t)\right)\left(Y(T)\cdot h\left(X(t)\right)\right)_X \right] \\
&\quad + K\left(T, X(t)\right),
\end{aligned}
$$

$$\left(div\left(a_X(t)\right)\right)_j = \sum_{i,j=1}^{n} \left(a_{ij}\right)_{X_i}, \ j = 1,\cdots,n$$

$$tr\left(a_{XX}\right) = \sum_{i,j=1}^{n} \left(a_{ij}\right)_{X_i X_j}$$

These assumptions imply uniform bounds for A and B, depending on the sup norm $\|Y\|$ on $\left[0, T_1\right]$, but not on ε. Taking log transform: $Z\left(T, X(t)\right) = \varepsilon \log p\left(T, X(t)\right)$, which satisfies the nonlinear parabolic PDE:

$$\frac{\partial Z}{\partial T} = \frac{\varepsilon}{2} tr\left(Z_{XX}\right) + A\cdot Z_X + \frac{1}{2} Z_X \cdot Z_X + B, \qquad (9)$$

with initial condition $Z_X\left(0, X(t)\right) = -\varphi\left(X(t)\right)$. The optimal risk-sensitive filtering problem consists in found the estimate $C(t)$, of the state $X(t)$ through verification that

$$
\begin{aligned}
Z\left(T, X(t)\right) &= \frac{1}{2}\left(X(t) - C(t)\right)^T Q(t)\left(X(t) - C(t)\right) \\
&\quad + \rho(T) - Y(T)\cdot h\left(X(t)\right),
\end{aligned} \qquad (10)
$$

is a viscosity solution of (9).

Where $X(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, $Y(t)$, $v(t) \in \mathbb{R}^p$, f, $h \in \mathbb{R}^n$ with f_X, h_x bounded is assumed throughout. Here h_x is the matrix of partial derivatives of h and the same form for Z_X.

3. Optimal Risk-Sensitive Filtering Problem

3.1. Optimal Risk-Sensitive Filtering for Stochastic System of Second Degree

Taking $f\left(X(t)\right) = A(t) + A_1(t)X(t) + A_2(t)\cdot X^T(t)X(t)$, $h\left(X(t)\right) = E(t) + E_1(t)X(t)$ with $A(t) \in \mathbb{R}^n$, $A_1(t) \in M_{n\times n}$, $A_2(t) \in T_{n\times n\times n}$, $E(t) \in E_1(t) \in \mathbb{R}^p$, $M_{n\times p}$ where $M_{i\times j}$ denotes the field of matrixes of dimension $i\times j$ and $T_{i\times j\times k}$ denotes the field of tensors of dimension $i\times j\times k$ n. The following stochastic equations system is obtained:

$$
\begin{aligned}
dX(t) &= A(t) + A_1(t)X(t) + A_2(t)X^T(t)X(t) \\
&\quad + \sqrt{\tilde{\varepsilon}} dB(t), \qquad (11) \\
dY(t) &= E(t) + E_1(t)X(t) + \sqrt{\tilde{\varepsilon}} d\tilde{B}(t),
\end{aligned}
$$

where $\tilde{\varepsilon} = \varepsilon / \left(2\gamma^2\right)$. The optimal filtering problem consists in to obtain the estimate of the state $X(t)$ given the observations equations, which minimizes the exponential mean-square cost criterion, taking $Z\left(T, X(t)\right)$ (10) as solution of the nonlinear parabolic partial differential Equation (9).

Theorem 1. The solution to the filtering problem, for the system (11) with criterion (3) takes the form:

$$
\begin{aligned}
\dot{C}(t) &= A(t) + A_1(t)C(t) - \dot{Q}(t)Q^{-1}(t)C(t) \\
&\quad + Q^{-1}(t)\cdot E_1(t)\left(dy - E(t)\right) + Q(t)C(t) \\
&\quad + 2\varepsilon A_2(t)Q^{-1}(t), \qquad (12) \\
\dot{Q}(t) &= -A_1(t)Q(t) - Q(t)A_1^T(t) + Q^T(t)Q(t) \\
&\quad - E_1^T(t)E_1(t).
\end{aligned}
$$

where $C(t)$ is the state estimate vector with initial conditions with initial condition $C(0) = C_0$, and $Q(t)$ is a symmetric matrix negative defined, where the initial condition $Q(0) = q_0$ is derived from initial conditions for Z. If $\varphi\left(X(t)\right) = X^T(t)KX(t)$, $Q(0) = -K$.

Proof: The value function is proposed

$$
\begin{aligned}
Z\left(T, X(t)\right) &= \frac{1}{2}\left(X(t) - C(t)\right)^T Q(t)\left(X(t) - C(t)\right) \\
&\quad + \rho(T) - Y(T)\cdot h\left(X(t)\right),
\end{aligned} \qquad (13)
$$

$Z_X\left(0, X(t)\right) = -\varphi\left(X(t)\right)$, $\left(C(t),\ Q(t),\ \rho(t)\right)$ are functions defined on $[0, T]$, $C(t) \in \mathbb{R}^n$, $Q(t)$ is a symmetric matrix of dimension $n\times n$ and $\rho(t)$ is a scalar function) as a viscosity solution of the nonlinear parabolic PDE (9). Z_X, Z_{XX} are the partial derivatives of Z respect to $X(t)$ and ∇Z is the gradient of Z. Then the partial derivatives of Z are given by:

$$
\begin{aligned}
Z_T &= \frac{1}{2}\left(X(t) - C(t)\right)^T \dot{Q}\left(X(t) - C(t)\right) \\
&\quad - \left(X(t) - C(t)\right)^T Q(t)\dot{C}(t) + \dot{\rho}(t) \\
&\quad - \dot{Y}(t)\left(E(t) - E_1(t)X(t)\right) \\
Z_X &= \frac{1}{2} Q(t)\left(X(t) - C(t)\right) \\
&\quad + \frac{1}{2}\left(X(t) - C(t)\right)^T Q(t) - Y(t)E_1(t), \\
Z_{XX} &= Q(t).
\end{aligned} \qquad (14)
$$

Let us consider:

$$
\begin{aligned}
A &= -A(t) - A_1(t)X(t) - A_2(t)X^T(t)X(t) \\
&\quad + Y(T)E_1(t)
\end{aligned} \qquad (15)
$$

$$
\begin{aligned}
B &= -2\varepsilon A_2(t)X(t) - \varepsilon A_1(t) + \frac{1}{2}\left(Y(T)E_1(t)\right)^2 \\
&\quad \times\left(-A(t) - A_1(t)X(t) - A_2(t)X^T(t)X(t)\right. \\
&\quad \left. + Y(T)E_1(t)\right) - \frac{1}{2}\left|E(t) + E_1(t)X(t)\right|^2
\end{aligned}
$$

Substituting (14) and the expressions for *A*, *B* in (9); we obtain:

$$
\begin{aligned}
0 = &-\frac{1}{2}\left(X(t)-C(t)\right)^T \dot{Q}(t)\left(X(t)-C(t)\right)^T Q(t)\dot{C}(t) \\
&-\dot{\rho}(t)+\dot{Y}(t)\left(E(t)+E_1(t)X(t)\right)\frac{\varepsilon}{2}tr\left(Q(t)\right) \\
&+\left(-A(t)-A_1(t)X(t)-A_2(t)X^T(t)X(t)+Y(t)\right. \\
&\times E_1(t)\Big)\left(\frac{1}{2}Q(t)\cdot\left(X(t)-C(t)\right)+\frac{1}{2}\left(X(t)-C(t)\right)\right)^T \\
&\times Q(t)-Y(t)E_1(t)+\frac{1}{2}\left(\frac{1}{2}Q(t)\left(X(t)-C(t)\right)\right. \\
&\left.+\frac{1}{2}\left(X(t)-C(t)\right)^T Q(t)-Y(t)E_1(t)\right)-\varepsilon A_1(t) \\
&+\frac{1}{2}\left(Y(t)E_1(t)\right)^2\left(-A(t)-A_1(t)X(t)-A_2(t)X^T(t)\right. \\
&\left.\times X(t)+Y(t)E_1(t)\right)-\frac{1}{2}\left|E(t)+E_1(t)X(t)\right|^2.
\end{aligned} \tag{16}
$$

Collecting the $X^T(t)X(t), X^T(t)X(t)X^T(t)$ and $X^T(t)X(t)X^T(t)X(t)$ terms, and replacing $X(t)$ by $C(t)$; we obtain the matrix equation for $\dot{Q}(t)$. Collecting the $X(t)$ terms, the vectorial equations for $\dot{C}(t)$ are obtained (12).

3.2. Optimal Risk-Sensitive Filtering for Stochastic System of Third Degree

Taking $f\left(X(t)\right)=A(t)+A_1(t)X(t)+A_2(t)X^T(t)\cdot X(t)+A_3(t)X^T(t)X(t)X^T(t),\ h\left(X(t)\right)=E(t)+E_1(t)X(t)$ with $A(t)\in\mathbb{R}^n,\ A_1(t)\in M_{n\times n},\ A_2(t)\in T_{n\times n\times n},\ A_3(t)\in T_{n\times n\times n\times n},\ E(t)\in\mathbb{R}^p,\ E_1(t)\in M_{n\times p}$ where $M_{i\times j}$ denotes the field of matrixes of dimension $i\times j$, $T_{i\times j\times k}$ denotes the field of tensors of dimension $i\times j\times k$ and $T_{i\times j\times k\times l}$ denotes the field of tensors of dimension $i\times j\times k\times l$. The following stochastic equations system is obtained:

$$
\begin{aligned}
dX(t) &= A(t)+A_1(t)X(t)+A_2(t)X^T(t)X(t) \\
&\quad +A_3(t)X^T(t)X(t)X^T(t)+\sqrt{\tilde{\varepsilon}}dB(t), \qquad (17)\\
dY(t) &= E(t)+E_1(t)X(t)+\sqrt{\tilde{\varepsilon}}d\tilde{B}(t),\ X(0)=X_0,
\end{aligned}
$$

where $\tilde{\varepsilon}=\varepsilon/\left(2\gamma^2\right)$.

Theorem 2. The solution to the filtering problem, for the system (17) with criterion (3) takes the form:

$$
\begin{aligned}
\dot{C}(t) &= A(t)-A_1(t)C(t)-\dot{Q}(t)Q^{-1}(t)C(t) \\
&\quad -Q^{-1}(t)\cdot E_1(t)\left(dy-\frac{1}{2}E(t)\right)+Q(t)C(t),
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\dot{Q}(t) &= A_1(t)Q(t)+Q(t)A_1^T(t)-A_2(t)Q(t)C(t) \\
&\quad -\left(A_2(t)Q(t)C(t)\right)^T+A_2(t)Q(t)C(t) \\
&\quad -A_3(t)C(t)C^T(t)Q(t) \\
&\quad -\left(A_3(t)C(t)C^T(t)Q(t)\right)^T \\
&\quad +A_3(t)Q(t)C(t)C^T(t) \\
&\quad +\left(A_3(t)Q(t)C(t)C^T(t)\right)^T \\
&\quad +Q^T(t)Q(t)-\varepsilon div\left[f(C)-Y(t)E_1(t)\right] \\
&\quad -E_1^T(t)E_1(t)
\end{aligned}
$$

where $C(t)$ is the state estimate vector with initial conditions with initial condition $C(0)=C_0$, and $Q(t)$ is a symmetric matrix negative defined, where the initial condition $Q(0)=q_0$ is derived from initial conditions for *Z*. If $\varphi\left(X(t)\right)=X^T(t)KX(t),\ Q(0)=-K$.

Proof: In similar form to Theorem 1.

4. Applications

4.1. Application for Systems of Second-Degree. Optimal Risk-Sensitive Filtering Equations

Consider the following dynamical stochastic system associated to a continuous stirred tank reactor in which is a chemical reaction occurs. This reaction is in liquid phase and has isothermal character between multicomponents [15].

$$
\begin{aligned}
\dot{X}_1(t) &= -\left(1+D_{a1}\right)X_1+u(t)+\sqrt{\frac{\varepsilon}{2\gamma^2}}dW_1(t), \\
\dot{X}_2(t) &= D_{a1}X_1-X_2-D_{a2}X_2^2+\sqrt{\frac{\varepsilon}{2\gamma^2}}dW_2(t).
\end{aligned} \tag{19}
$$

where $X_1(t)$ represents the unnormalized concentration P/C_{Po} of a certain specie *P* of the reactor, $X_2(t)$ represents unnormalized concentration C_Q/C_{Po} of a certain specie *Q*. The control variable *u* is defined as the relation between the alimentation molar rate by volumetric unit of *P*, designated by N_{PF} and the nominal concentration C_{Po}, this is $u=N_{PF}/FC_{Po}$, where *F* is the volumetric flow of alimentation on m^3s^{-1}. $D_{a1}=k_1V/F$, $D_{a2}=k_2VC_{Po}/F$ where *V* is the volume of reactor in m^3, k_1 and k_2 are constants of first degree given in s^{-1}. It can take that D_{a1} and D_{a2} are considering by $D_{a1}=1$ and $D_{a2}=1$. *Q* is highly sour while *P* is neuter. Then, the following dynamical stochastic system is obtained:

$$
\begin{aligned}
\dot{X}_1(t) &= -2X_1+u(t)+\sqrt{\frac{\varepsilon}{2\gamma^2}}dW_1(t), \\
\dot{X}_2(t) &= X_1-X_2-X_2^2+\sqrt{\frac{\varepsilon}{2\gamma^2}}dW_2(t).
\end{aligned} \tag{20}
$$

Applying the equations (12) to the system (20), the equations of the optimal risk-sensitive filtering are obtained:

$$\dot{Q}_{11} = 4Q_{11} + Q_{11}^2 Q_{12}^2 - 1,$$
$$\dot{Q}_{12} = 3Q_{12} + Q_{11} + Q_{11}Q_{12} - Q_{12}Q_{22},$$
$$\dot{Q}_{22} = 2Q_{22} - 2Q_{12} + Q_{12}^2 + Q_{22}^2 - 1,$$
$$\dot{C}_1 = \frac{-\dot{Q}_{11}}{Q_{11}Q_{22} - Q_{12}^2}\Big[\dot{Q}_{11}\left(Q_{22}C_1 - Q_{12}C_2\right) + \dot{Q}_{12}\left(Q_{11}C_2 - Q_{12}C_1\right) - \dot{Y}_1 Q_{22} + \dot{Y}_2 Q_{12}\Big] - 2C_1 + Q_{11}C_1 + Q_{12}C_2, \tag{21}$$
$$\dot{C}_2 = \frac{-\dot{Q}_{12}}{Q_{11}Q_{22} - Q_{12}^2}\Big[\dot{Q}_{12}\left(Q_{22}C_1 - Q_{12}C_2\right) + \dot{Q}_{22}\left(Q_{11}C_2 - Q_{12}C_1\right) - \dot{Y}_1 Q_{12} + \dot{Y}_2 Q_{11} - 2\varepsilon Q_{11}\Big] + C_1 - C_2 + Q_{12}C_1 + Q_{22}C_2.$$

The initial conditions for the risk-sensitive filtering equations are: $X_1(0) = 20, X_2(0) = 10, Y_1(0) = 2, Y_2(0) = 1,$ $C_1(0) = 1,$ $C_2(0) = 5,$ $Q_{11}(0) = 6,$ $Q_{12}(0) = 0.0001,$ $Q_{22}(0) = -7$, the final time is $T = 2s$. The system formed by the equations (20) and (21), is simulated using Simulink in MatLab7. The performance of the designed equations is compared versus the equations of the polynomial filtering [1] and the equations of the extended Kalman-Bucy filtering [16], applied to the system (20), that is optimal with respect to the conventional exponential mean-square cost criterion.

4.1.1. Polynomial Filtering Equations

The corresponding equations for the polynomial filtering [1] are given by:

$$\dot{P}_{11} = 4P_{11} + \frac{\varepsilon}{2\gamma^2} - \frac{2\gamma^2}{\varepsilon}\left(P_{12}^2 + P_{11}^2\right),$$
$$\dot{P}_{12} = P_{11} - 3P_{12} - 2m_2P_{12} - \frac{2\gamma^2}{\varepsilon}\left(P_{11}P_{12} + P_{12}P_{22}\right),$$
$$\dot{P}_{22} = 2P_{22} + 2P_{12} - 4m_2P_{22} + \frac{\varepsilon}{2\gamma^2} - \frac{2\gamma^2}{\varepsilon}\left(P_{12}^2 + P_{22}^2\right), \tag{22}$$
$$\dot{m}_1 = -2m_1 + \frac{2\gamma^2}{\varepsilon}\Big[P_{11}\left(\dot{Y}_1 - m_1\right) + P_{12}\left(\dot{Y}_2 - m_2\right)\Big],$$
$$\dot{m}_2 = m_1 - m_1 - m_2^2 - P_{22} + \frac{2\gamma^2}{\varepsilon}\Big[P_{12}\left(\dot{Y}_1 - m_1\right) + P_{22}\left(\dot{Y}_2 - m_2\right)\Big].$$

where the initial conditions are $X_1(0) = 20,$ $X_2(0) = 10,$ $Y_1(0) = 2,$ $Y_2(0) = 1,$ $m_1(0) = 1,$ $P_{11}(0) = 100,$ $P_{12}(0) = -1,$ $P_{22}(0) = 1\times10^7$.

4.1.2. Extended Kalman-Bucy Filtering Equations

The equations of the extended Kalman-Bucy [13] filtering are given by:

$$\dot{P}_{11} = 4P + \frac{\varepsilon}{2\gamma^2} - \frac{2\gamma^2}{\varepsilon}\left(P_{12}^2 + P_{11}^2\right),$$
$$\dot{P}_{12} = P_{11} - 3P_{12} - \frac{2\gamma^2}{\varepsilon}\left(P_{11}P_{12} + P_{12}P_{22}\right),$$
$$\dot{P}_{22} = 2P_{12} - 2P_{22} + \frac{\varepsilon}{2\gamma^2} - \frac{2\gamma^2}{\varepsilon}\left(P_{12}^2 + P_{22}^2\right), \tag{23}$$
$$\dot{m}_1 = -2m_1 + \frac{2\gamma^2}{\varepsilon}\Big[P_{11}\left(\dot{Y}_1 - m_1\right) + P_{12}\left(\dot{Y}_2 - m_2\right)\Big],$$
$$\dot{m}_2 = m_1 - m_1 + \frac{2\gamma^2}{\varepsilon}\Big[P_{12}\left(\dot{Y}_1 - m_1\right) + P_{22}\left(\dot{Y}_2 - m_2\right)\Big].$$

1) Consider the stochastic dynamical system associated to a continuous stirred tank reactor and the following initial conditions for the state and observations equations: $X_1(0) = 20,$ $X_2(0) = 10,$ $Y_1(0) = 2,$ $Y_2(0) = 1,$ the final time is $T = 2s$. The initial conditions for the filtering equations in which case are given by:

a) For risk-sensitive filtering equations:
$C_1(0) = 1,$ $C_2(0) = 5,$ $Q_{11}(0) = 6,$ $Q_{12}(0) = 0.0001,$ $Q_{22}(0) = -7.$

b) For polynomial filtering equations:
$m_1(0) = 1,$ $m_2(0) = 5,$ $P_{11}(0) = 100,$ $P_{12}(0) = -1,$ $P_{22}(0) = 1\times10^7.$

c) For Extended Kalman-Bucy filtering equations:
$m_1(0) = 1,$ $m_2(0) = 5,$ $P_{11}(0) = 5,$ $P_{12}(0) = 3,$ $P_{22}(0) = 5.$

Table 1 presents comparison between the exponential mean square cost criterion J for the three types of filtering equations; you can see that the J_{R-S} values are the smallest for all values of the intensity parameter ε.

2) Consider the stochastic dynamical system associated to a continuous stirred tank reactor and the following initial conditions for the state and observations equations: $X_1(0) = 50,$ $X_2(0) = 1,$ $Y_1(0) = 2,$ $Y_2(0) = 1,$ the final time is $T = 2s$. The initial conditions for the filtering equations in which case are given by:

a) For risk-sensitive filtering equations:
$C_1(0) = 1,$ $C_2(0) = 5,$ $Q_{11}(0) = -7,$ $Q_{12}(0) = 0.0001,$ $Q_{22}(0) = -7.5.$

b) For polynomial filtering equations:
$m_1(0) = 1,$ $m_2(0) = 5,$ $P_{11}(0) = 85,$ $P_{12}(0) = -10,$ $P_{22}(0) = 2\times10^7.$

c) For Extended Kalman-Bucy filtering equations:
$m_1(0) = 1,$ $m_2(0) = 5,$ $P_{11}(0) = 2,$ $P_{12}(0) = 5,$ $P_{22}(0) = 10.$

Table 2 presents comparison between the exponential mean square cost criterion J for the three types of filtering equations; you can see that the J_{R-S} values are the smallest for all values of the intensity parameter ε.

3) Consider the stochastic dynamical system associated to a continuous stirred tank reactor and the following initial conditions for the state and observations equations: $X_1(0)=0.05,\ X_2(0)=50,\ Y_1(0)=2,\ Y_2(0)=1,$ the final time is $T=2s$. The initial conditions for the filtering equations in which case are given by:

a) For risk-sensitive filtering equations:

$C_1(0)=1,\ C_2(0)=5,\ Q_{11}(0)=-6,\ Q_{12}(0)=0.0001,$

$Q_{22}(0)=-7.$

b) For polynomial filtering equations:

$m_1(0)=1,\ m_2(0)=5,\ P_{11}(0)=100,\ P_{12}(0)=-5,$

$P_{22}(0)=1\times 10^7.$

c) For Extended Kalman-Bucy filtering equations:

$m_1(0)=1,\ m_2(0)=5,\ P_{11}(0)=1.85,\ P_{12}(0)=3,$

$P_{22}(0)=5.$

Table 3 presents comparison between the exponential mean square cost criterion J for the three types of filtering equations; you can see that the J_{R-S} values are the smallest for all values of the intensity parameter ε.

With these tables, showed that the filter risk-sensitive is the best, because the values obtained are lower.

The **Figures 1**, **2** and **3** show the $Error_1$ and $Error_2$ which are defined as $Error_1 = X_1(t) - C_1(t)$ (in same form for $Error_2$); and the exponential mean-square cost criterion values in $T=2s$.

Table 1. Comparison of exponential mean-square cost criterion values *J*(3) in *T* = 2*s* for risk-sensitive, polynomial and extended Kalman-Bucy filtering equations.

ε	J_{R-S}	J_{Pol}	J_{K-B}
0.1	53.4293	69.2292 ($t=0.17$s)	69.0816($t=0.14$s)
1	53.5165	145.7323	277.3136
10	53.7994	157.2172	235.5110
100	54.7621	858.7622	189.6937
1000	58.5054	58230	185.7343

Table 2. Comparison of exponential mean-square cost criterion values *J*(3) in *T* = 2*s* for risk-sensitive, polynomial and extended Kalman-Bucy filtering equations.

ε	J_{R-S}	J_{Pol}	J_{K-B}
1	505.8493	705.1152 ($t=1.64$s)	686.3813 ($t=0.28$s)
10	513.3591	712.2522	527.0787
100	537.8120	1430.4728	587.0328
1000	622.1946	59067	641.1202
10000	960.423	5597700	673.6555

Table 3. Comparison of exponential mean-square cost criterion values *J*(3) in *T* = 2*s* for risk-sensitive, polynomial and extended Kalman-Bucy filtering equations.

ε	J_{R-S}	J_{Pol}	J_{K-B}
0.1	42.0377	55.6339	70.3916 ($t=0.36$s)
1	41.9340	56.1153	144.2845
10	41.6143	70.7942	317.9369
100	40.6851	763.7829	678.2942
1000	38.5611	57957	812.9141
10000	36.1603	55918000	859.8006

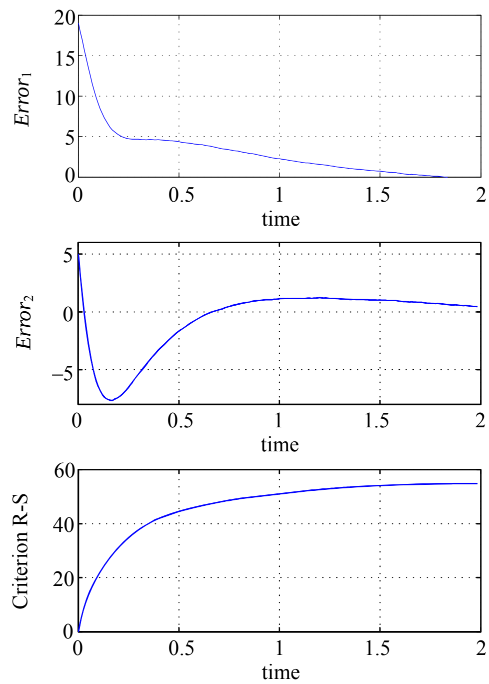

Figure 1. Graphs of the $Error_1$, $Error_2$, and exponential mean square cost criterion corresponding to the risk-sensitive optimal filtering equations for a continuous stirred tank reactor for $\varepsilon=10$, $X_1(0)=20$, $X_2(0)=10$, $Y_1(0)=2$, $Y_2(0)=1$.

4.2. Application for Polynomial System of Third Degree

4.2.1. Optimal Risk-Sensitive Filtering Equations

The risk-sensitive control equations for third degree polynomial systems will be applied to the problem of orientation of a monoaxial satellite [15]. The description is as follows: a satellite rotates around a fixed axis without gravity. The rotation torques is produced by a system of mini-engines through a controlled explosion of gases in the opposite direction. The state equations for this model are given by:

$$\dot{X}_1(t) = 0.5\left(1+X_1^2(t)\right)X_2(t) + \sqrt{\frac{\varepsilon}{2\gamma^2}}dW_1(t)$$

$$\dot{X}_2(t)=\sqrt{\frac{\varepsilon}{2\gamma^2}}dW_2(t),\ \ \dot{Y}(t)=X_1(t)+\sqrt{\frac{\varepsilon}{2\gamma^2}}dW(t).\quad(24)$$

where $X_1(t)$ represents the orientation angle of the satellite, measured with respect of a secondary axis which does not coincide with the principal one. $X_2(t)$ represents the angular velocity with respect to the principal axis. Applying the system of equations (18) to the system (24), the following optimal risk-sensitive filtering equations are obtained:

$$\begin{aligned}
\dot{Q}_{11}&=Q_{12}-C_1C_2Q_{11}-C_2^2Q_{12}+C_2Q_{22}+Q_{11}\\
&\quad+Q_{22}-\varepsilon C_1C_2,\\
\dot{Q}_{12}&=0.5Q_{22}+Q_{11}Q_{12}+Q_{12}Q_{22},\\
\dot{Q}_{22}&=Q_{12}^2+Q_{22}^2-\varepsilon C_1C_2,\\
\dot{C}_1&=\frac{-\dot{Q}_{22}}{Q_{11}Q_{22}-Q_{12}^2}\left(C_1\dot{Q}_{11}+C_2\dot{Q}_{12}+\dot{Y}\right)\\
&\quad+\frac{-\dot{Q}_{12}}{Q_{11}Q_{22}-Q_{12}^2}\left(C_1\dot{Q}_{12}+C_2\dot{Q}_{22}\right)-\frac{1}{2}C_2\\
&\quad+C_1Q_{11}+C_2Q_{12},\\
\dot{C}_2&=\frac{-\dot{Q}_{12}}{Q_{11}Q_{22}-Q_{12}^2}\left(C_1\dot{Q}_{11}+C_2\dot{Q}_{12}+\dot{Y}\right)\\
&\quad+\frac{-\dot{Q}_{11}}{Q_{11}Q_{22}-Q_{12}^2}\left(C_1\dot{Q}_{12}+C_2\dot{Q}_{22}\right)+C_1Q_{12}\\
&\quad+C_2Q_{22}.
\end{aligned}\quad(25)$$

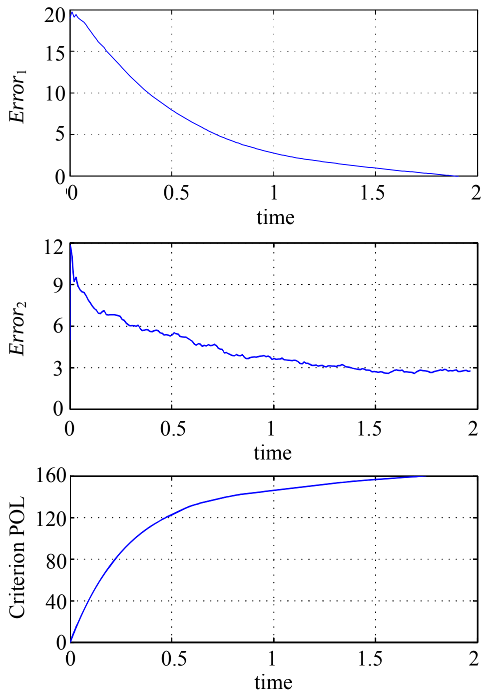

Figure 2. Graphs of the $Error_1$, $Error_2$, and exponential mean square cost criterion corresponding to the polynomial filtering equations for a continuous stirred tank reactor for $\varepsilon=10$, $X_1(0)=20$, $X_2(0)=10$, $Y_1(0)=2$, $Y_2(0)=1$.

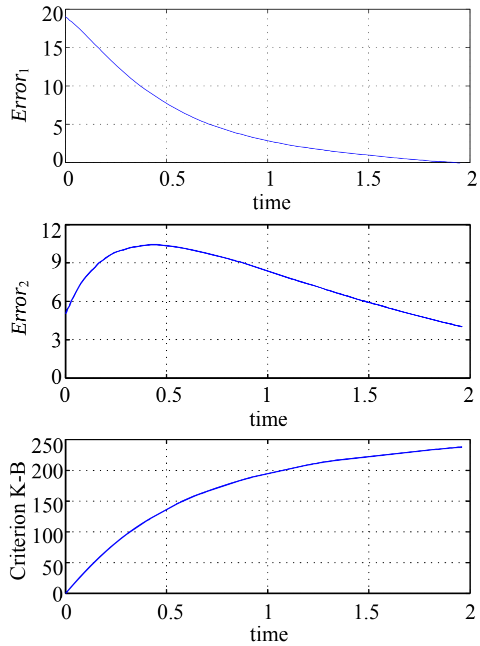

Figure 3. Graphs of the $Error_1$, $Error_2$, and exponential mean square cost criterion corresponding to the extended Kalman-Bucy filtering equations for a continuous stirred tank reactor for $\varepsilon=10$, $X_1(0)=20$, $X_2(0)=10$, $Y_1(0)=2$, $Y_2(0)=1$.

The initial conditions are: $X_1(0)=0.115$, $X_2(0)=0.073$, $Y_1(0)=0$, $C_1(0)=0.92$, and $C_2(0)=0.5$, $Q_{11}(0)=-4500$, $Q_{12}(0)=100$, $Q_{22}(0)=-8500$, $T=1.8s$.

4.2.2. Polynomial Filtering Equations

The corresponding equations for the polynomial filter [1] are given by:

$$\begin{aligned}
\dot{P}_{11}&=P_{12}+3P_{12}P_{11}+3P_{12}P_{22}+3P_{11}m_1m_2\\
&\quad+3P_{12}m_2^2+\frac{\varepsilon}{2\gamma^2}-\frac{2\gamma^2P_{11}^2}{\varepsilon},\\
\dot{P}_{12}&=0.5P_{22}-\frac{2\gamma^2P_{11}P_{12}}{\varepsilon},\ \dot{P}_{22}=\frac{\varepsilon}{2\gamma^2}-\frac{2\gamma^2P_{12}^2}{\varepsilon},\\
\dot{m}_1&=0.5m_2+1.5m_2P_{11}+0.5m_1^2m_2\\
&\quad+\frac{2\gamma^2P_{11}}{\varepsilon}\left(\dot{Y}_1-m_1\right),\\
\dot{m}_2&=\frac{2\gamma^2P_{12}}{\varepsilon}\left(\dot{Y}_1-m_1\right).
\end{aligned}\quad(26)$$

1) Consider the stochastic dynamical system associated to a problem of orientation of a monoaxial satellite and the following initial conditions for the state and observations equations: $X_1(0)=0.09$, $X_2(0)=0.65$, $Y_1(0)=2$, $Y_2(0)=1$, the final time is $T=1s$. The initial conditions for the filtering equations in which case are given by:

a) For risk-sensitive filtering equations:
$C_1(0)=0.92,\ C_2(0)=0.5,\ Q_{11}(0)=-400,\ Q_{12}(0)=100,$
$Q_{22}(0)=-850.$

b) For polynomial filtering equations:
$m_1(0)=0.92,\ m_2(0)=0.5,\ P_{11}(0)=10,\ P_{12}(0)=20,$
$P_{22}(0)=5.$

Table 4 presents comparison between the exponential mean square cost criterion *J* for the two types of filtering equations; it can be saw, that the J_{R-S} values are the smallest for all values of the intensity parameter ε.

2) Consider the stochastic dynamical system associated to a problem of orientation of a monoaxial satellite and the following initial conditions for the state and observations equations: $X_1(0)=0.115,\ X_2(0)=0.073,$ $Y_1(0)=2,\ Y_2(0)=1$, the final time is $T=1\text{s}$. The initial conditions for the filtering equations in which case are given by:

a) For risk-sensitive filtering equations:
$C_1(0)=0.92,\ C_2(0)=0.5,\ Q_{11}(0)=-4500,$
$Q_{12}(0)=100,\ Q_{22}(0)=-8500.$

b) For polynomial filtering equations:
$m_1(0)=0.92,\ m_2(0)=0.5,\ P_{11}(0)=1500,$
$P_{12}(0)=-879.21,\ P_{22}(0)=1000.$

Table 5 presents comparison between the exponential mean square cost criterion *J* for the two types of filtering equations; it can be saw, that the J_{R-S} values are the smallest for all values of the intensity parameter ε.

The system Equations (24), (25) and (26) is simulated using Simulink in MatLab7. The performance of the designed equations is compared versus the equations of the polynomial filter [1], with respect to the exponential mean-square exponential criterion *J*.

The **Figures 4** and **5** show the $Error_1$ and $Error_2$ which are defined as $Error_1 = X_1(t)-C_1(t)$ (in same form for $Error_2$); and the exponential mean- square cost criterion values.

Table 4. Comparison of mean-square exponential criterion *J*(3) for r-s filtering equations and polynomial filtering equations.

ε	J_{R-S}	J_{Pol}
0.01	0.3239	2.0321
0.1	0.3232	1.1319
1	0.3198	0.6655
10	0.3063	0.3319
100	0.2800	26.8974

Table 5. Comparison of mean-square exponential criterion *J*(3) for r-s filtering equations and polynomial filtering equations.

ε	J_{R-S}	J_{Pol}
0.01	0.3842	*0.5895*
0.1	0.3835	0.4287
1	0.3691	0.4013
10	0.2841	0.3054
100	0.1454	0.2454

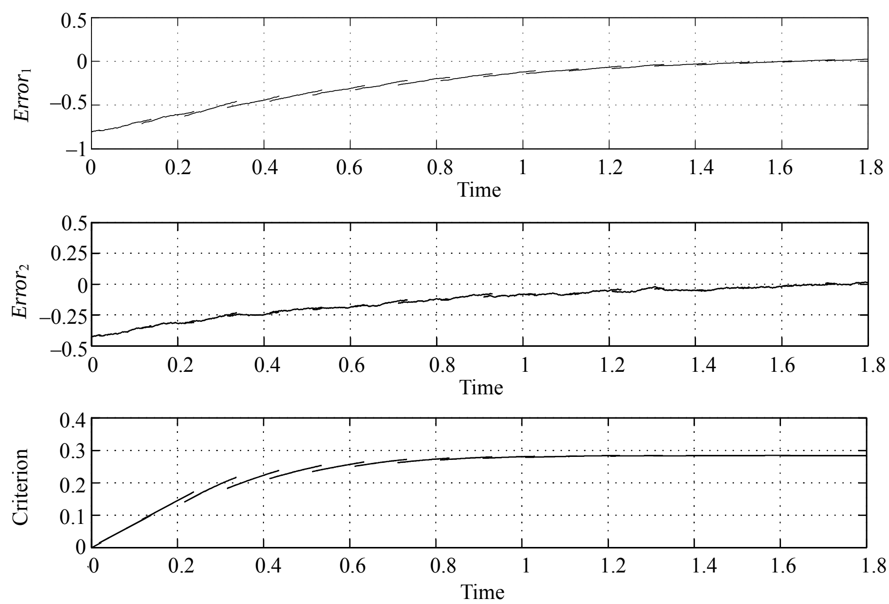

Figure 4. Graphs of the $Error_1$, $Error_2$, and exponential mean square cost criterion corresponding to the risk-sensitive optimal filtering equations for satellite monoaxial for $\varepsilon=10,\ X_1(0)=0.115,\ X_2(0)=0.073,\ Y_1(0)=2,\ Y_2(0)=1$.

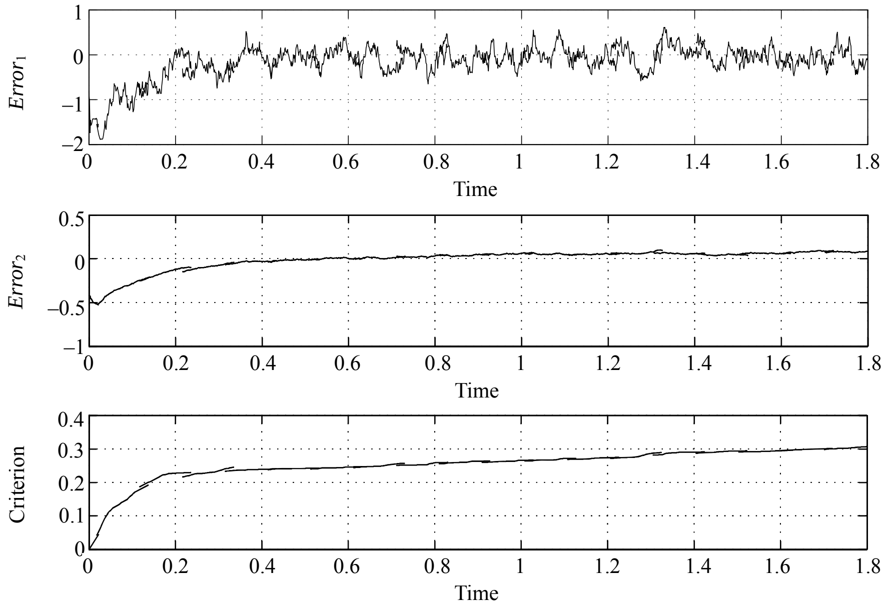

Figure 5. Graphs of the $Error_1$, $Error_2$, and exponential mean square cost criterion corresponding to the polynomial filtering equations for satellite monoaxial for $\varepsilon = 10$, $X_1(0) = 0.115$, $X_2(0) = 0.073$, $Y_1(0) = 2$, $Y_2(0) = 1$.

5. Conclusions

In this paper the equations have been obtained for the optimal risk-sensitive filtering problem, when the system is polynomial of second and third degree, with presence of Gaussian white noise, exponential mean-square cost criterion to be minimized, with parameter ε multiplying the Gaussian white noise in the state and observations equations, and taking into account a value function as a viscosity solution of the nonlinear parabolic PDE.

Numerical application is solved for risk-sensitive and polynomial filtering equations for system of second and third degree (and Kalman-Bucy for system of second degree) for some values of parameter $\tilde{\varepsilon}$. The performance for optimal risk-sensitive filtering equations is verified through of the comparison between the values of the exponential mean-square cost criterion J for polynomial and extended Kalman Bucy filtering equations.

It can be seen that the values of the mean square cost criterion J_{R-S} in final time, are smaller than J_{Pol} and J_{K-B} for all values given to the intensity parameter ε.

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