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The Solutions for the Eco-Epidemic Model with Homotopy Analysis Method

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ABSTRACT

In this paper, the Homotopy Analysis Method (HAM) has been used to solve an eco-epidemic model equation. The algorithm of approximate analytical solution is obtained. HAM contains the auxiliary parameter *h* which provides us with a convenient way to adjust and control convergence region and rate of solution series. The results obtained show that these algorithms are accurate and efficient for the model.

Keywords: Nonlinear Partial Differential Equations; Homotopy Analysis Method; Eco-Epidemic Equations

1. Introduction

There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models. Finding approximate solutions of these nonlinear equations is interesting and important. Many methods have been developed to solve nonlinear partial differential equations (NPDEs) such as biologically-based technologies [1], Adomian decomposition method [2], homotopy perturbation method [3], and so on.

The HAM first envisioned by Liao [4], is another powerful analytical method for nonlinear problem. Many authors have applied the HAM for solving nonlinear equations in [5], for solving solitary waves with discontinuity in [6], for series solutions of nano boundary layer flows in [7], for nonlinear equations in [8] and many other subjects. The technique employed here is very powerful and has been already successfully applied to various complicated problems [9-13]. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible.

The application of the HAM in engineering problems in highly considered by scientists, because HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. The purpose of this paper is to implement HAM to eco-epidemic model equations [1]:

$$\begin{cases} x' = r_1 x - a x^2 - b_1 xy - b_2 xz \\ y' = r_2 y + cxy - d_1 y(y+z) - \beta yz \\ z' = \beta yz - d_2 z(y+z) \end{cases} \tag{1}$$

where $x(t), y(t), z(t)$ are the density of a wild plant species, a susceptible pest and infected pest which live on the crop. All parameters are positive constants.

2. The Solution by Ham

Note $a_1 = a, a_2 = b_1, a_3 = b_2, b_1 = c, b_2 = d_1,\quad b_3 = d_1 + \beta$ $d_1 = \beta - d_2$, this leads to the dimensionless equations

$$\begin{cases} x' = r_1 x - a_1 x^2 - a_2 xy - a_3 xz \\ y' = r_2 y + b_1 xy - b_2 y^2 - b_3 yz \\ z' = d_1 yz - d_2 z^2 \end{cases} \tag{2}$$

with the initial conditions

$$x(0) = A_1, y(0) = A_2, z(0) = A_3$$

Due to the governing Equation (2), we choose the auxiliary linear operators as follows:

$$L_1[X(t;p)] = \frac{\partial X(t;p)}{\partial t} \tag{3.1}$$

$$L_2[Y(t;p)] = \frac{\partial Y(t;p)}{\partial t} \tag{3.2}$$

$$L_3[Z(t;p)] = \frac{\partial Z(t;p)}{\partial t} \tag{3.3}$$

which satisfy

$$L_1[C_1] = 0,\quad L_1[C_2] = 0,\quad L[C_3] = 0 \tag{4}$$

where C_1, C_2 and C_3 are integral constants, and the $X(t;p), Y(t;p), Z(t;p)$ are real functions. Furthermore, due to (2), we define the non-linear operators

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$$N_1[X,Y,Z] = X' - r_1 X + a_1 X^2 + a_2 XY + a_3 XZ \quad (5.1)$$

$$N_2[X,Y,Z] = Y' - r_2 Y - b_1 XY + b_2 Y^2 + b_3 YZ \quad (5.2)$$

$$N_3[X,Y,Z] = Z' - d_1 YZ + d_2 Z^2 \quad (5.3)$$

Then, introducing a non-zero auxiliary h, we construct the zero-order deformation equations

$$(1-p)L_1[X(t;p) - x_0(t)] = h_1 H_1(t) p N_1[X,Y,Y] \quad (6.1)$$

$$(1-p)L_2[Y(t;p) - y_0(t)] = h_2 H_2(t) p N_2[X,Y,Y] \quad (6.2)$$

$$(1-p)L_3[Z(t;p) - z_0(t)] = h_3 H_3(t) p N_3[X,Y,Y] \quad (6.3)$$

Obviously, when $p=0$ and $p=1$, we have the solutions

$$X(t;0) = x_0(t), Y(t;0) = y_0(t), Z(t;0) = z_0(t), \quad (7)$$

$$X(t;1) = x(t), Y(t;1) = y(t), Z(t;1) = z(t) \quad (8)$$

Therefore, as the embedding parameter p increases from 0 to 1, $X(t;p), Y(t;p)$ and $Z(t;p)$ vary from the initials $x_0(t), y_0(t)$ and $z_0(t)$ to the exact solution $x(t)$, $y(t)$ and $z(t)$ governed by (2). This is the basic idea of the homotopy and this kind of variation is called deformations in topology.

Thus, by Taylor's theorem and (7), we can express

$$X(x,t,p) = x_0(t) + \sum_{k=1}^{\infty} x_k(t) p^k \quad (9.1)$$

$$Y(x,t,p) = y_0(t) + \sum_{k=1}^{\infty} y_k(t) p^k \quad (9.2)$$

$$Z(x,t,p) = z_0(t) + \sum_{k=1}^{\infty} z_k(t) p^k \quad (9.3)$$

where

$$x_k(t) = \frac{1}{k!} \frac{\partial^k X(t;p)}{\partial p^k}\Big|_{p=0} \quad (10.1)$$

$$y_k(x,t) = \frac{1}{k!} \frac{\partial^k Y(t;p)}{\partial p^k}\Big|_{p=0} \quad (10.2)$$

$$z_k(x,t) = \frac{1}{k!} \frac{\partial^k Z(t;p)}{\partial p^k}\Big|_{p=0} \quad (10.3)$$

If the auxiliary linear parameter, the initial conditions, and the auxiliary parameters h_1, h_2, h_3, $H_1(t), H_2(t)$, $H_3(t)$ are chosen, the above series converge at $p=1$, and one has

$$x(t) = x_0(t) + \sum_{k=1}^{\infty} x_k(t) \quad (11.1)$$

$$y(t) = y_0(t) + \sum_{k=1}^{\infty} y_k(t) \quad (11.2)$$

$$z(t) = z_0(t) + \sum_{k=1}^{\infty} z_k(t) \quad (11.3)$$

Let $h_1 = h_2 = h_3 = h$ and $H_1(t) = H_2(t) = H_3(t) = 1$. Differentiating the zero-order deformation Equations (6) m times with respect to p and then dividing them by $m!$ and finally setting $p=0$, the mth-order deformation equations read

$$L_1[x_m(t) - \chi_m x_{m-1}(t)] = hR_{1m}[x_{m-1}, y_{m-1}, z_{m-1}] \quad (12.1)$$

$$L_2[y_m(t) - \chi_m y_{m-1}(t)] = hR_{2m}[x_{m-1}, y_{m-1}, z_{m-1}] \quad (12.2)$$

$$L_3[z_m(t) - \chi_m z_{m-1}(t)] = hR_{3m}[x_{m-1}, y_{m-1}, z_{m-1}] \quad (12.3)$$

with the initial conditions ($m \ge 1$)

$$x_m(0) = 0, y_m(0) = 0, y_m(0) = 0 \quad (13)$$

where

$$R_{1m}(t) = x'_{m-1} - r_1 x_{m-1} + a_1 \sum_{j=1}^{m-1} x_j x_{m-1-j} + a_2 \sum_{j=1}^{m-1} x_j y_{m-1-j} + a_3 \sum_{j=1}^{m-1} x_j z_{m-1-j} \quad (14.1)$$

$$R_{2m}(t) = y'_{m-1} - r_r y_{m-1} - b_1 \sum_{j=1}^{m-1} x_j y_{m-1-j} + b_2 \sum_{j=1}^{m-1} y_j y_{m-1-j} + b_3 \sum_{j=1}^{m-1} x_j z_{m-1-j} \quad (14.2)$$

$$R_{3m}(t) = z'_{m-1} - d_1 \sum_{j=1}^{m-1} y_j z_{m-1-j} + d_2 \sum_{j=1}^{m-1} z_j z_{m-1-j} \quad (14.3)$$

and

$$\chi_m = \begin{cases} 1 & m > 1 \\ 0 & m \le 1 \end{cases} \quad (15)$$

Now, the solutions of the mth-order deformation Equations (12) for $m \ge 1$ become

$$x_m = \chi_m x_{m-1} + h\int_0^t R_{1m}(\tau) d\tau \quad (16.1)$$

$$y_m = \chi_m y_{m-1} + h\int_0^t R_{2m}(\tau) d\tau \quad (16.2)$$

$$z_m = \chi_m z_{m-1} + h\int_0^t R_{3m}(\tau) d\tau \quad (16.3)$$

Mathematica software is used to solve the linear Equations (16) under the initial conditions up to first few order of approximations. We have:

$$x_0(t) = A_1; \quad y_0(t) = A_2; \quad z_0(t) = A_3 \quad (17)$$

$$x_1(t) = hA_1 + htE_1 \quad (18.1)$$

$$y_1(t) = hA_2 + htF_1 \quad (18.2)$$

$$z_1(t) = hA_3 + htH_1 \quad (18.3)$$

$$x_2(t) = h(1+h)A_1 + h(1+h)tE_1 + h^2 tE_{21} + h^2 t^2 E_{22} \quad (19.1)$$

$$y_2(t) = h(1+h)A_2 + h(1+h)tF_1 + h^2tF_{21} + h^2t^2F_{22} \quad (19.2)$$

$$z_2(t) = h(1+h)A_3 + h(1+h)tH_1 + h^2tH_{21} + h^2t^2H_{22} \quad (19.3)$$

……

where $E_1 = -r_1A_1 + a_1A_1^2 + a_2A_1A_2 + a_3A_1A_3$;

$$F_1 = -r_2A_2 + b_2A_2^2 + b_1A_1A_2 + b_3A_2A_3;$$

$$H_1 = -d_1A_2A_3 + d_2A_3^2;$$

$$E_{21} = -r_1A_1 + 2a_1A_1^2 + 2a_2A_1A_2 + 2a_3A_1A_3;$$

$$E_{22} = -r_1E_1 + 2a_1A_1E_1 + a_2(A_2E_1 - A_1F_1) + a_3(A_1H_1 + A_3E_1);$$

$$F_{21} = -r_2A_2 + 2b_2A_2^2 + 2b_1A_1A_2 + 2b_3A_2A_3;$$

$$F_{22} = -r_2F_1 + 2b_2A_2F_1 + b_1(A_1F_1 + A_2E_1) + b_3(A_2H_1 + A_3F_1);$$

etc. Therefore,

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots \quad (20.1)$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots \quad (20.2)$$

$$z(t) = z_0(t) + z_1(t) + z_2(t) + \cdots \quad (20.3)$$

3. Results and Analysis

It is important to ensure that the solution series (20) is convergent. Note that the solution series (20) contain the auxiliary parameter h, which we can choose properly by plotting the so-called h -curves to ensure solution series converge, as suggested by Liao [5]. The valid region of h is a horizontal line segment. Thus the valid region of h is shown in **Figure 1** when $A_1 = 0.85$, $A_2 = 0.15$, $r_2 = 0.1$, $a_1 = 1$,

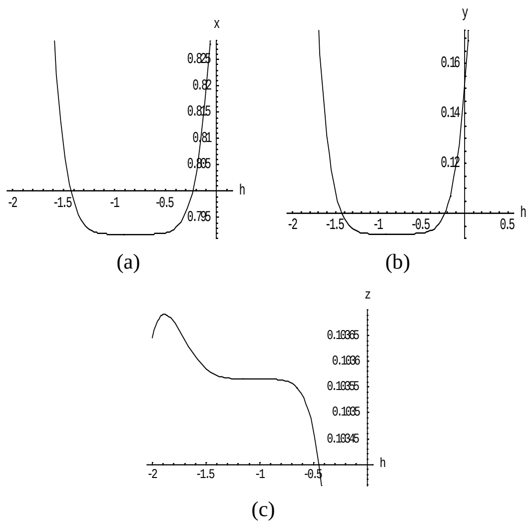

Figure 1. The *h*-curves of the 6-th order approximation for *x*, *y* and *z* when *t* = 0.5: (a) *x* − *y*, (b) *y* − *h*, (c) *z* − *h*.

$a_2 = 0.5, a_3 = 0.4, b_1 = 0.4, b_2 = 0.1, b_3 = 0.69, d_1 = 0.49$
$d_2 = 0.1,\ A_3 = 0.1, r_1 = 0.8.$

4. Conclusion

The model equations which arise from problems are usually nonlinear and such biological equations are difficult to estimate numerically or analytically. In this paper, the HAM was applied to solve the eco-epidemic model equations. HAM contains the auxiliary parameter which provides us with a convenient way to adjust and control convergence region and rate of solution series. In this regard the HAM is found to be a very useful analytic technique to get highly accurate and purely analytic solution to such kind of problems.

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