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Notes on the Global Attractors for Semigroup

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ABSTRACT

First we introduce two necessary and sufficient conditions which ensure the existence of the global attractors for semigroup. Then we recall the concept of measure of noncompactness of a set and recapitulate its basic properties. Finally, we prove that these two conditions are equivalent directly.

Keywords: Natural Global Attractors; Measure of Noncompactness; Asymptotic Compactness; ω-Limit Compact

1. Introduction

It is well known that many mathematical physics problems can be put into the perspective of infinite dimensional systems, which can be equivalently described by C^0 semigroups in proper function spaces. One important object to describe the long time dynamics of an infinite dimensional system is the global attractor, which is a connected and compact invariant set in some function space, and which attracts all bounded sets.

To show the existence of the global attractor, one normally needs to verify:

1) there exists an absorbing set, and

2) the semigroup is uniformly compact.

However, it is difficult or even impossible to verify the uniform compactness of the semigroup for many problems. In [1], the authors use the measure of noncompactness of a set to introduce a new concept of compactness called ω-limit compact, then they show that there exists a global attractor for a C^0 semigroup if and only if:

1) there is an absorbing set, and

2) the semigroup is ω-limit compact.

A well-known result (see [2-6]) is that a continuous semigroup has a global attractor if and only if:

1) it has a bounded absorbing set, and

2) it is asymptotically compact.

Furthermore, in [7], the author introduce the concept of asymptotically null and show that a lattice system has a global attractor if and only if:

1) it has a bounded absorbing set, and

2) it is asymptotically null.

Our main motivation of this paper is to prove that asymptotically compact $\Leftrightarrow$ ω-limit compact, and then we prove that the conditions in [1,7] are equivalent directly in $\ell^p(\phi)$.

The concept of pullback random attractors for random dynamical systems, which is an extension of the attractors theory of deterministic systems, was introduced by the authors in [8-10]. We point out that our work in this paper also can be extended to pullback attractors.

2. Measure of Noncompactness and Its Properties

In this section, we recall the concept of measure of noncompactness and recapitulate its basic properties; see [11].

Definition 2.1 Let *M* be a metric space and *A* be a bounded subset of *M*. The measure of noncompactness $\gamma(A)$ of A is defined by

$$\gamma(A) = \inf\{\delta > 0 \,|\, A \text{ admits a finite cover by sets of diameter} \le \delta\}.$$

Lemma 2.1 *Let M be a complete metric space, and* γ *be the measure of noncompactness of a set.*

1) $\gamma(B) = 0$ if and only if $\bar{B}$ is compact;

2) If *M* is a Banach space, then $\gamma(B_1 + B_2) \le \gamma(B_1) + \gamma(B_2)$;

3) $\gamma(B_1) \le \gamma(B_2)$ whenever $B_1 \subset B_2$;

4) $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$;

5) $\gamma(\bar{B}) = \gamma(B)$.

Proof. 1) (a) If $\bar{B}$ is compact, then B is precompact. M is a complete metric space, thus for any $\epsilon > 0$, there exists a finite subset B_0 of B such that the balls of radii ϵ centered at B_0 form a finite covering of B. By Definition 2.1, B admits a finite cover by sets of diameter $\leq 2\epsilon$. The arbitrariness of ϵ implies that $\gamma(B)=0$.

(b) On the other hand, if $\gamma(B)=0$, then by Definition 2.1, we have that for any $\epsilon > 0$, B admits a finite cover by sets of diameter $\leq 2\epsilon$. So for any $\epsilon > 0$, B always has a finite ϵ-net. Then B is totally bounded. M is complete, thus B is precompact, and $\bar{B}$ is compact.

2) If $\{C_n^1\}$ is a finite cover of B_1, and $\{C_m^2\}$ is a finite cover of B_2, then $\{C_n^1\}+\{C_m^2\}$ is a finite cover of B_1+B_2, thus $\gamma(B_1+B_2)\leq\gamma(B_1)+\gamma(B_2)$.

3) If $B_1 \subset B_2$, then the finite cover of B_2 must be a finite cover of B_1, so $\gamma(B_1)\leq\gamma(B_2)$.

4) (a) The finite cover of $B_1 \cup B_2$ must be a finite cover of both of B_1 and B_2. So we have $\gamma(B_1)\leq\gamma(B_1\cup B_2)$ and $\gamma(B_2)\leq\gamma(B_1\cup B_2)$. Thus $\max\{\gamma(B_1),\gamma(B_2)\}\leq\gamma(B_1\cup B_2)$.

(b) For any $\delta > \max\{\gamma(B_1),\gamma(B_2)\}$, we can find finite covers C_1 of B_1 and C_2 of B_2 with the diameter of C_1 and C_2 less than δ. But $C_1 \cup C_2$ is a cover of $B_1 \cup B_2$ and the diameter of $C_1 \cup C_2$ is less than δ. Hence $\gamma(B_1\cup B_2)<\delta$. So $\max\{\gamma(B_1),\gamma(B_2)\}\geq\gamma(B_1\cup B_2)$.

5) Since $B\subset\bar{B}$, then $\gamma(B)\leq\gamma(\bar{B})$. For any $\delta>\gamma(B)\geq 0$, B has a finite cover by sets of diameter $\leq\delta$. For any $\epsilon>0$, $\bar{B}$ has a finite cover by sets of diameter $\leq\delta+\epsilon$. From the arbitrariness of ϵ and Definition 2.1, we have $\gamma(\bar{B})\leq\delta$. Thus $\gamma(B)\geq\gamma(\bar{B})$. So $\gamma(B)=\gamma(\bar{B})$.

3. Main Results

In this section, firstly we recall some basic definitions in [1,7], then we show that the two necessary and sufficient conditions for the existence of global attractors for semigroups are equivalent directly.

Definition 3.1 Let M be a complete metric space. A one parameter family $\{S(t)\}_{t\geq 0}$ of maps $S(t): M\to M$, $t\geq 0$ is called a C^0 semigroup if

1) $S(0)$ is the identity map on *M*,

2) $S(t+s)=S(t)S(s)$ for all $t,s\geq 0$,

3) the function $S(t)x$ is continuous at each point $(t,x)\in[0,\infty)\times M$.

Definition 3.2 Let $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup in a complete metric space M. A subset B_0 of M is called an absorbing set in M, if for any bounded subset B of M, there exists some $t_1\geq 0$ such that $S(t)B\subset B_0$, for all $t\geq t_1$.

Definition 3.3 A C^0 semigroup $\{S(t)\}_{t\geq 0}$ in a comple te metric space M is called ω-limit compact, if for every bounded subset B of M and any $\epsilon>0$, there exists $t_0>0$ such that

$$\gamma\left(\bigcup_{t\geq t_0} S(t)B\right)\leq\epsilon.$$

Definition 3.4 A C^0 semigroup $\{S(t)\}_{t\geq 0}$ in a complete metric space M is called asymptotically compact if, for every bounded subset $B\subset M$, for any $\{u_n\}\subset B$ and any $t_n\to\infty$, $\{S(t_n)u_n\}$ has a convergent subsequence.

Let ϕ be a positive smooth function on R and $0<p<\infty$. Then define a weighted ℓ^p space as

$$\ell^p(\phi)=\left\{u=(u_i)_{i\in Z}:\sum_{i\in Z}\phi(i)|u_i|^p<\infty\right\}$$

with norm $\|u\|_{\ell^p(\phi)}=\left(\sum_{i\in Z}\phi(i)|u_i|^p\right)^{\frac{1}{p}}$.

Definition 3.5 $\{S(t)\}_{t\geq 0}$ is said to be asymptotically null in $\ell^p(\phi)$ if for any $u_n=(u_{n,i})_{i\in Z}$ bounded in $\ell^p(\phi)$ and $t_n\to\infty$, the following holds

$$\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{|i|\geq k}\phi(i)\left|\left(S(t_n)u_n\right)_i\right|^p=0.$$

Theorem 3.1 *Let* $\{S(t)\}_{t\geq 0}$ *be a* C^0 *semigroup in a complete metric space M, then we can have*:

$\{S(t)\}_{t\geq 0}$ is ω-limit compact $\Leftrightarrow$ $\{S(t)\}_{t\geq 0}$ is asymptotically compact.

Proof. First, we prove the necessity.

It suffices to prove that for every bounded subset $B\subset M$, for any $\epsilon>0$, there exists $t_0>0$, such that

$$\gamma\left(\bigcup_{t\geq t_0} S(t)B\right)\leq\epsilon.$$

Assume otherwise, then there exists a bounded subset $B\subset M$ and $\epsilon_0>0$, such that for every $t_0>0$ we have

$$\gamma\left(\bigcup_{t\geq t_0} S(t)B\right)>\epsilon_0.$$

We take $t_0^1=1$, then $\gamma\left(\bigcup_{t\geq 1}S(t)B\right)>\epsilon_0$. Let $t_1=1$ and take $S(t_1)u_1\in\bigcup_{t\geq 1}S(t)B$.

Let $t_0^2=\max\{2,t_1\}$, then $\gamma\left(\bigcup_{t\geq t_0^2}S(t)B\right)>\epsilon_0$. By the definition of measure of noncompactness, $\bigcup_{t\geq t_0^2}S(t)B$ has no finite covering of balls of radii $\frac{\epsilon_0}{2}$.

Thus there exists $t_2\geq t_0^2$ and $S(t_2)u_2\in\bigcup_{t\geq t_0^2}S(t)B$ such that

$$d\left(S(t_1)u_1,S(t_2)u_2\right)>\frac{\epsilon_0}{2}.$$

Otherwise $\{S(t_1)u_1\}$ is the finite $\frac{\epsilon_0}{2}$-net of

$\bigcup_{t\ge t_0^2} S(t)B$.

Next we take $t_0^3 = \max\{3, t_2\}$, hence $\gamma\left(\bigcup_{t\ge t_0^3} S(t)B\right) > \epsilon_0$. That is to say $\bigcup_{t\ge t_0^3} S(t)B$ has no finite $\frac{\epsilon_0}{2}$-net. Thus there exists $t_3 \ge t_0^3$ and $S(t_3)u_3 \in \bigcup_{t\ge t_0^2} S(t)B$ such that

$$d\left(S(t_i)u_i, S(t_3)u_3\right) > \frac{\epsilon_0}{2}, \quad i = 1, 2.$$

Otherwise $\{S(t_1)u_1, S(t_2)u_2\}$ is the finite $\frac{\epsilon_0}{2}$-net of $\bigcup_{t\ge t_0^3} S(t)B$.

Repeat the previous procedure, then we have the sequence $\{S(t_i)u_i\}$ which satisfies

$$d\left(S(t_i)u_i, S(t_j)u_j\right) > \frac{\epsilon_0}{2}, \quad \forall i \ne j. \qquad (1)$$

By the way of taking t_0^i, and $t_i \ge t_0^i$, we have $t_i \to +\infty$. Since $\{u_i\} \subset B$ and B is a bounded subset of M, $\{S(t)\}_{t\ge 0}$ is asymptotically compact. Therefore $\{S(t_i)u_i\}$ has a convergent subsequence. This gives contradiction to (1).

Thus $\{S(t)\}_{t\ge 0}$ is ω-limit compact.

Next, we prove the sufficiency.

We need to prove that for every bounded subset $B \subset M$, for any $\{u_n\} \subset B$ and any $t_n \to \infty$, $\{S(t_n)u_n\}$ has a convergent subsequence.

Since $\{S(t)\}_{t\ge 0}$ is ω-limit compact, then for the bounded subset $B \subset M$ above, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$\gamma\left(\bigcup_{t\ge t_\epsilon} S(t)B\right) < \epsilon.$$

For $t_n \to \infty$, there exists $N > 0$, such that $t_n \ge t_\epsilon$ when $n \ge N$. $\{u_n\} \subset B$ implies

$$\bigcup_{n\ge N} S(t_n)u_n \subset \bigcup_{t\ge t_\epsilon} S(t)B.$$

Property (3) of the measure of noncompactness in Lemma 2.1 shows that

$$\gamma\left(\bigcup_{n\ge N} S(t_n)u_n\right) \le \gamma\left(\bigcup_{t\ge t_\epsilon} S(t)B\right) < \epsilon.$$

So $\gamma\left(\bigcup_{n\ge N} S(t_n)u_n\right) < \epsilon$. Notice that $\bigcup_{n=1}^{N-1} S(t_n)u_n$ contains only a finite number of elements (where N is fixed such that $t_n \ge t_\epsilon$ as $n \ge N$).

Using properties in Lemma 2.1, we have

$$\gamma\left(\bigcup_{n=1}^{N-1} S(t_n)u_n\right) = \gamma\left(\overline{\bigcup_{n=1}^{N-1} S(t_n)u_n}\right) = 0.$$

Thus

$$\gamma\left(\bigcup_{n\ge 1} S(t_n)u_n\right) = \max\left\{\gamma\left(\bigcup_{n=1}^{N-1} S(t_n)u_n\right), \gamma\left(\bigcup_{n\ge N} S(t_n)u_n\right)\right\} = \gamma\left(\bigcup_{n\ge N} S(t_n)u_n\right) < \epsilon.$$

From the arbitrariness of ϵ, it has

$$\gamma\left(\bigcup_{n\ge 1} S(t_n)u_n\right) = 0.$$

Hence $\{S(t_n)u_n\}$ is precompact. Thus $\{S(t_n)u_n\}$ has a convergent subsequence. Therefore $\{S(t)\}_{t\ge 0}$ is asymptotically compact. This completes the proof of Theorem.

Corollary 1 *Let $\{S(t)\}_{t\ge 0}$ be a semigroup of continuous operators in $\ell^p(\phi)$. Then $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set and it is asymptotically null in $\ell^p(\phi) \Leftrightarrow \{S(t)\}_{t\ge 0}$ has a bounded absorbing set and it is ω-limit compact.*

Proof. By Corollary 3.4 in [7], we have $\{S(t)\}_{t\ge 0}$ is asymptotically compact in $\ell^p(\phi)$ if and only if $\{S(t)\}_{t\ge 0}$ is asymptotically null in $\ell^p(\phi)$ and $\{S(t_n)u_n\}_{n=1}^{\infty}$ is bounded in $\ell^p(\phi)$ provided $\{u_n\}_{n=1}^{\infty}$ is bounded and $t_n \to \infty$.

Using the Theorem 3.1 above, we have $\{S(t)\}_{t\ge 0}$ is ω-limit compact in $\ell^p(\phi)$ if and only if $\{S(t)\}_{t\ge 0}$ is asymptotically null in $\ell^p(\phi)$ and $\{S(t_n)u_n\}_{n=1}^{\infty}$ is bounded in $\ell^p(\phi)$ provided $\{u_n\}_{n=1}^{\infty}$ is bounded and $t_n \to \infty$. Thus the necessity of the corollary is obvious.

If $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set, $\{u_n\}_{n=1}^{\infty}$ is bounded and $t_n \to \infty$, then there exists N such that $\{S(t_n)u_n\}_{n=N}^{\infty}$ is contained in the bounded absorbing set. $\{S(t_n)u_n\}_{n=1}^{N}$ is a finite set in $\ell^p(\phi)$, so it is bounded. Thus $\{S(t_n)u_n\}_{n=1}^{\infty}$ is bounded. Now we can have the sufficiency immediately. This completes the proof of Corollary.

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