

Finite-Difference Solution of the Helmholtz Equation Based on Two Domain Decomposition Algorithms

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ABSTRACT

In this paper, wave simulation with the finite difference method for the Helmholtz equation based on the domain decomposition method is investigated. The method solves the problem by iteratively solving subproblems defined on smaller subdomains. Two domain decomposition algorithms both for nonoverlapping and overlapping methods are described. More numerical computations including the benchmark Marmousi model show the effectiveness of the proposed algorithms. This method can be expected to be used in the full-waveform inversion in the future.

Keywords: Finite Difference; Domain Decomposition; Nonoverlapping; Overlapping; Helmholtz Equation; Preconditioner

1. Introduction

The numerical solution of acoustic wave equation is an important problem. The Helmholtz equation is the version of acoustic wave equation in the frequency domain. It has applications in seismic wave propagation, imaging and inversion. In the geophysical frequency-domain inversion, one needs to do forward modeling which means solving the Helmholtz equation. During the inversion process, the synthetic model is continuously updated until a convergence is reached. Thus numerical methods for solving the Helmholtz equation have been under active research during the past few decades. The finite element method and the finite difference method have been used successfully for this problem. The discretization of the 2D Helmholtz for mid-frequency and high-frequency problems may lead to a large linear system because of the requirement of ten points per wavelength. This makes the problem even harder to solve. Direct methods easily suffer from unacceptable computational work. So many iterative methods for the Helmholtz equation have been developed, for instance, see [1-6]. As the resulting system is non-Hermitian and indefinite, a good preconditioner is necessary for the iterative methods. Various preconditioners have been proposed [5-11], for example, a tensor product preconditioner [6], the incomplete factorization preconditioner [7] and the Laplacian preconditioner [8,9]. We will use the shifted-Laplacian preconditioner in this paper [9].

The domain decomposition method (DDM) is an effective technique for solving large-scale problems [12-

22]. It splits the whole computational domain into several smaller subdomains and solves a sequence of similar subproblems on these subdomains. The number and size of subdomains can now be chosen so as to enable direct methods to solve the subproblems. Between adjacent subdomains the boundary conditions are adjusted iteratively by transmission conditions. For the boundary of whole computational domain, absorbing boundary conditions (ABCs) are required. There exist several ABC methods, for instance, the paraxial approximation method [23] and the perfectly matched layer method [24]. In this paper, we use the former as our computational domain is a rectangular domain.

In this paper, we focus on solving the Helmholtz with the finite difference method based on the nonoverlapping and overlapping DDM algorithms. More numerical computations demonstrate the correctness of the algorithms presented in this paper. The method will be used in the frequency-domain inversion in the future.

2. Theory

2.1. Finite Difference Scheme

The 2-D acoustic wave equation can be written as

$$\frac{1}{v(x, z)^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial z^2} = g(x, z), \quad (1)$$

with the absorbing boundary conditions

$$\frac{\partial u}{\partial n} + \frac{1}{v(x, z)} \frac{\partial u}{\partial t} = 0, \quad (2)$$

where $v(x, z)$ is the velocity of media and $u(x, z, t)$ is the wavefield, $g(x, z)$ is the source term. In the frequency domain (1) can be written as

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial z^2} - k^2(x, z)u = g(x, z), \quad (3)$$

where $k = \omega / v(x, z)$ is the wave number and $\omega = 2\pi f$ is the angular frequency. The boundary condition in the frequency domain is

$$\frac{\partial u}{\partial n} - ik(x, y)u = 0, \quad (4)$$

where n is the outward normal of the boundary and $i = \sqrt{-1}$ is the imagine unit. We use the second-order difference scheme to discrete (3) and (4) and the result can be written as a linear system

$$Au = b, \quad A \in C^{N \times N}, \quad u, b \in C^N, \quad (5)$$

where $N = N_x N_y$ is the total number of wavefield u on the computational domain Ω_h , and N_x and N_y are the discretization number along x and z directions respectively. The matrix A is a complex matrix as the boundary condition contains complex number. Moreover, it is non-positive and non-Hermitian matrix. Usually we use the Krylov iterative methods to solve (5) as A is a large-scale sparse matrix. The Bi-CGSTAB algorithm [25,26] is a good choice. The following is the Bi-CGSTAB algorithm for solving $Ax = b$.

Algorithm 1 [Bi-CGSTAB algorithm]

Step 1. Give the matrix A , vector b and initial value x_0 , the maximal iterative number k_{\max} and the tolerance error ε_{tol} , the preconditioned matrix M , compute $r_0 = b - Ax_0$ and set $k = 1$ and $\tilde{r}_0 = r_0$;

Step 2. If $k \leq k_{\max}$ and $\varepsilon > \varepsilon_{tol}$, turn to step 3; otherwise stop and output x_k ;

Step 3. Compute $\rho_{k-1} = r_{k-1}^T r_{k-1}$, if $\rho_{k-1} = 0$ or $\omega_{k-1} = 0$ then stop; otherwise turn to step 4;

Step 4. If $k = 1$ then $p_1 = r_0$, otherwise compute

$$\beta_{k-1} = \frac{\rho_{k-1} \alpha_{k-1}}{\rho_{k-2} \omega_{k-1}}, \quad p_k = r_{k-1} + \beta_{k-1}(p_{k-1} - \omega_{k-1} V_{k-1}); \quad (6)$$

Step 5. Solve the system $M\hat{p} = p$ and compute

$$V_k = A\hat{p}, \quad \alpha_{k-1} = \frac{\rho_{k-1}}{r_{k-1}^T V_k}, \quad s_k = r_{k-1} - \alpha_{k-1} V_k; \quad (7)$$

Step 6. Set $\varepsilon = \|s_k\|$, if $\varepsilon > \varepsilon_{tol}$ then $x_k = x_{k-1} + \alpha_{k-1} \hat{p}$, otherwise stop and output x_k ;

Step 7. Solve the systems $M\hat{s} = s$ and $t = A\hat{s}$, then compute

$$\omega_k = \frac{t^T s \rho_{k-1}}{t^T t}, \quad x_k = x_{k-1} + \alpha_k p_k + \omega_k \hat{s}, \quad (8)$$

and set $\varepsilon = \|r_k\|$, let $k = k + 1$, turn to step 2.

For the preconditioner matrix M , we adopt the shifted-

Laplace preconditioned method [9]

$$M_{SL} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \alpha k^2(x, y), \quad (9)$$

where $\text{Re}(\alpha) > 0$ and $\text{Im}(\alpha) > 0$. A typical choice for α is $\alpha = i$, named complex shifted-Laplace preconditioner.

2.2. Two Domain Decomposition Algorithms

In this subsection we discuss how to solve the problem (3) and (4) with DDM, including the nonoverlapping algorithm and overlapping algorithm. First of all, we consider the nonoverlapping problem. We divide the computational domain Ω into N non-overlapped subdomains Ω_m , $m = 1, \dots, N$. Denote $u^{n,m}$ be the value of u at n th iteration and on the m th subdomain Ω_m . Obviously the division satisfies

$$\bar{\Omega} = \bigcup \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j, \quad (i \neq j).$$

Given the iteration value $u^{0,m}$, $m = 1, \dots, N$, solve the following system iteratively:

$$-\frac{\partial^2 u^{n,m}}{\partial x^2} - \frac{\partial^2 u^{n,m}}{\partial z^2} - k^2 u^{n,m} = g(x_{ij}, y_{ij}), \quad (x, y) \in \Omega_m \quad (10)$$

$$\frac{\partial u^{n,m}}{\partial n_m} - iku^{n,m} = 0, \quad (x, y) \in \partial\Omega \cap \partial\Omega_m, \quad (11)$$

$$\frac{\partial u^{n,m}}{\partial n_m} - iku^{n,m} = -\frac{\partial u^{n-1,r}}{\partial n_r} - iku^{n-1,r}, \quad (12)$$

$$(x, y) \in \Gamma_{mr} = \partial\Omega_m \cap \partial\Omega_r.$$

The last Equation (12) is the interface equation. Using the standard five-point difference scheme, we obtain the following system

$$A^{n,m} u^{n,m} = b^{n,n}, \quad m = 1, \dots, N. \quad (14)$$

In the iterations, the $u^{n-1,m}$ is assumed to be known and is used in the interface equation. Thus we can give the following nonoverlapping DDM algorithm.

Algorithm 2 [Nonoverlapping DDM algorithm]

Step 1. Select initial value $u^{0,m}$ and set $n = 0$.

Step 2. Obtain $b^{n,m}$ by solving the interface equation discretized from (12).

Step 3. Solve the system (14).

Step 4. Set $n = n + 1$, turn to step 2.

In the following we consider the overlapping DDM. We still divide the computational domain Ω into N subdomains Ω_i , $i = 1, \dots, N$:

$$\bar{\Omega} = \bigcup \bar{\Omega}_i, \quad (15)$$

but now any two adjacent subdomains are overlapped, i.e.,

$$\Omega_m \cap \Omega_r \neq \emptyset. \quad (16)$$

For simplicity we consider the case of two subdomains, *i.e.*, $N=2$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$. Notice that the boundary

$\Gamma_i = \partial\Omega_i \setminus (\partial\Omega \cap \partial\Omega_i)$ (17) doesn't belong to the subdomain Ω_i . When the iteration for the problem converges, the following conditions are true obviously

$$u^{n,m} - u^{n-1,m} \rightarrow 0, \quad (18)$$

$$\frac{\partial u^{n,m}}{\partial n_i} - \frac{\partial u^{n-1,m}}{\partial n_i} \rightarrow 0, \quad n \rightarrow +\infty, \quad (19)$$

To keep the symmetry of the matrix on the subdomains, we construct the auxiliary equation

$$\frac{\partial u^{n,m}}{\partial n_m} - iku^{n,m} = \frac{\partial u^{n-1,r}}{\partial n_r} - iku^{n-1,r}. \quad (20)$$

Thus we get the similar system

$$\tilde{A}^{n,m} u^{n,m} = \tilde{b}^{n,n}, \quad m=1, \dots, N. \quad (21)$$

Now we can give the following overlapping DDM algorithm.

Algorithm 3 [Overlapping DDM algorithm]

Step 1. Set $n=0$ and select initial value $u^{n,m}$;

Step 2. Solve the auxiliary Equation (21);

Step 3. Extrapolate $u^{n,m}$ according to the following formulation:

$$u^{n,m} |_{\Omega_m} = u^{n,m}, \quad u^{n,m} |_{\Omega \setminus \Omega_m} = u^{n,m},$$

$$u^n = \frac{1}{N} \sum_{i=1}^N u^{n,m}; \quad (22)$$

Step 4. Set $n = n+1$ and turn to step 2.

3. Numerical Computations

For testing the correctness of the discretized finite difference schemes, we solve the problem without using DDM first. The first model is a homogeneous model with constant wave number. The computational domain is a square $\Omega = (0, 1)^2$, and the source $g(x, y)$ is defined as the following δ function:

$$\delta(x, y) = \begin{cases} 1, & x = 1/2, y = 1, 2. \\ 0, & \text{other.} \end{cases}$$

We use the preconditioned Bi-CGSTAB method to solve the problem. **Figure 1**, **Figures 2** and **3** are the simulation results for wave number 20, 30 and 40 respectively. We can see that the obtained waveform in these figures is very clear. We also can see that the wave has more vibration as k increases. Next we consider a three-layered model shown in **Figure 4**. The velocity from top to bottom is 2700 m/s, 1500m/s and 2100 m/s respectively. The computational domain is a rectangle domain: $\Omega = (0, 600) \times (0, 1000)$, and $g(x, y)$ is the following δ function

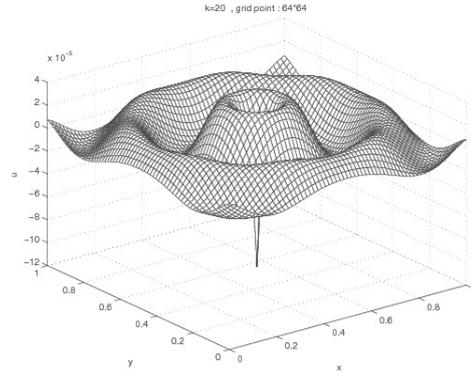


Figure 1. Wavefield for $k = 20$ obtained by the preconditioned Bi-CGSTAB method. The DDM is not used.

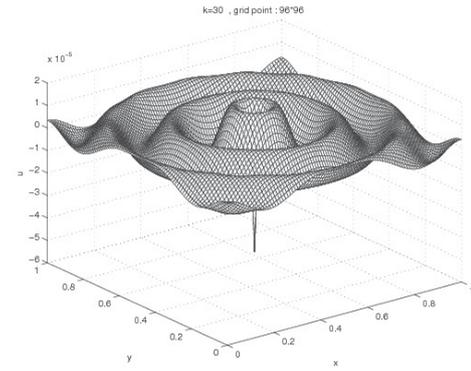


Figure 2. Wavefield for $k = 30$ obtained by the preconditioned Bi-CGSTAB method. The DDM is not used.

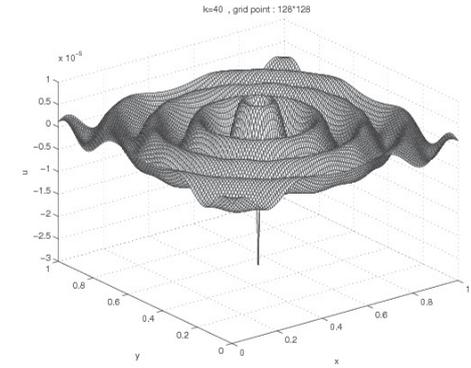


Figure 3. Wavefield for $k = 40$ obtained by the preconditioned Bi-CGSTAB method. The DDM is not used.

$$\delta(x, y) = \begin{cases} 1, & x = 300, y = 1, 2. \\ 0, & \text{other.} \end{cases}$$

Figures 5 and **6** are the simulation results for this model with wave number 20 and 30 respectively. Our analysis shows the results are right.

Now we solve the problem with the nonoverlapping DDM. We consider a square domain $\Omega = (0, 1)^2$. We divide this domain into two subdomains. **Figure 7** is the

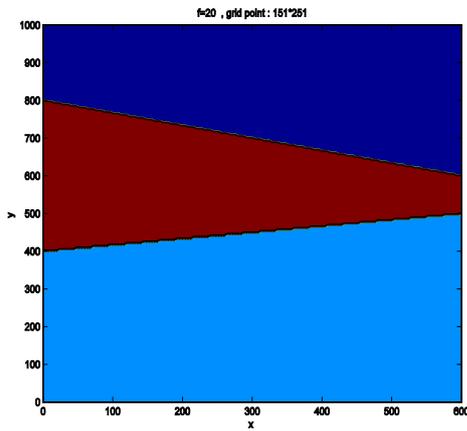


Figure 4. A three-layered model with velocity 2700 m/s, 1500 m/s and 2100 m/s from top to bottom.

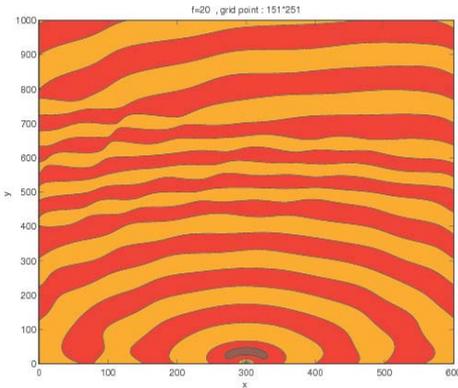


Figure 5. Wavefield contour for $f = 20$ Hz obtained by the preconditioned Bi-CGSTAB method. The DDM is not used.

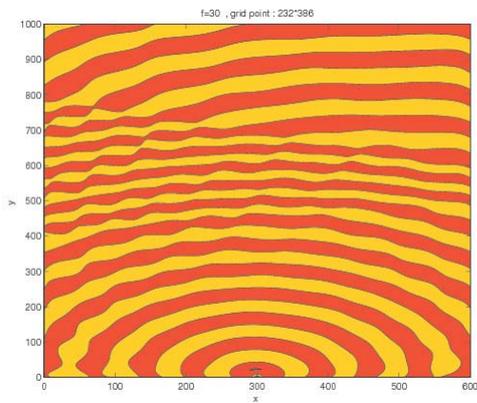


Figure 6. Wavefield contour $f = 30$ Hz obtained by the preconditioned Bi-CGSTAB method. The DDM is not used.

wavefield result for $f = 4$ Hz obtained by the nonoverlapping DDM algorithm with two subdomains. For a L-type model we divide it into three subdomains. **Figure 8** is the corresponding simulation result for $f = 4$ Hz obtained by the nonoverlapping DDM algorithm with three sub-

domains. **Figure 9** is result for $f = 4$ Hz obtained by the nonoverlapping DDM method with four equal square subdomains for a square domain.

Next we solve the problem with the overlapping DDM algorithm. The velocity media is 2100 m/s. The location of the pulse is at $(x, y) = (3, 2.5)$. The frequency is

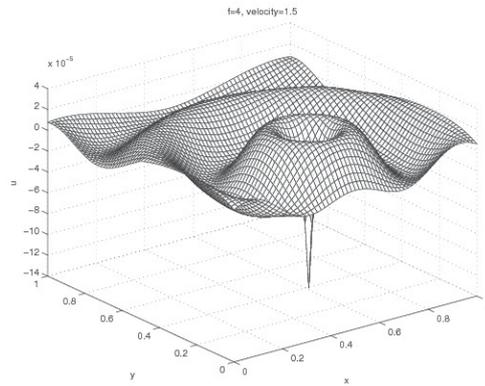


Figure 7. Wavefield for $f = 4$ Hz obtained by the nonoverlapping DDM algorithm. The square computational domain is divided into two subdomains up and down.

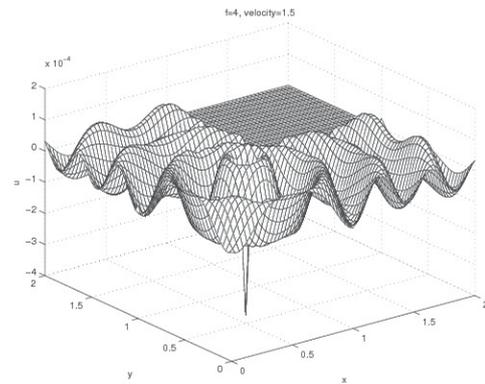


Figure 8. Wavefield for $f = 4$ Hz obtained by the nonoverlapping DDM algorithm. The L-type computational domain is divided into three square subdomains.

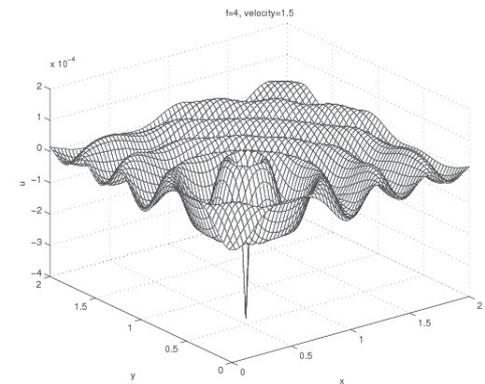


Figure 9. Wavefield for $f = 4$ Hz obtained by the nonoverlapping DDM algorithm. The square computational domain is divided into four equal square subdomains.

$f = 1$ Hz. **Figure 10** is the wavefield obtained by the overlapping DDM algorithm with two equal square subdomains up and down. **Figure 11** is the wavefield obtained by the overlapping DDM algorithm with three square subdomains for an L-type domain. **Figure 12** is the

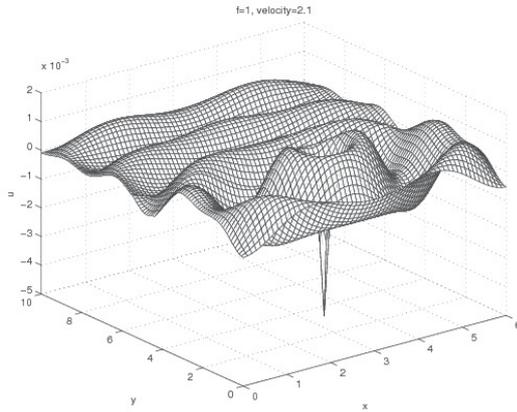


Figure 10. Wavefield for $f = 1$ Hz obtained by the overlapping DDM algorithm. The computational domain is divided into two equal subdomains up and down.

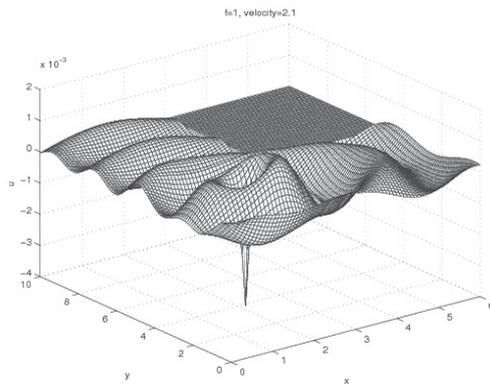


Figure 11. Wavefield for $f = 1$ Hz obtained by the overlapping DDM algorithm. The square computational domain is divided into three equal square subdomains.

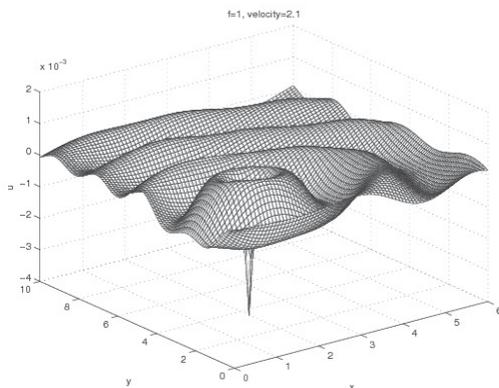


Figure 12. Wavefield for $f = 1$ Hz obtained by the overlapping DDM algorithm. The square computational domain is divided into four equal subdomains.

wavefield obtained by the overlapping DDM algorithm with four equal square subdomains for a square domain.

Finally we consider a typical inhomogeneous model named Marmousi model which is usually used to test the ability of seismic migration and inversion [27]. The velocity is shown in **Figure 13**. The velocity varies from 1500 m/s to 5500 m/s. We select a part of this model to simulate wave propagation. **Figure 14** is the wavefield contour obtained by the preconditioned Bi-CGSTAB method for $f = 5$ Hz and the DDM algorithm is not used. **Figure 15** is the wavefield contour obtained by the non-overlapping DDM algorithm with two subdomains. **Figure 16** is the wavefield contour obtained by the overlapping DDM algorithm with two subdomains. **Figure 17** is the similar result but for $f = 20$ Hz. Comparing **Figures 15** and **16** with **Figure 14**, we know that they almost the same which are just we expect.

4. Conclusion

The acoustic wave equation in the frequency domain is solved by the finite difference method based on the domain decomposition method. The discretization of the

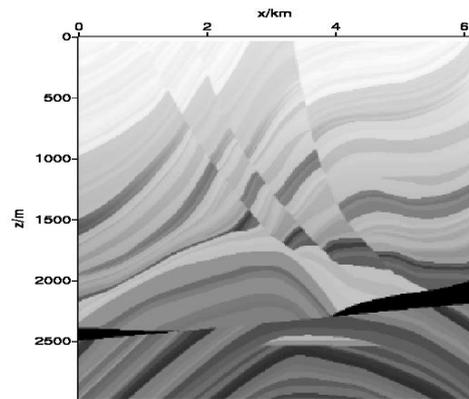


Figure 13. Marmousi model.

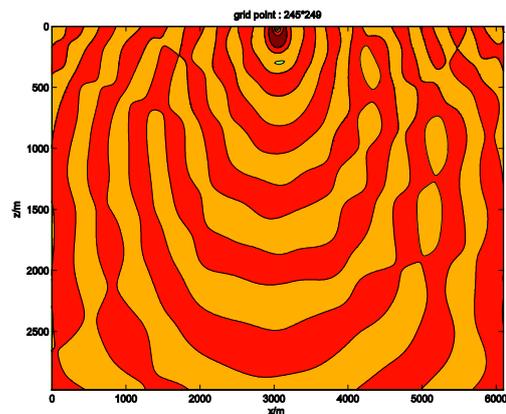


Figure 14. Wavefield contour for $f = 5$ Hz obtained by the preconditioned Bi-CGSTAB algorithm. The DDM algorithm is not used.

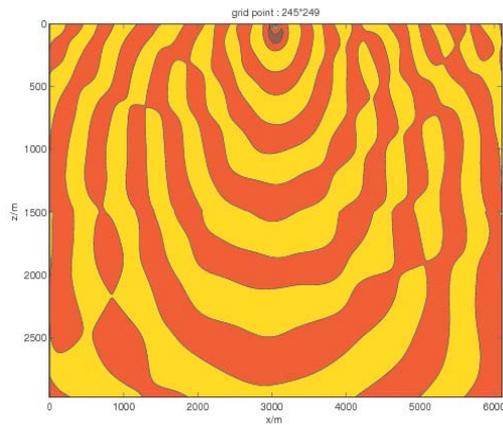


Figure 15. Wavefield contour for $f = 5$ Hz by Bi-CGSTAB preconditioned method. The nonoverlapping DDM algorithm with two subdomains is used.

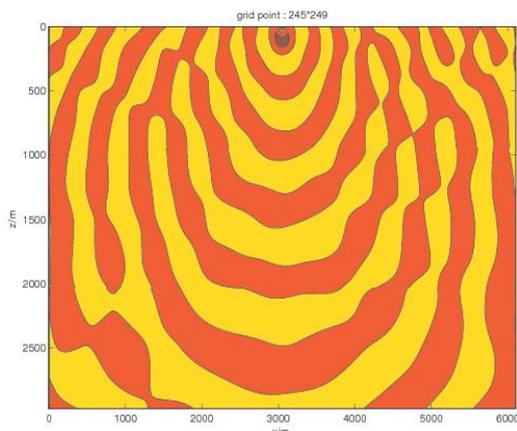


Figure 16. Wavefield contour for $f = 5$ Hz obtained by Bi-CGSTAB preconditioned method. The overlapping DDM algorithm with two subdomains is used.

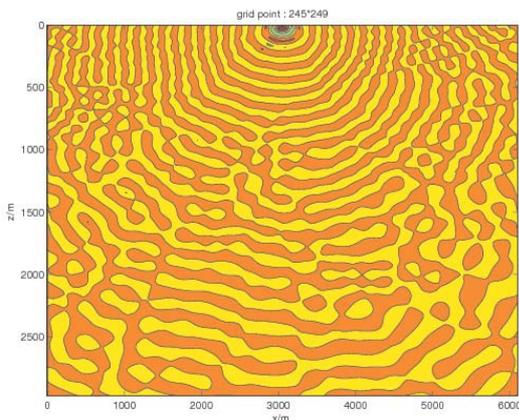


Figure 17. Wavefield contour for $f = 20$ Hz obtained by Bi-CGSTAB preconditioned method. The overlapping DDM algorithm with two subdomains is used.

problem leads to a sparse system which is solved by the complex shifted-Laplace preconditioned Bi-CGSTAB

iteration method. Two DDM algorithms both for non-overlapping and overlapping method are given. Many numerical computational examples including the complex Marmousi model are implemented which show the correctness and effectiveness of the algorithms presented in this paper. This method can be used in the full-waveform inversion. It can sometimes reduce the computational complexity.

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