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Boundedness of Hyper-Singular Parametric Marcinkiewicz Integrals with Variable Kernels

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ABSTRACT

In this article, we consider the boundedness of $\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha}$ on Hardy type space $H^p_{\boldsymbol{b}}(R^n)$. Where $0<\rho<n$, $\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha}(f)(x)=\left(\int_0^\infty \left|F^{\rho}_{\Omega,\boldsymbol{b},t}(f)(x)\right|^2 \frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}}$, $F^{\rho}_{\Omega,\boldsymbol{b},t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\prod_{i=1}^{m}\left[b_i(x)-b_i(y)\right]f(y)\mathrm{d}y$. $\boldsymbol{b}=(b_1,b_2,\cdots,b_m), b_i\in\dot{\Lambda}_{\beta_i}(R^n), 1\le i\le m, \beta_i>0, \sum_{i=1}^m\beta_i=\beta, 0\le\alpha<\beta<1$

Keywords: Hyper-Singular Marcinkiewicz Integral; Variable Kernel; Multilinear Commutator; Hardy Type Space

1. Introduction

A function $\Omega(x,z)$ defined on $R^n\times R^n$ is said to belong to $L^\infty(R^n)\times L^r(S^{n-1})$, if it satisfies the following three conditions:

1) $\Omega(x,\lambda z)=\Omega(x,z)$, for any $x,z\in R^n$ and any $\lambda>0$;

2) $\|\Omega\|_{L^\infty(R^n)\times L^r(S^{n-1})} =: \sup_{\rho\ge0,y\in R^n}\left(\int_{S^{n-1}}\left|\Omega(\rho z'+y,z')\right|^r \mathrm{d}\sigma(z')\right)^{1/r}<\infty$

3) $\int_{S^{n-1}}\Omega(x,z')\mathrm{d}\sigma(z')=0$, for any $x\in R^n$.

In [1], the authors considered the hyper-singular parametric Marcinkiewicz integral with variable kernel as follows:

$$\mu^{\rho}_{\Omega,\alpha}(f)(x)=\left(\int_0^\infty\left|F^{\rho}_{\Omega,t}(f)(x)\right|^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}},$$

where

$$F^{\rho}_{\Omega,t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}f(y)\mathrm{d}y.$$

When $\alpha=0$, we set $\mu^{\rho}_{\Omega,0}=\mu^{\rho}_{\Omega}$, which is the parametric Marcinkiewicz integral with variable kernels considered in [2].

For $\beta>0$, the homogenous Lipschitz space $\dot{\Lambda}_\beta(R^n)$ is the space of function f such that

$$\|f\|_{\dot{\Lambda}_\beta}=\sup_{x,h\in R^n,h\ne0}\frac{\left|\Delta_h^{[\beta]+1}f(x)\right|}{|h|^\beta}<\infty,$$

where Δ_h^k denotes k-th difference operator (see [3]).

In 2006, Lu and Xu studied the boundedness of the commutator of $\mu^m_{\Omega,b}$ in [4]. They proved that:

Theorem A [4]. *Suppose* $\Omega\in\dot{\Lambda}_\alpha(S^{n-1})$ for $0<\alpha\le\frac{1}{2}, b\in\dot{\Lambda}_\beta(R^n), 0<\beta\le\frac{\alpha}{m}$. If $\frac{n}{n+\alpha}\le p\le1$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, then $\mu^m_{\Omega,b}$ maps $H^p_{b^m,s}(R^n)$ continuously into $L^q(R^n)$. Here $\mu^m_{\Omega,b}$ is defined as

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follows:

$$\mu_{\Omega,b}^{m}(f)(x)=\left(\int_0^\infty\left|F_{\Omega,b,t}^{m}(f)(x)\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}, \qquad (1)$$

where

$$F_{\Omega,b,t}^{m}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\left[b(x)-b(y)\right]^m f(y)\mathrm{d}y.$$

Let $\boldsymbol{b}=(b_1,b_2,\cdots,b_m)$, $b_i\in\dot{\Lambda}_{\beta_i}(R^n)$, $1\le i\le m$, $\beta_i>0$, $\sum_{i=1}^m\beta_i=\beta, 0\le\alpha<\beta<1.$ In this article, we mainly consider the commutator $\mu_{\Omega,\boldsymbol{b},\alpha}^{\rho}$ defined by

$$\mu_{\Omega,\boldsymbol{b},\alpha}^{\rho}(f)(x)=\left(\int_0^\infty\left|F_{\Omega,\boldsymbol{b},t}^{\rho}(f)(x)\right|^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}}, \qquad (2)$$

where

$$F_{\Omega,\boldsymbol{b},t}^{\rho}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\prod_{i=1}^m\left[b_i(x)-b_i(y)\right]f(y)\mathrm{d}y.$$

Given any positive integer m, for all $1\le j\le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(j)\}$ of $\{1,2,\cdots,m\}$ of j different elements. For any $\sigma\in C_j^m$, we associate the complementary sequence σ' given by $\sigma'=\{1,2,\cdots,m\}\setminus\sigma$, (see [5]).

For any $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(j)\}\in C_j^m$, we will denote $\boldsymbol{b}_\sigma=\left(b_{\sigma(1)},b_{\sigma(2)},\cdots,b_{\sigma(j)}\right)$ and the product $b_\sigma=b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(j)}$. When $\sigma\in C_0^m$, we have $\sigma=\varnothing$, by definition, we have $b_\sigma=1$. Similarly, when $\sigma\in C_m^m$, we have $\sigma'=\varnothing$ and $b_{\sigma'}=1$. With this notation, if $\beta_{\sigma(1)}+\cdots+\beta_{\sigma(j)}=\beta_\sigma$, we write

$$\|\boldsymbol{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}=\|b_{\sigma(1)}\|_{\dot{\Lambda}_{\beta_{\sigma(1)}}}\cdots\|b_{\sigma(j)}\|_{\dot{\Lambda}_{\beta_{\sigma(j)}}}.$$

When $\sum_{i=1}^m\beta_i=\beta$, we write

$$\|\boldsymbol{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}=\prod_{i=1}^m\|b_{\sigma(i)}\|_{\dot{\Lambda}_{\beta_{\sigma(i)}}}.$$

Definition 1.1. Let $0<p\le1$, $\boldsymbol{b}$ be defined as above such that $b_i\in\dot{\Lambda}_{\beta_i}(R^n), 1\le i\le m, 0<\beta_i<1, \sum_{i=1}^m\beta_i=\beta$. A function $a(x)$ on R^n is called a $(p,q,\boldsymbol{b})$-atom if

1) $\operatorname{supp}a\subset B(x_0,r):=\{x\in R^n:|x-x_0|<r\}$, for some $x_0\in R^n$ and $r>0$;

2) $\|a\|_q\le|B(x_0,r)|^{\frac{1}{q}-\frac{1}{p}}$

3) $\int_{R^n}a(x)b_{\sigma'}(x)\mathrm{d}x=0$ for any $\sigma\in C_j^m, \sigma'=\{1,\cdots,m\}\setminus\sigma$ and $j=1,2,\cdots,m.$

Definition 1.2. Let $0<p\le1$, we say that a distribution f on R^n belongs to $H_{\boldsymbol{b}}^p(R^n)$ if and only if f can be written as $f=\sum_{i=-\infty}^{\infty}\lambda_i a_i$ in the distributional sense, where each a_i is a $(p,q,\boldsymbol{b})$-atom and $\sum_{i=-\infty}^{\infty}|\lambda_i|^p<\infty.$ Moreover,

$$\|f\|_{H_{\boldsymbol{b}}^p}\sim\inf\left\{\left(\sum_{i=-\infty}^{\infty}|\lambda_i|^p\right)^{\frac{1}{p}}\right\}$$

with the infimum taken over all the above decompositions of f as above

Definition 1.3. A function $\Omega(x,z)\in L^\infty(R^n)\times L^r(S^{n-1})(r\ge1)$ is said to satisfy the $L^{r,\beta}$-Dini condition, if

$$\int_0^1\frac{\omega_r(\delta)}{\delta^{1+\beta}}\mathrm{d}\delta<\infty, 0\le\beta\le1, \qquad (3)$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order r of Ω defined by

$$\omega_r(\delta)=\sup_{x\in R^n,\rho\ge0}\left(\int_{S^{n-1}}\sup_{y'\in S^{n-1},|y'-z'|\le\delta}\left|\Omega(x+\rho z',y')-\Omega(x+\rho z',z')\right|^r\mathrm{d}\sigma(z')\right)^{\frac{1}{r}}.$$

We will denote simply L^r-Dini condition for $L^{r,\beta}$-Dini condition when $\beta=0$.

2. Main Theorem

Now let us formulate our main results as follows.

Theorem 2.1. *Suppose that* $\mu_{\Omega,\boldsymbol{b},\alpha}^{\rho}$ *is the commutator* (2), *and let* $0\le\alpha<\beta<1, 1<p<\frac{n}{\beta-\alpha}, \frac{1}{q}=\frac{1}{p}-\frac{\beta-\alpha}{n}.$ $\Omega(x,z)\in L^\infty(R^n)\times L^r(S^{n-1})$, *then* $\mu_{\Omega,\boldsymbol{b},\alpha}^{\rho}$ *is bounded from* $L^p(R^n)$ *into* $L^q(R^n)$. *That is,*

$$\left\|\mu_{\Omega,\boldsymbol{b},\alpha}^{\rho}\right\|_{L^q(R^n)}\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(R^n)}.$$

Theorem 2.2. *Suppose that* $\mu_{\Omega,\boldsymbol{b},\alpha}^{\rho}$ *is the commutator* (2), *and let* $\frac{n}{n+\beta-\alpha}<p\le1, \frac{1}{q}=\frac{1}{p}-\frac{\beta-\alpha}{n}.$ *If* $\Omega(x,z)$ *satisfies the following two conditions*:

1) $\Omega(x,z)$ satisfies $L^{r,\beta}$-Dini condition (3);

2) there exists

$r > \max\left\{\frac{n}{n+\beta-\alpha-\frac{n}{p}}, \frac{n}{n-\beta+\alpha}, \frac{n}{n+\frac{1}{2}-\frac{n}{p}}\right\}$ such that $\Omega(x,z) \in L^\infty(R^n) \times L^r(S^{n-1})$ then $\mu^\rho_{\Omega,\boldsymbol{b},\alpha}$ is bounded from $H^p_{\boldsymbol{b}}(R^n)$ into $L^q(R^n)$. That is

$$\left\|\mu^\rho_{\Omega,\boldsymbol{b},\alpha}(f)\right\|_{L^q(R^n)} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\|f\|_{H^p_{\boldsymbol{b}}(R^n)}.$$

Remark Obviously, $\mu^\rho_{\Omega,\boldsymbol{b},\alpha}$ is the commutator of the operator $\mu^\rho_{\Omega,\alpha}$ in [1]. At the same time, we change the course of the statement in [4].

In order to prove our Theorems, we need several preliminary lemmas.

Lemma 2.1. [6] *Let* $0 \le \mu < n$ *and* suppose $\Omega(x,z) \in L^\infty(R^n) \times L^r(S^{n-1})(r \ge 1)$. If there exists a constant $0 < R < \frac{1}{2}$ such that $|y| < RK$, then for any $x_0 \in R^n$,

$$\left(\int_{K<|x|\le 2K}\left|\frac{\Omega(x_0+x,x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x_0+x,x)}{|x|^{n-\mu}}\right|^r \mathrm{d}x\right)^{1/r} \le CK^{\frac{n}{r}-(n-\mu)}\left(\frac{|y|}{K} + \int_{2|y|/K<\delta\le 4|y|/K}\frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right).$$

where the constant $C > 0$ is independent of K and y.

lemma 2.2. [7] *Let* $0 < \mu < n$, $1 < p < \frac{n}{\mu}, \frac{1}{q} = \frac{1}{p} - \frac{\mu}{n}$ and $T_{\mu,\Omega}$ be defined as $T_{\mu,\Omega}(f)(x) = \int_{R^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}f(y)\mathrm{d}y.$ If there exists $r > p'$, such that $\Omega(x,z) \in L^\infty(R^n) \times L^r(S^{n-1})$, then $T_{\mu,\Omega}$ is bounded from $L^p(R^n)$ into $L^q(R^n)$. That is

$$\left\|T_{\mu,\Omega}(f)\right\|_{L^q(R^n)} \le C\|\Omega\|_{L^\infty(R^n)\times L^r(S^{n-1})}\|f\|_{L^p(R^n)}.$$

3. Proofs

3.1. Proof of Theorem 2.1.

Applying the Minkowski' inequality, we can get

$$\begin{aligned}
\mu^\rho_{\Omega,\boldsymbol{b},\alpha}(f)(x) &= \left(\int_0^\infty \left|F^\rho_{\boldsymbol{b},t}(f)(x)\right|^2 \frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty\left(\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\prod_{i=1}^m\left[b_i(x)-b_i(y)\right]f(y)\mathrm{d}y\right)^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}} \\
&\le \int_{R^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho}}\prod_{i=1}^m\left|b_i(x)-b_i(y)\right|\left|f(y)\right|\left(\int_{|x-y|}^\infty\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}}\mathrm{d}y \\
&\le C\prod_{i=1}^m\sup_{\substack{x,y\in R^n\\ x\ne y}}\frac{|b_i(x)-b_i(y)|}{|x-y|^{\beta_i}}\int_{R^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho}}|x-y|^\beta\times|f(y)|\frac{1}{|x-y|^{\rho+\alpha}}\mathrm{d}y \\
&= C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\int_{R^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\beta+\alpha}}|f(y)|\mathrm{d}y = \|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}T_{|\Omega|,\beta-\alpha}|f|(x).
\end{aligned}$$

By Lemma 2.2 , we have

$$\begin{aligned}
&\left\|\mu^\rho_{\Omega,\boldsymbol{b},\alpha}(f)\right\|_{L^q(R^n)} \\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\left\|T_{|\Omega|,\beta-\alpha}|f|\right\|_{L^q(R^n)} \\
&\le C\|\Omega\|_{L^\infty(R^n)\times L^r(S^{n-1})}\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{L^p(R^n)}.
\end{aligned}$$

This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2.

Noting that $r > \frac{n}{n-\beta+\alpha}$, we can choose l_1 such that $r' < l_1 < \frac{n}{\beta-\alpha}$. It is easy to see that $r > l_1'$. Next , we choose l_2 such that $\frac{1}{l_2} = \frac{1}{l_1} - \frac{\beta-\alpha}{n}$. It follows from

Theorem 2.1 that $\mu_{\Omega,\boldsymbol{b},\alpha}$ is bounded from $L^{l_1}(R^n)$ into $L^{l_2}(R^n)$. That is

$$\left\|\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} f\right\|_{L^{l_2}(R^n)} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\|f\|_{L^{l_1}(R^n)}. \tag{4}$$

By the atomic decomposition theory on Hardy type space, it suffices to prove that there is a constant $C>0$ such that for all $(p,l_1,\boldsymbol{b})$-atom the following holds

$$\left\|\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a\right\|_{L^q(R^n)} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}.$$

Without loss of generality we may assume that $\operatorname{supp} a\subset B=B(x_0,d)$. We write $B^*=8\sqrt{n}B=B(x_0,8\sqrt{n}d):=B(x_0,d')$. We split $\left\|\mu_{\Omega,\boldsymbol{b},\alpha} a\right\|_{L^q(R^n)}$ into two parts as follows:

$$\left\|\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a\right\|_{L^q(R^n)} \le \left(\int_{B^*}\left|\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a(x)\right|^q \mathrm{d}x\right)^{1/q} + \left(\int_{(B^*)^C}\left|\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a(x)\right|^q \mathrm{d}x\right)^{1/q} := I_1+I_2.$$

We can easily see that $l_2>q$. By (4) and the size condition of atom a, we have

$$I_1 \le \left(\int_{B^*}\left|\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a(x)\right|^{l_2}\mathrm{d}x\right)^{\frac{1}{l_2}}\left|B^*\right|^{\frac1q-\frac1{l_2}} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\left|B^*\right|^{\frac1q-\frac1{l_2}}\|a\|_{L^{l_1}(R^n)} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\left|B^*\right|^{\frac1q-\frac1{l_2}}|B|^{\frac1{l_1}-\frac1p} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}.$$

Next we estimate I_2. Let us consider $\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a(x)$:

$$\begin{aligned}\mu^{\rho}_{\Omega,\boldsymbol{b},\alpha} a(x) \le &\left(\int_0^{|x-x_0|+2d}\left(\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\prod_{i=1}^m\left[b_i(x)-b_i(y)\right]a(y)\mathrm{d}y\right)^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{1/2}\\ &+\left(\int_{|x-x_0|+2d}^{\infty}\left(\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\prod_{i=1}^m\left[b_i(x)-b_i(y)\right]a(y)\mathrm{d}y\right)^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{1/2}\\ &:= I_{21}+I_{22}.\end{aligned}$$

$|x-y|\sim|x-x_0|\sim|x-x_0|+2d$ for $y\in B(x_0,d)$, $x\in(B^*)^C$. By the mean value theorem, we have

$$\left||x-y|^{-2\rho-2\alpha}-\left(|x-x_0|+2d\right)^{-2\rho-2\alpha}\right|\le Cd\,|x-y|^{-2\rho-1-2\alpha}.$$

Thus, by the Minkowski's inequality for integrals,

$$\begin{aligned}|I_{21}| &\le \int_{R^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho}}\prod_{i=1}^m|b_i(x)-b_i(y)||a(y)|\left(\int_{|x-y|}^{|x-x_0|+2d}\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{1/2}\mathrm{d}y\\ &\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\int_B\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho}}|x-y|^{\beta}|a(y)|\times\left(\frac{1}{|x-y|^{2\rho+2\alpha}}-\frac{1}{\left(|x-x_0|+2d\right)^{2\rho+2\alpha}}\right)^{1/2}\mathrm{d}y\\ &\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\int_B\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho-\beta}}\frac{d^{\frac12}}{|x-y|^{\rho+\alpha+1/2}}|a(y)|\mathrm{d}y \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}d^{\frac12}|x-x_0|^{\beta-\alpha-n-\frac12}\int_B|\Omega(x,x-y)||a(y)|\mathrm{d}y.\end{aligned}$$

Applying the Hölder inequality and the size condition of a, we have

$$\begin{aligned}&\int_B|\Omega(x,x-y)||a(y)|\mathrm{d}y\\ &\le\left(\int_B|\Omega(x,x-y)|^r\mathrm{d}y\right)^{1/r}\left(\int_B|a(y)|^{r'}\mathrm{d}y\right)^{1/r'} \le\left(\int_{\frac12|x-x_0|\le|x-y|\le2|x-x_0|}|\Omega(x,x-y)|^r\mathrm{d}y\right)^{1/r}\left(\int_B|a(y)|^{l_1}\mathrm{d}y\right)^{1/l_1}|B|^{1/r'-1/l_1}\\ &\le\left(\int_{\frac12|x-x_0|}^{2|x-x_0|}t^{n-1}\int_{S^{n-1}}|\Omega(x,z')|^r\mathrm{d}\sigma(z')\mathrm{d}t\right)^{1/r}\times\|a\|_{L^{l_1}(R^n)}|B|^{1/r'-1/l_1} \le C\|\Omega\|_{L^\infty(R^n)\times L^r(S^{n-1})}|x-x_0|^{n/r}|B|^{1-1/r-1/p}.\end{aligned}$$

So we can get

$$\left|I_{21}\right| \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)}\|\Omega\|_{L^\infty\left(R^n\right)\times L^r\left(S^{n-1}\right)} d^{1/2-n/r-n/p+n}\left|x-x_0\right|^{\beta-\alpha-1/2+n/r-n}.$$

Noting that $r > \dfrac{n}{n-\beta+\alpha}$, we have

$$J_{21} := \left(\int_{(B*)^C}\left|I_{21}\right|^q\right)^{1/q} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)}\|\Omega\|_{L^\infty\left(R^n\right)\times L^r\left(S^{n-1}\right)} d^{1/2-n/r-n/p+n}\times\left(\int_{(B*)^C}\left|x-x_0\right|^{(\beta-\alpha-1/2+n/r-n)q}\mathrm{d}x\right)^{1/q}$$

$$\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)} d^{1/2-n/r-n/p+n}\left(\int_{d'}^\infty \rho^{(\beta-\alpha-1/2+n/r-n)q}\rho^{n-1}\mathrm{d}\rho\right)^{1/q} \le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)} d^{1/2-n/r-n/p+n} d^{n/r+n/p-n-1/2} = C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)}.$$

For I_{22}, we write

$$G_{\boldsymbol{b},t}^\rho(a)(x) := \int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\prod_{i=1}^m\left[b_i(x)-b_i(y)\right]a(y)\mathrm{d}y.$$

$$\begin{aligned}
G_{\boldsymbol{b},t}^\rho(a)(x) &= \sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}\left(\boldsymbol{b}(x)-\boldsymbol{b}(x_0)\right)_\sigma\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\times\left(\boldsymbol{b}(y)-\boldsymbol{b}(x_0)\right)_{\sigma'}a(y)\mathrm{d}y\\
&= \prod_{i=1}^m\left[b_i(x)-b_i(x_0)\right]\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}a(y)\mathrm{d}y\\
&\quad+\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}\left(\boldsymbol{b}(x)-\boldsymbol{b}(x_0)\right)_\sigma\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\times\left(\boldsymbol{b}(y)-\boldsymbol{b}(x_0)\right)_{\sigma'}a(y)\mathrm{d}y.
\end{aligned}$$

So I_{22} is dominated by

$$\begin{aligned}
I_{22} &\le \left(\int_{|x-x_0|+2d}^\infty\left|\prod_{i=1}^m\left[b_i(x)-b_i(x_0)\right]\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}a(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac12}\\
&\quad+\left(\int_{|x-x_0|+2d}^\infty\left|\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}\left(\boldsymbol{b}(x)-\boldsymbol{b}(x_0)\right)_\sigma\times\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}\left(\boldsymbol{b}(y)-\boldsymbol{b}(x_0)\right)_{\sigma'}a(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac12}\\
&:= K_1+K_2.
\end{aligned}$$

Now let us estimate K_1. By the vanishing condition of a, we have

$$\begin{aligned}
K_1 &= \left(\int_{|x-x_0|+2d}^\infty\left|\prod_{i=1}^m\left[b_j(x)-b_j(x_0)\right]\int_{|x-y|\le t}D(x,x_0,y)a(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac12}\\
&\le \prod_{i=1}^m\left|b_j(x)-b_j(x_0)\right|\int_{R^n}\left|D(x,x_0,y)\right|\left|a(y)\right|\left(\int_{|x-x_0|+2d}^\infty\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac12}\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)}\left|x-x_0\right|^\beta\int_B\left|D(x,x_0,y)\right|\left|a(y)\right|\frac{1}{\left(\left|x-x_0\right|+2d\right)^{\rho+\alpha}}\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta\left(R^n\right)}\left|x-x_0\right|^{\beta-\alpha-\rho}\int_B\left|D(x,x_0,y)\right|\left|a(y)\right|\mathrm{d}y.
\end{aligned}$$

where $D(x,x_0,y) = \dfrac{\Omega(x,x-y)}{|x-y|^{n-\rho}}-\dfrac{\Omega(x,x-x_0)}{|x-x_0|^{n-\rho}}.$

Since $\dfrac{n}{p}-n-1 < \dfrac{n}{p}-n-\beta < 0$, we get from

Hölder's inequality and Lemma 2.1,

$$
\begin{aligned}
&\left(\int_{(B*)^C}|K_1|^q\right)^{\frac{1}{q}}\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\left(\int_{(B*)^C}\left|\int_B|D(x,x_0,y)||x-x_0|^{\beta-\alpha-\rho}|a(y)|\mathrm{d}y\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\int_B|a(y)|\left(\int_{(B*)^C}|D(x,x_0,y)|^q|x-x_0|^{(\beta-\alpha-\rho)q}\mathrm{d}x\right)^{\frac{1}{q}}\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\int_B|a(y)|\sum_{j=0}^{\infty}\left[\int_{2^jd'<|x-x_0|\le 2^{j+1}d'}|D(x,x_0,y)|^q|x-x_0|^{(\beta-\alpha-\rho)q}\mathrm{d}x\right]^{\frac{1}{q}}\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\int_B|a(y)|\sum_{j=0}^{\infty}\left(2^jd'\right)^{\beta-\alpha-\rho}\left(\int_{2^jd'<|x-x_0|\le 2^{j+1}d'}|D(x,x_0,y)|^q\mathrm{d}x\right)^{\frac{1}{q}}\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\int_B|a(y)|\sum_{j=0}^{\infty}\left(2^jd'\right)^{\beta-\alpha-\rho+\frac{n}{q}-\frac{n}{r}}\left(\int_{2^jd'<|x-x_0|\le 2^{j+1}d'}|D(x,x_0,y)|^r\mathrm{d}x\right)^{\frac{1}{r}}\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\int_B|a(y)|\sum_{j=0}^{\infty}\left(2^jd'\right)^{\beta-\alpha-\rho+\frac{n}{q}-\frac{n}{r}}\left(2^jd'\right)^{\frac{n}{r}-(n-\rho)}\left[\frac{1}{2^j}+\int_{\frac{|y-x_0|}{2^{j-1}d'}}^{\frac{|y-x_0|}{2^{j-2}d'}}\frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right]\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}\int_B|a(y)|\sum_{j=0}^{\infty}\left(2^jd'\right)^{\frac{n}{p}-n}\left[\frac{1}{2^j}+\left(\frac{|y-x_0|}{2^{j-1}d'}\right)^{\beta}\int_{\frac{|y-x_0|}{2^{j-1}d'}}^{\frac{|y-x_0|}{2^{j-2}d'}}\frac{\omega_r(\delta)}{\delta^{1+\beta}}\mathrm{d}\delta\right]\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}d^{\frac{n}{p}-n}\int_B|a(y)|\sum_{j=0}^{\infty}\left[2^{j\left(\frac{n}{p}-n-1\right)}+2^{j\left(\frac{n}{p}-n-\beta\right)}\int_{\frac{|y-x_0|}{2^{j-1}d'}}^{\frac{|y-x_0|}{2^{j-2}d'}}\frac{\omega_r(\delta)}{\delta^{1+\beta}}\mathrm{d}\delta\right]\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}d^{\frac{n}{p}-n}\|a\|_{L^{l_1}(R^n)}|B|^{1-\frac{1}{l_1}}\left(1+\int_0^1\frac{\omega_r(\delta)}{\delta^{1+\beta}}\mathrm{d}\delta\right)\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}d^{\frac{n}{p}-n+\frac{n}{l_1}-\frac{n}{p}+n-\frac{n}{l_1}}\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta}.
\end{aligned}
$$

Now we estimate K_2. Applying Minkowski's inequality, the size condition of a, we obtain

$$
\begin{aligned}
K_2&\le C\int_{R^n}|a(y)|\left|\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}\left(\boldsymbol{b}(x)-\boldsymbol{b}(x_0)\right)_\sigma\left(\boldsymbol{b}(y)-\boldsymbol{b}(x_0)\right)_{\sigma'}\right|\times\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho}}\left(\int_{|x-x_0|+2d}^{\infty}\frac{\mathrm{d}t}{t^{2\rho+1+2\alpha}}\right)^{\frac{1}{2}}\mathrm{d}y\\
&\le C\int_B|a(y)|\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}\left|\left(\boldsymbol{b}(x)-\boldsymbol{b}(x_0)\right)_\sigma\right|\left|\left(\boldsymbol{b}(y)-\boldsymbol{b}(x_0)\right)_{\sigma'}\right|\times\frac{|\Omega(x,x-y)|}{|x-y|^{n-\rho}}\frac{1}{\left(|x-x_0|+2d\right)^{\rho+\alpha}}\mathrm{d}y\\
&\le C\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i\in\sigma}\sup_{x\ne x_0}\frac{|b_i(x)-b_i(x_0)|}{|x-x_0|^{\beta_i}}|x-x_0|^{\beta_\sigma}\prod_{i\in\sigma'}\sup_{x\ne x_0}\frac{|b_i(y)-b_i(x_0)|}{|y-x_0|^{\beta_i}}\times\int_B|y-x_0|^{\beta_{\sigma'}}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}}|a(y)|\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|x-x_0|^{\beta_\sigma}\int_B|y-x_0|^{\beta_{\sigma'}}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}}|a(y)|\mathrm{d}y\\
&\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|x-x_0|^{\beta_\sigma-n-\alpha}d^{\beta_{\sigma'}}\int_B|\Omega(x,x-y)||a(y)|\mathrm{d}y.
\end{aligned}
$$

So we have

$$
K_2\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\|\Omega\|_{L^\infty(R^n)\times L^r(S^{n-1})}\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}|x-x_0|^{\beta_\sigma-n-\alpha+\frac{n}{r}}d^{\beta_{\sigma'}+n-\frac{n}{r}-\frac{n}{p}}.
$$

Thus

$$\left(\int_{(B*)^C}|K_2|^q\right)^{\frac{1}{q}}$$
$$\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}\|\Omega\|_{L^\infty(R^n)\times L^r(S^{n-1})}d^{n-\frac{n}{r}-\frac{n}{p}}$$
$$\times\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}d^{\beta_{\sigma'}}\left(\int_{(B*)^C}|x-x_0|^{\left(\beta_\sigma-n-\alpha+\frac{n}{r}\right)q}\,\mathrm{d}x\right)^{\frac{1}{q}}$$
$$\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}d^{n-\frac{n}{r}-\frac{n}{p}}\sum_{j=0}^{m-1}\sum_{\sigma\in C_j^m}d^{\beta_{\sigma'}+\beta_\sigma-\alpha-n+\frac{n}{r}+\frac{n}{q}}$$
$$\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}.$$

So when $|x-x_0|>Cd$, we have

$$J_{22}:=\left(\int_{(B*)^C}|I_{22}|^q\right)^{\frac{1}{q}}\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)},$$
$$I_2\le J_{21}+J_{22}\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}.$$

Combining the estimates for I_1 and I_2, we have

$$\left\|\mu_{\Omega,\boldsymbol{b},\alpha}a\right\|_{L^q(R^n)}\le C\|\boldsymbol{b}\|_{\dot{\Lambda}_\beta(R^n)}.$$

This completes the proof of Theorem 2.2.

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