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Permanence and Globally Asymptotic Stability of Cooperative System Incorporating Harvesting

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ABSTRACT

The stability of a kind of cooperative models incorporating harvesting is considered in this paper. By analyzing the characteristic roots of the models and constructing suitable Lyapunov functions, we prove that nonnegative equilibrium points of the models are globally asymptotically stable. Further, the corresponding nonautonomous cooperative models have a unique asymptotically periodic solution, which is uniformly asymptotically stable. An example is given to illustrate the effectiveness of our results.

Keywords: Cooperative System; Equilibrium; Stability; Asymptotically Periodic Solution

1. Introduction

Permanence, stability and periodic solution for Lotka-Volterra models had been extensively investigated by many authors (see [1-8] and the references therein). Jorge Rebaza [1] had discussed the dynamic behaviors of predator-prey model with harvesting and refuge

$$\begin{cases} \dot{x} = x(1-x) - \dfrac{a(1-m)xy}{1+c(1-m)x} - H(x), \\ \dot{y} = y\left(-d + \dfrac{b(1-m)x}{1+c(1-m)x}\right), \end{cases} \quad (1)$$

he obtained that harvesting and refuge affected the stability of some coexistence equilibrium and periodic solutions of model (1), where $H(x)$ was a continuous threshold policy harvesting function. Motivated by Jorge's work, we consider the following cooperative system incorporating harvesting

$$\begin{cases} \dot{x} = x\left(r_1 - b_1 x - \dfrac{a_1 x}{y+k_1}\right) - Eqx, \\ \dot{y} = y\left(r_2 - b_2 y - \dfrac{a_2 y}{x+k_2}\right), \end{cases} \quad (2)$$

where x and y denote the densities of two populations at time t. The parameters $r_1, r_2, a_1, a_2, b_1, b_2, k_1, k_2, E, q$ are all positive constants.

Definition 1 [2] $f(t)$ is called asymptotically T-periodic function, if $f \in C(R_+, R)$ and it satisfies $f(t) = g(t) + \alpha(t)$, where $g(t)$ is continuous periodic function with periodic T and $\lim_{t\to+\infty} \alpha(t) = 0$.

We will discuss our problems in the region

$$R_+^2 = \{(x, y) | x \ge 0, y \ge 0\},$$

where $R_+ = [0, +\infty)$.

2. Permanence of System

Definition 2 [2] If there are positive constants $m, M > 0$ such that each positive solution $(x(t), y(t))$ of system (2) satisfies

$$0 < m \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M,$$

$$0 < m \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le M.$$

Then system (2) is persistent. If the system is not persistent, then system (2) is called non-persistent.

Lemma 1 If $r_1 > Eq, k_1 b_1 > a_1, k_2 b_2 > a_2$, then system (2) is persistent.

Proof. By the first equation of (2) and the comparison theorem, we get $\dot{x}(t) \le x[r_1 - b_1 x - Eq]$, it implies that

$$\limsup_{t\to+\infty} x(t) \le \frac{r_1 - Eq}{b_1} := A.$$

For any $\varepsilon > 0$, there exists a $T_1 > 0$, as $t > T_1$, it then follows

$$x(t) \le A + \varepsilon.$$

Similarly, we have $\limsup_{t\to+\infty} y(t) \le \frac{r_2}{b_2} := B$. By the discussion above, for any $\varepsilon > 0$, there exists a $T_2 > T_1$, as $t > T_2$, it yields that $y(t) \le B + \varepsilon$.

On the other hand, we have

$$\dot{x}(t) \ge x\left(r_1 - b_1 x - Eq - \frac{a_1(A+\varepsilon)}{k_1}\right),$$

$$\dot{y}(t) \ge y\left(r_2 - b_2 y - \frac{a_2(B+\varepsilon)}{k_2}\right).$$

By the comparison theorem, and letting $\varepsilon \to 0$, one gets that

$$\liminf_{t\to+\infty} x(t) \ge \frac{(r_1 - Eq)(k_1 b_1 - a_1)}{b_1^2 k_1} := C,$$

$$\liminf_{t\to+\infty} y(t) \ge \frac{r_2(k_2 b_2 - a_2)}{b_2^2 k_2} := D.$$

By Definition 2, system (2) is persistent. □

3. Equilibrium Points and Stability

If $r_1 > Eq$, then the equilibrium points of (2) are

$$H_0 = (0,0),$$

$$H_1 = \left(0, \frac{r_2 k_2}{a_2 + k_2 b_2}\right),$$

$$H_2 = \left(\frac{(r_1 - Eq)k_1}{a_1 + k_1 b_1}, 0\right),$$

$$H_3 = (x^*, y^*),$$

where

$$x^* = \frac{-(k_2 P - F) + \sqrt{(k_2 P - F)^2 + 4PQM}}{2P},$$
$$y^* = \frac{r_2(x^* + k_2)}{b_2 x^* + a_2 + k_2 b_2}, \tag{3}$$

$$P = r_2 b_1 + k_1 b_1 b_2 + a_1 b_2,$$

$$Q = r_1 - Eq,$$

$$F = r_2 Q + b_2 k_1 Q - k_1 a_2 b_1 - a_1 a_2,$$

$$M = r_2 k_2 + k_1 k_2 b_2 + a_2 k_1.$$

The general Jacobian matrix of (2) is given by

$$J = \begin{pmatrix} r_1 - Eq - 2b_1 x - \frac{2a_1 x}{y + k_1} & \frac{a_1 x^2}{(y+k_1)^2} \\ \frac{a_2 y^2}{(x+k_2)^2} & r_2 - 2b_2 y - \frac{2a_2 y}{x + k_2} \end{pmatrix}.$$

The characteristic equation of system (2) at H_0 is $(\lambda - r_1 + Eq)(\lambda - r_2) = 0$, this immediately indicates that H_0 is always unstable.

The characteristic equation of system (2) at H_1 is $(\lambda - r_1 + Eq)(\lambda + r_2) = 0$, by the condition $r_1 > Eq$, one then gets that H_1 is a saddle point.

The characteristic equation of system (2) at H_2 is $(\lambda + r_1 - Eq)(\lambda - r_2) = 0$, we derive that H_2 is a saddle point.

The characteristic equation of system (2) at H_3 takes the form

$$\lambda^2 + \left(b_1 x^* + \frac{a_1 x^*}{y^* + k_1} + b_2 y^* + \frac{a_2 y^*}{x^* + k_2}\right)\lambda$$
$$+ \left(b_1 x^* + \frac{a_1 x^*}{y^* + k_1}\right)\left(b_2 y^* + \frac{a_2 y^*}{x^* + k_2}\right)$$
$$- \frac{a_1 a_2 (x^* y^*)^2}{(y^* + k_1)^2 (x^* + k_2)^2} = 0,$$

it is easy to check that $\lambda_1 + \lambda_2 < 0$, $\lambda_1 \lambda_2 > 0$, then $\lambda_1 < 0$, $\lambda_2 < 0$, thus H_3 is locally asymptotically stable.

Theorem 1 *If* $k_1 b_1 > a_1$, $k_2 b_2 > a_2$,

$$b_1 + \frac{a_1}{y^* + k_1} > \frac{Ba_2}{2(C + k_2)(x^* + k_2)} + \frac{Aa_1}{2(D + k_1)(y^* + k_1)},$$

$$b_2 + \frac{a_2}{x^* + k_2} > \frac{Ba_2}{2(C + k_2)(x^* + k_2)} + \frac{Aa_1}{2(D + k_1)(y^* + k_1)},$$

then the positive equilibrium point H_3 *of system* (2) *is globally asymptotically stable, where* A, B, C, D *can be found in Lemma* 1.

Proof. Define a Lyapunov function

$$V(x,y) = x - x^* - x^* \ln\frac{x}{x^*} + y - y^* - y^* \ln\frac{y}{y^*},$$

it then yields that

$$\dot{V}(x,y)=(x-x^*)\left(r_1-b_1x-\frac{a_1x}{y+k_1}-Eq\right)$$
$$+(y-y^*)\left(r_2-b_2y-\frac{a_2y}{x+k_2}\right)$$
$$=(x-x^*)^2\left[-b_1-\frac{a_1}{y^*+k_1}\right]+a_1x\frac{(x-x^*)(y-y^*)}{(y+k_1)(y^*+k_1)}$$
$$+(y-y^*)^2\left[-b_2-\frac{a_2}{x^*+k_2}\right]+a_2y\frac{(x-x^*)(y-y^*)}{(x+k_2)(x^*+k_2)}$$
$$\le\left[-\left(b_1+\frac{a_1}{y^*+k_1}\right)+\frac{Ba_2}{2(C+k_2)(x^*+k_2)}+\frac{Aa_1}{2(D+k_1)(y^*+k_1)}\right](x-x^*)^2$$
$$+\left[-\left(b_2+\frac{a_2}{x^*+k_2}\right)+\frac{Ba_2}{2(C+k_2)(x^*+k_2)}+\frac{Aa_1}{2(D+k_1)(y^*+k_1)}\right](y-y^*)^2,$$

by the conditions of theorem 1, thus, $\dot{V}(x,y)<0$. The positive equilibrium point H_3 of system (2) is globally asymptotically stable. □

4. Existence and Uniqueness of Solutions

Next, we will discuss a nonautonomous system

$$\begin{cases}\dot{x}=x\left(r_1(t)-b_1(t)x-\dfrac{a_1(t)x}{y+k_1(t)}\right)-E(t)q(t)x,\\ \dot{y}=y\left(r_2(t)-b_2(t)y-\dfrac{a_2(t)y}{x+k_2(t)}\right),\end{cases}\tag{4}$$

where $r_i(t),a_i(t),b_i(t),k_i(t)\ (i=1,2),\quad E(t),q(t)$ are positive continuous bounded asymptotically periodic functions with period T. The initial data of (4) is given by

$$x(0)>0,y(0)>0.\tag{5}$$

The solution of (4) with initial data (5) is denoted by $X(t)=X(t,t_0,X(0))$, $X(0)=(x(0),y(0))$, $t_0\in R_+$. For a continuous function $f(t)$ defined on R_+, define

$$f^l=\inf_{t\in R_+}f(t)>0,\ f^u=\sup_{t\in R_+}f(t)<+\infty.$$

Definition 3 [2] If there exists a $B>0$, for any $t_0\ge 0$, $X(0)=(x(0),y(0))$, there exists a

$$T=T(t_0,X(0))>0$$

such that $|X(t)|\le B$ for $t\ge t_0+T$, then the solution $X(t)$ is called ultimately bounded.

Let us consider the following asymptotically periodic system

$$\dot{x}(t)=f(t,x_t)\tag{6}$$

where $x\in R_+^n,\ f:R_+\times C([-\tau,0],R_+^n)\to R_+^n$. Set

$$x_t(\theta)=x(t+\theta),|x|=\sum_{i=1}^n|x_{ik}|,\|\phi\|=\sup_{-\tau\le\theta\le 0}|\phi(\theta)|,$$
$$C_H=\{\phi\in C([-\tau,0],R_+^n),\|\phi\|<H\},$$
$$S_H=\{x\in R_+^n,|x|<H\}.$$

In order to discuss the existence and uniqueness of asymptotically periodic solution of system (6), we can consider the adjoint system

$$\dot{x}(t)=f(t,x_t),\dot{y}(t)=f(t,y_t).\tag{7}$$

Lemma 2 If

$$(r_1^l-E^uq^u)k_1^lb_1^l>a_1^u(r_1^u-E^lq^l)$$

and

$$r_2^lk_2^lb_2^l>a_2^ur_2^u,\quad r_1^u>E^lq^l,$$

then the solution of system (4) is ultimately boundedness.

Proof. By the first equation of system (4) and the comparison theorem, one gets that

$$\dot{x}(t)\le x(r_1^u-b_1^lx-E^lq^l),$$

it then implies that

$$\limsup_{t\to+\infty}x(t)\le\frac{r_1^u-E^lq^l}{b_1^l}:=M_1>0.$$

Similarly, we have

$$\limsup_{t\to+\infty}y(t)\le\frac{r_2^u}{b_2^l}:=M_2>0.$$

By the same discussion, one thus gets that

$$\dot{x}(t)\ge x\left(r_1^l-b_1^ux-E^uq^u-\frac{a_1^u(M_1+\varepsilon)}{k_1^l}\right),$$
$$\dot{y}(t)\ge y\left(r_2^l-b_2^uy-\frac{a_2^u(M_2+\varepsilon)}{k_2^l}\right).$$

Letting $\varepsilon\to 0$, we have

$$\liminf_{t\to+\infty}x(t)\ge\frac{(r_1^l-E^uq^u)k_1^lb_1^l-a_1^u(r_1^u-E^lq^l)}{b_1^uk_1^lb_1^l}$$
$$:=m_1>0,$$

$$\liminf_{t\to+\infty} y(t) \geq \frac{r_2^l k_2^l b_2^l - a_2^u r_2^u}{b_2^u k_2^l b_2^l} := m_2 > 0 .$$

By the Definition 3, the solution of system (4) is ultimately bounded. □

Lemma 3 [2] If $V(t,x,y) \in C(R_+ \times S_H \times S_H, R_+)$ satisfies the following conditions:

1) $a(|x-y|) \leq V(t,x,y) \leq b(|x-y|)$, where $a(r)$ and $b(r)$ are continuously positively increasing functions;

2) $|V(t,x_1,y_1) - V(t,x_2,y_2)| \leq l(|x_1-x_2| + |y_1-y_2|)$, where $l>0$ is a constant;

3) there exists a continuous non-increasing function $P(s)$, such that for $s > 0$, $P(s) > s$. And as $\theta \in [-\tau, 0]$,

$$P(V(t,\phi(0),\phi(0))) > V(t+\theta,\phi(\theta),\phi(\theta)),$$

it then follows that

$$\dot{V}(t,\phi(0),\phi(0)) \leq -\delta V(t,\phi(0),\phi(0)),$$

where $\delta > 0$ is a constant; furthermore, system (6) has a solution $\varsigma(t)$ for $t \geq t_0$ and satisfies $\|\varsigma_t\| \leq H$.

Then system (6) has a unique asymptotically periodic solution, which is uniformly asymptotically stable.

Theorem 2 *If conditions*

$$b_1^l + \frac{a_1^l}{M_2 + k_1^u} > \frac{a_2^u M_2}{(m_1 + k_2^l)^2}$$

and

$$b_2^l + \frac{a_2^l}{M_1 + k_2^u} > \frac{a_1^u M_1}{(m_2 + k_1^l)^2}$$

hold, the conditions of Lemma 2 *are satisfied, then system* (4) *has a unique asymptotically periodic solution, which is uniformly asymptotically stable.*

Proof. By Lemma 2, the solutions of system (4) is ultimately bounded. We consider the adjoint system

$$\begin{cases} \dot{x} = x\left(r_1(t) - b_1(t)x - \dfrac{a_1(t)x}{y + k_1(t)}\right) - E(t)q(t)x, \\ \dot{y} = y\left(r_2(t) - b_2(t)y - \dfrac{a_2(t)y}{x + k_2(t)}\right), \\ \dot{u} = u\left(r_1(t) - b_1(t)u - \dfrac{a_1(t)u}{v + k_1(t)}\right) - E(t)q(t)u, \\ \dot{v} = v\left(r_2(t) - b_2(t)v - \dfrac{a_2(t)v}{u + k_2(t)}\right), \end{cases} \quad (8)$$

Let

$$x^* = \ln x, y^* = \ln y, u^* = \ln u, v^* = \ln v$$

and (x,y,u,v) be the solution of (8). By the fact

$$|x-u| = |\exp x^* - \exp u^*| = \exp \eta^* |x^* - u^*| = \eta |x^* - u^*|,$$

$$|y-v| = |\exp y^* - \exp v^*| = \exp \xi^* |y^* - v^*| = \xi |y^* - v^*|,$$

where η lies between x and u, ξ lies between y and v, it then follows

$$\begin{aligned} m_1 |x^* - u^*| &\leq |x-u| \leq M_1 |x^* - u^*|, \\ m_2 |y^* - v^*| &\leq |y-v| \leq M_2 |y^* - v^*|. \end{aligned} \quad (9)$$

Define Lyapunov function $W(t) = |x^* - u^*| + |y^* - v^*|$, taking

$$a(r) = b(r) = |x^* - u^*| + |y^* - v^*|,$$

By suing of the inequality $\||a| - |b|\| \leq |a-b|$, it is easy to check that 1) and 2) of Lemma 3 are valid. Computing the derivative of $W(t)$ along the solution of system (8), by (9) and $m_1 \leq x,u \leq M_1, m_2 \leq y,v \leq M_2$, we get that

$$\begin{aligned} \dot{W}(t) &= \left[\frac{\dot{x}}{x} - \frac{\dot{u}}{u}\right] sign|x-u| + \left[\frac{\dot{y}}{y} - \frac{\dot{v}}{v}\right] sign|y-v| \\ &\leq -\left[b_1(t) + \frac{a_1(t)}{y + k_1(t)}\right]|x-u| \\ &\quad + \frac{a_1(t)u}{[v + k_1(t)][y + k_1(t)]}|y-v| \\ &\quad - \left[b_2(t) + \frac{a_2(t)}{x + k_2(t)}\right]|y-v| \\ &\quad + \frac{a_2(t)v}{[u + k_2(t)][x + k_2(t)]}|x-u| \\ &\leq -\left[b_1^l + \frac{a_1^l}{M_2 + k_1^u}\right]|x-u| + \frac{a_1^u M_1}{(m_2 + k_1^l)^2}|y-v| \\ &\quad - \left[b_2^l + \frac{a_2^l}{M_1 + k_2^u}\right]|y-v| + \frac{a_2^u M_2}{(m_1 + k_2^l)^2}|x-u| \\ &\leq -\left[b_1^l + \frac{a_1^l}{M_2 + k_1^u} - \frac{a_2^u M_2}{(m_1 + k_2^l)^2}\right] m_1 |x^* - u^*| \\ &\quad - \left[b_2^l + \frac{a_2^l}{M_1 + k_2^u} - \frac{a_1^u M_1}{(m_2 + k_1^l)^2}\right] m_2 |y^* - v^*| \\ &:= -Q_1 |x^* - u^*| - Q_2 |y^* - v^*|, \end{aligned}$$

taking $\delta = \min\{Q_1, Q_2\} > 0$, it yields $\dot{W}(t) \leq -\delta W(t)$, then, system (4) has a unique positive asymptotically periodic solution, which is uniformly asymptotically stable. □

5. Examples and Numerical Simulations

Now, let us consider a autonomous cooperative system incorporating harvesting

$$\dot{x} = x\left(3 - x - \frac{0.2x}{y+1}\right) - x,\ \ \dot{y} = y\left(2 - y - \frac{0.5y}{x+1}\right), \quad (10)$$

it is easy to check that

$$A = 2, B = 2, C = 1.6, D = 1, P = 3.2, Q = 2, F = 5.4\ ,$$

$$M = 3.5, x^* = 1.8622, y^* = 1.7026\ ,$$

$$b_1 + \frac{a_1}{y^* + k_1} = 1.0740\ ,\ \ b_2 + \frac{a_2}{x^* + k_2} = 1.1747\ ,$$

$$\frac{Ba_2}{2(C + k_2)(x^* + k_2)} + \frac{Aa_1}{2(D + k_1)(y^* + k_1)} = 0.1042\ ,$$

the conditions of Theorem 1 are valid, then the positive equilibrium point $H_3 = (1.8622, 1.7026)$ of system (2) is globally asymptotically stable in **Figures 1** and **2**.

6. Conclusions

By analyzing the characteristic roots of a kind of cooperative models (2) incorporating harvesting, the stability of positive equilibrium point H_3 to model (2) is obtained by constructing a suitable Lyapunov function. Our results have shown that the harvesting coefficient Eq affects the stability and the existence of equilibrium point to model (2).

The related non-autonomous asymptotically periodic cooperative model (4) has been discussed later. Under some conditions, which also depend on model parameters (see Theorem 2), model (4) has a unique asymptotically periodic solution $x(t), y(t)$, which is uniformly asymptotically stable. Example model (10) shows the effectiveness of our results.

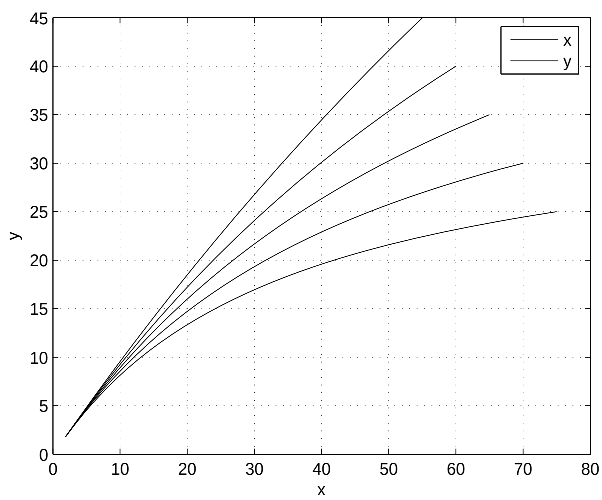

Figure 1. Positive equilibrium point H_3 of (2) is globally asymptotically stable.

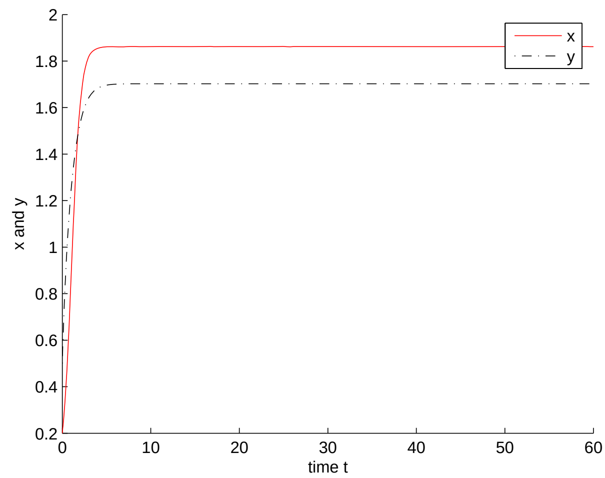

Figure 2. Solution of (2) is uniformly asymptotically stable.

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