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A Modified Wallman Method of Compactification

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ABSTRACT

Closed $\wp_x$- and basic closed C^*_D-filters are used in a process similar to Wallman method for compactifications of the topological spaces $\mathbf{Y}$, of which, there is a subset D of $\mathbf{C}^*(\mathbf{Y})$ containing a non-constant function, where $\mathbf{C}^*(\mathbf{Y})$ is the set of bounded real continuous functions on $\mathbf{Y}$. An arbitrary Hausdorff compactification (Z,h) of a Tychonoff space $\mathbf{X}$ can be obtained by using basic closed C^*_D-filters from $D=\{{}^\circ f\circ h \mid {}^\circ f\in{}^\circ D=C(Z)\}$ in a similar way, where $C(Z)$ is the set of real continuous functions on Z.

Keywords: Closed $\wp_x$-Filter; Open and Closed C^*_D-Filter Bases; Basic Open and Closed C^*_D-Filters; Compactification; Stone-Čech and Wallman Compactifications

1. Introduction

Throughout this paper, $[T]^{<\omega}$ will denote the collection of all finite subsets of the set T. For the other notations and the terminologies in general topology which are not explicitly defined in this paper, the readers will be referred to the reference [1].

Let $\mathbf{C}^*(\mathbf{Y})$ be the set of bounded real continuous functions on a topological space $\mathbf{Y}$. For any subset D of $\mathbf{C}^*(\mathbf{Y})$, we will show in Section 2 that there exists a unique r_f in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ for each f in D so that for any

$$H\in[D]^{<\omega},\varepsilon>0,\phi\neq\cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon,r_f+\varepsilon\right)\right)\subset\cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon,r_f+\varepsilon\right]\right).$$

Let K be the set

$$\left\{\cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon,r_f+\varepsilon\right]\right) \mid \cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon,r_f+\varepsilon\right]\right)\neq\phi \text{ for any } H\in[D]^{<\omega},\varepsilon>0\right\}$$

and let V be the set

$$\left\{\cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon,r_f+\varepsilon\right)\right) \mid \cap_{f\in H}f^{-1}((r_f-\varepsilon,r_f+\varepsilon))\neq\phi \text{ for any } H\in[D]^{<\omega},\varepsilon>0\right\}$$

K and V are called a ***closed C^*_D-filter base*** and an ***open C^*_D-filter base on*** $\mathbf{Y}$, respectively. A closed filter (or an open filter) on $\mathbf{Y}$ generated by a K (or a V) is called a ***basic closed C^*_D-filter*** (or a ***basic open C^*_D-filter***), denoted by $\mathcal{E}$ (or Å). If $r_f=f(x)$ for all f in D at some x in $\mathbf{Y}$, then K, V, $\mathcal{E}$ and Å are denoted by K_x, V_x, $\mathcal{E}_x$ and Å$_x$, respectively. Let $\mathbf{Y}$ be a topological space, of which, there is a subset D of $\mathbf{C}^*(\mathbf{Y})$ containing a non-constant function. A compactification $(Y^w,\mathfrak{I})$ of $\mathbf{Y}$ is obtained by using closed $\wp_x$- and basic closed C^*_D-filters in a process similar to the Wallman method, where $Y^w=Y_E\cup Y_F$, Y_E is the set $\{\mathsf{N}_x \mid \mathsf{N}_x$ is a closed $\wp_x$-filter, x is in $\mathbf{Y}\}$, Y_F is the set of all basic closed C^*_D-filter that does not converge in $\mathbf{Y}$, $\mathfrak{I}$ is the topology induced by the base $\tau=\{F^* \mid F$ is a nonempty closed set in $\mathbf{Y}\}$ for the closed sets of Y^w and F^* is the set of all $\mathfrak{C}$ in Y^w such that $F\cap T\neq\phi$ for all T in $\mathfrak{C}$. Similarly, an arbitrary Hausdorff compactification $(\mathbf{Z},\mathrm{h})$ of a Tychonoff space $\mathbf{X}$ can be obtained by using the basic closed C^*_D-filters on $\mathbf{X}$ from $D=\{{}^\circ f\circ h \mid {}^\circ f\in{}^\circ D\}$, where ${}^\circ D$ is the set $\mathbf{C}^*(\mathbf{Z})$.

2. Open and Closed C^*_D-Filter Bases, Basic Open and Closed C^*_D-Filters

For an arbitrary topological space $\mathbf{Y}$, let D be a subset of $\mathbf{C}^*(\mathbf{Y})$.

Theorem 2.1 *Let $\mathcal{F}$ be a filter on* **Y**. *For each f in D there exists a r_f in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ such that*

$$f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right)\cap F\neq\phi$$

for any F in $\mathcal{F}$ and any $\varepsilon>0$ (*See* **Thm. 2.1** *in* [2, p.1164]).

Proof. If the conclusion is not true, then there is an f in D such that for each r_t in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ there exist an F_t in $\mathcal{F}$ and an $\varepsilon_t>0$ such that

$$F_t\cap f^{-1}\left(\left(r_t-\varepsilon_t, r_t+\varepsilon_t\right)\right)=\phi.$$

Since $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ is compact and $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ is contained in

$$\cup\left\{\left(r_t-\varepsilon_t, r_t+\varepsilon_t\right)\mid r_t \text{ is in } \mathrm{Cl}(\mathrm{f}(\mathbf{Y}))\right\},$$

there exist $r_1,\cdots,r_n$ in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ such that **Y** is contained in

$$\cup\left\{f^{-1}\left(\left(r_i-\varepsilon_i, r_i+\varepsilon_i\right)\right)\mid i=1,\cdots,n\right\}.$$

Let $F_\circ=\cap\left\{F_i\mid i=1,\cdots,n\right\}$, then $F_\circ$ is in $\mathcal{F}$ and

$$F_\circ\subseteq\cup\left\{F_i\cap f^{-1}\left(\left(r_i-\varepsilon_i, r_i+\varepsilon_i\right)\right)\mid i=1,\cdots,n\right\}=\phi,$$

contradicting that ϕ is not in $\mathcal{F}$.

Corollary 2.2 *Let $\mathcal{F}$* (*or* Q) *be a closed* (*or an open*) *ultrafilter on* **Y**. *For each f* in D, *there exists a unique r_f in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ such that* (1) *for any $H\in[D]^{<\omega}$, any $\varepsilon>0$,*

$$\cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right)\in\mathcal{F}$$

$$\left(\text{or }\cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right)\in \mathrm{Q}\right)$$

and (2) *for any $H\in[D]^{<\omega}$, any $\varepsilon>0$,*

$$\cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right)\neq\phi$$

$$\left(\text{or }\cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right)\neq\phi\right).$$

(*See* **Cor. 2.2 & 2.3** *in* **[2**, p.1164**]**.)

Therefore, for a given closed ultrafilter $\mathcal{F}$ (or open ultrafilter Q), there exists a unique r_f in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$ for each f in D such that for any $H\in[D]^{<\omega},\varepsilon>0$,

$$\cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right)\neq\phi$$

$$\left(\text{or }\cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right)\neq\phi\right).$$

Let K be the set

$$\left\{\cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right)\mid \cap_{f\in H}f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right)\neq\phi \text{ for any } H\in[D]^{<\omega},\varepsilon>0\right\}$$

and let V be the set

$$\left\{\cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right)\mid \cap_{f\in H}f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right)\neq\phi \text{ , for any } H\in[D]^{<\omega},\varepsilon>0\right\}$$

K and V are called a ***closed*** and an ***open C*$_D$-filter bases***, respectively. If for all f in D, $\mathrm{r_f}=\mathrm{f}(\mathrm{x})$ for some x in **Y**, then K and V are called the ***closed*** and ***open C*$_D$-filter bases at x***, denoted by K_x and V_x, respectively. Let $\mathcal{E}$ and $\mathcal{E}_x$ (or Å and Å$_x$) be the closed (or open) filters generated by K and K_x (or V and V_x), respectively, then $\mathcal{E}$ and $\mathcal{E}_x$ (or Å and Å$_x$) are called a ***basic closed C*$_D$-filter*** and the ***basic closed C*$_D$-filter at x*** (or a ***basic open C*$_D$-filter*** and the ***basic open C*$_D$-filter at x***), respectively.

Corollary 2.3 *Let $\mathcal{F}$ and* Q *be a closed and an open ultrafilters on a topological space* **Y**, *respectively. Then there exist a unique basic closed* **C***$_D$*-filter $\mathcal{E}$ and a unique basic open* **C***$_D$*-filter* Å *on* **Y** *such that $\mathcal{E}$* is contained in $\mathcal{F}$ and Å is contained in Q.

3. A Closed $\wp_x$-Filter and a Modified Wallman Method of Compactification

Let **Y** be a topological space, of which, there is a subset D of $\mathbf{C}^*(\mathbf{Y})$ containing a non-constant function. For each x in **Y**, let N_x be the union of $\{\{\mathrm{x}\}\}$ and $\mathcal{E}_x$, if V_x is an open nhood filter base at x; let N_x be the union of $\{\{\mathrm{x}\}\}$ and $\{F\mid F \text{ is closed, } x \text{ is in } F\}$, if V_x is not an open nhood filter base at x. For each x in **Y**, N_x is a $\wp$-filter with $\wp$ being N_x. (See 12**E**. in **[1**, p.82**]** for definition and convergence). N_x is called a closed ***$\wp_x$-filter***. It is clear that K_x is contained in $\mathcal{E}_x$ and $\mathcal{E}_x$ is contained in N_x, N_x converges to x for each x in **Y.** Let Y_E be the set of all N_x, x in **Y**. Let Y_F be the set of all basic closed ***C*$_D$***-filter $\mathcal{E}$ that does not converge in **Y** and let $Y^w=Y_E\cup Y_F$.

Definition 3.4 *For each nonempty closed set* ***F*** *in* **Y**, *let* **F*** *be the set of* $\mathfrak{C}$ *in* Y^w *such that the intersection of* ***F*** *and* **T** *is not an empty set for all* **T** in $\mathfrak{C}$.

From the **Def. 3.4**, the following **Cor. 3.5** can be readily proved. We omit its proofs.

Corollary 3.5 *For a closed set* ***F*** *in* **Y**, (*i*) *x is in* ***F*** *if* N_x *is in* **F***; (*ii*) ***F*** is equal to **Y** *if* **F*** is equal to Y^w; (*iii*) *if* ***F*** *is in* $\mathfrak{C}$, *then* $\mathfrak{C}$ *is in* **F***; (*iv*) $\mathfrak{C}$ *is in* $\left(Y^w-\mathbf{F}^*\right)$ *if there is a* **T** *in* $\mathfrak{C}$ *such that* **T** *is contained in* **Y** – **F**.

Lemma 3.6 *For any two nonempty closed sets E and F in* **Y**,

(*i*) $E\subseteq F\Leftrightarrow E^*\subseteq F^*$,

(*ii*) $(E\cap F)^*\subseteq(E^*\cap F^*)$,

(*iii*) $(E\cup F)^*=(E^*\cup F^*)$.

Proof. (i) For [⇐]: If $E\not\subset F$, pick an x in $E-F$, by

Cor. 3.5 (i), N_x is in E^* and N_x is not in F^*; *i.e.*, $E^* \not\subset F^*$. For ($\Rightarrow$) is obvious. (ii) is clear from (i). (iii) For [$\subseteq$]: If $\mathfrak{C}$ belongs to $(E \cup F)^*$ and does not belong $E^* \cup F^*$, then pick T_1, T_2 in $\mathfrak{C}$ such that

$$E \cap T_1 = F \cap T_2 = \phi.$$

Since $T_1 \cap T_2$ is in $\mathfrak{C}$ and

$$(E \cup F) \cap (T_1 \cap T_2) \subset (E \cap T_1) \cup (F \cap T_2) = \phi.$$

Thus, $\mathfrak{C}$ does not belong to $(E \cup F)^*$, contradicting the assumption. For [$\supseteq$] is obvious from (i).

Proposition 3.7 $\tau = \{F^* | F$ is a nonempty closed set in $\mathbf{Y}\}$ *is a base for the closed sets of* Y^w.

Proof. Let $\mathcal{B}$ be the set $\{Y^w - F^* \mid F^* \in \tau\}$. We show that $\mathcal{B}$ is a base for Y^w. For (a) of **Thm. 5.3** in [**1**, p.38], if $\mathfrak{C} \in Y^w$, then there exist an f in D, a $\delta > 0$ such that

$$S = f^{-1}\left(\left[r_f - \delta, r_f + \delta\right]\right) \in \mathsf{K} \subset \mathcal{E} \subseteq \mathfrak{C}$$

and

$$E = \mathbf{Y} - f^{-1}\left(\left[r_f - 2\delta, r_f + 2\delta\right]\right) \neq \phi,$$

otherwise, if for all f in D, all $\delta > 0$, $E = \phi$, then for all f in D, $\mathrm{f}(\mathbf{Y}) = \{r_f\}$, contradicting that D contains a non-constant function. Thus $E \neq \phi$, E is closed, S is in $\mathfrak{C}$ and $S \cap E = \phi$ imply that $\mathfrak{C}$ is in $Y^w - E^*$. So,

$$X^w = \cup\left\{\left(Y^w - E^*\right) \mid E^* \in \tau\right\}.$$

For (b) of **Thm. 5.3**, if $\mathfrak{C}$ belongs to

$$\left(Y^w - E^*\right) \cap \left(Y^w - F^*\right),$$

then $E \cup F$ is closed, $(E \cup F)^* \in \tau$ and

$$\left(Y^w - E^*\right) \cap \left(Y^w - F^*\right) = Y^w - (E \cup F)^*$$

is in $\mathcal{B}$. Thus, $\mathfrak{C}$ is in

$$Y^w - (E \cup F)^* \subseteq \left(Y^w - E^*\right) \cap \left(Y^w - F^*\right).$$

Equip Y^w with the topology $\mathfrak{I}$ induced by τ. For each $f \in D$, define f*: $Y^w \to \mathbf{R}$ by $f^*(\mathfrak{C}) = r_f$, if

$$f^{-1}\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right) \in \mathsf{K} \subset \mathcal{E} \subseteq \mathfrak{C}$$

for all $\varepsilon > 0$. Since (i) if $\mathfrak{C}$ is equal to N_x for some N_x in Y_E, then

$$f^{-1}\left(\left[f(x) - \varepsilon, f(x) + \varepsilon\right]\right)$$

is in N_x for all $\varepsilon > 0$, (ii) if $\mathfrak{C}$ is $\mathcal{E}$ which is in Y_F, then

$$f^{-1}\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right)$$

is in $\mathcal{E}$ for all $\varepsilon > 0$, (iii) by **Cor. 2.2**, the r_f is unique for each f in D and (iv) the K that is contained in $\mathfrak{C}$ is unique. Thus, f^* is well-defined for each f in D. For all f in D, all x in $\mathbf{Y}$,

$$f^{-1}\left(\left[f(x) - \varepsilon, f(x) + \varepsilon\right]\right)$$

is in N_x for all $\varepsilon > 0$, thus $f^*(\mathsf{N}_x)$ is equal to $f(x)$ for all f in D and all x in $\mathbf{Y}$.

Lemma 3.8 *For each f in* D, *let r be in* $Cl(f(\mathbf{Y}))$, *then*

(*i*) $\left(f^{-1}\left([r - \delta, r + \delta]\right)\right)^* \subseteq f^{*-1}\left((r - \varepsilon, r + \varepsilon)\right)$

and

$(ii)\, f^{*-1}\left((r - \varepsilon, r + \varepsilon)\right) \subseteq \left(f^{-1}\left([r - \varepsilon, r + \varepsilon]\right)\right)^*$

for any $\varepsilon > \delta > 0$.

Proof. (i): If $\mathfrak{C}$ is in $\left(f^{-1}\left([r - \delta, r + \delta]\right)\right)^*$ and $f^*(\mathfrak{C})$ is t_f, then

$$f^{-1}\left([r - \delta, r + \delta]\right) \cap f^{-1}\left(\left[t_f - \gamma, t_f + \gamma\right]\right) \neq \phi$$

for all $\gamma > 0$, where $f^{-1}\left(\left[t_f - \gamma, t_f + \gamma\right]\right) \in \mathsf{K} \subset \mathfrak{C}$ for all $\gamma > 0$. Thus,

$$[r - \delta, r + \delta] \cap \left[t_f - \gamma, t_f + \gamma\right] \neq \phi$$

for all $\gamma > 0$; *i.e.*, $\mathrm{f}^*(\mathfrak{C})$ is

$$t_f \in [r - \delta, r + \delta] \subseteq (r - \varepsilon, r + \varepsilon),$$

so $\mathfrak{C}$ is in $f^{*-1}\left((r - \varepsilon, r + \varepsilon)\right)$. For (ii): If $\mathfrak{C}$ is in $f^{*-1}\left((r - \varepsilon, r + \varepsilon)\right)$ and $\mathrm{f}^*(\mathfrak{C})$ is t_f, then

$$t_f \in (r - \varepsilon, r + \varepsilon).$$

Pick a $\delta > 0$ such that

$$\left[t_f - \delta, t_f + \delta\right] \subset [r - \varepsilon, r + \varepsilon],$$

then

$$f^{-1}\left(\left[t_f - \delta, t_f + \delta\right]\right) \subset f^{-1}\left([r - \varepsilon, r + \varepsilon]\right).$$

Since

$$f^{-1}\left(\left[t_f - \delta, t_f + \delta\right]\right) \in \mathsf{K} \subset \mathfrak{C},$$

thus $f^{-1}\left([r - \varepsilon, r + \varepsilon]\right) \in \mathfrak{C}$. By **Cor. 3.5** (iii), $\mathfrak{C}$ is in $\left(f^{-1}\left([r - \varepsilon, r + \varepsilon]\right)\right)^*$.

Proposition 3.9 *For each f in* D, *f* is a bounded real continuous function on* Y^w.

Proof. For each f in D and each $\mathfrak{C}$ in Y^w, $\mathrm{f}^*(\mathfrak{C})$ is in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$. Thus $f^*(Y^w)$ is contained in $\mathrm{Cl}(\mathrm{f}(\mathbf{Y}))$; *i.e.*, f^* is bounded on Y^w. For the continuity of f^*: If $\mathfrak{C}$ is in Y^w and $f^*(\mathfrak{C})$ is t_f. We show that for any $\varepsilon > 0$, there is a E^* in τ such that $\mathfrak{C}$ is in

$$U = Y^w - E^* \subset f^{*-1}\left(\left(t_f - \varepsilon, t_f + \varepsilon\right)\right).$$

Let

$$E = f^{-1}\left(\left(-\infty, t_f - \varepsilon/2\right]\right) \cup f^{-1}\left(\left[t_f + \varepsilon/2, \infty\right)\right)$$

and $U = Y^w - E^*$. Since

$$P = f^{-1}\left(\left[t_f - \varepsilon/3, t_f + \varepsilon/3\right]\right) \in \mathsf{K} \subset \mathfrak{C}$$

and $P \subset Y - E$, by **Cor. 3.5** (iv), $\mathfrak{C} \in U$. Next, for any $\mathfrak{C}_s$ in U, if $\mathfrak{C}_s \neq \mathsf{N}_x$ for all x in **Y**, by **Cor. 3.5** (iv), pick a T in $\mathfrak{C}_s$ such that

$$T \subset Y - E \subset f^{-1}\left(\left[t_f - \varepsilon/2, t_f + \varepsilon/2\right]\right) = S,$$

then S is in $\mathfrak{C}_s$. By **Cor. 3.5** (iii) and **Lemma 3.8** (i), $\mathfrak{C}_s$ is in $S^* \subset f^{*-1}\left(\left(t_f - \varepsilon, t_f + \varepsilon\right)\right)$. If $\mathfrak{C}_s$ is N_x for some x in **Y**, by **Cor. 3.5** (i), N_x in U if $x \notin E$, thus

$$f^*(\mathsf{N}_x) = f(x) \in \left(t_f - \varepsilon/2, t_f + \varepsilon/2\right);$$

i.e., $\mathfrak{C}_s$ is N_x which is in $f^{*-1}\left(\left(t_f - \varepsilon, t_f + \varepsilon\right)\right)$.

Lemma 3.10 *Let k: $\mathbf{Y} \to Y^w$ be defined by $k(x) = \mathsf{N}_x$. Then,* (*i*) *k is an embedding from* **Y** *into Y^w;* (*ii*) *for all f in D, $f^* \circ k = f$; and* (*iii*) *$k(\mathbf{Y})$ is dense in Y^w.*

Proof. (i) By the setting, $\mathsf{N}_x = \mathsf{N}_y$ if $x = y$. Thus k is well-defined and one-one. Let k^{-1} be a function from $k(\mathbf{Y})$ into **Y** defined by $k^{-1}(k(x)) = x$. To show the continuity of k and k^{-1}, for any F^* in τ, (a): x is in

$$k^{-1}\left(\left[\left(Y^w - F^*\right)\right] \cap k(\mathbf{Y})\right)$$

iff (b): $k(x) = \mathsf{N}_x$ is in $\left(Y^w - F^*\right)$. By **Cor. 3.5** (i), (b) iff (c): x is not in F. So,

$$\mathbf{Y} - F = k^{-1}\left[\left(Y^w - F^*\right) \cap k(\mathbf{Y})\right];$$

i.e.,

$$k(Y - F) = k(\mathbf{Y}) \cap \left(Y^w - F^*\right).$$

So, k and k^{-1} are continuous. (ii) is obvious. (iii) For any F^* in τ such that $Y^w - F^* \neq \phi$, pick a $\mathfrak{C}$ in $Y^w - F^*$. By **Cor. 3.5** (iv), there is a T in $\mathfrak{C}$ such that $T \subset Y - F$. Pick an x in T, by **Cor. 3.5** (i), $k(x) = \mathsf{N}_x$ which is not in F^*, so $\mathsf{N}_x = k(x)$ is in both $k(\mathbf{Y})$ and $\left(Y^w - F^*\right)$; *i.e.*, $k(\mathbf{Y}) \cap \left(Y^w - F^*\right) \neq \phi$. Thus, $k(\mathbf{Y})$ is dense in Y^w.

Let $D^* = \{f^* \mid f \in D\}$. Then $D^* \subseteq \mathbf{C}^*\left(Y^w\right)$. Let

$$\mathsf{K}^* = \left\{\cap_{f^* \in H^*} f^{*-1}\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right) \mid \cap_{f^* \in H^*} f^{*-1}\right.$$
$$\left.\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right) \neq \phi \text{ for any } H^* \in [D^*]^{<\omega}, \varepsilon > 0\right\}$$

be a closed $\boldsymbol{C^*_{D^*}}$-filter base on Y^w and let $\mathcal{E}^*$ be the basic closed $\boldsymbol{C^*_{D^*}}$-filter on Y^w generated by K^*. Since k and k^{-1} are one-one, $f^* \circ k = f$ for all f in D and $k(\mathbf{Y})$ is dense in Y^w, so

$$\cap_{f^* \in H^*} f^{*-1}\left(\left(r_f - \varepsilon, r_f + \varepsilon\right)\right) \cap k(\mathbf{Y})$$
$$= k\left(\cap_{f \in H} f^{-1}\left(\left(r_f - \varepsilon, r_f + \varepsilon\right)\right)\right)$$

for any $H^* \in [D^*]^{<\omega}$, $H = \{f \mid f^* \in H^*\}$ (or any $H \in [D]^{<\omega}$, $H^* = \{f^* \mid f \cup H\}$ and all $\varepsilon > 0$. Thus,

$$\cap_{f^* \in H^*} f^{*-1}\left(\left(r_f - \varepsilon, r_f + \varepsilon\right)\right) \neq \phi$$

iff

$$\cap_{f \in H} f^{-1}\left(\left(r_f - \varepsilon, r_f + \varepsilon\right)\right) \neq \phi$$

and

$$\cap_{f^* \in H^*} f^{*-1}\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right) \neq \phi$$

iff

$$\cap_{f \in H} f^{-1}\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right) \neq \phi$$

for any $H^* \in [D^*]^{<\omega}$, $H = \{f \mid f^* \in H^*\}$ (or any $H \in [D]^{<\omega}$, $H^* = \{f^* \mid f \in H\}$ and all $\varepsilon > 0$. Therefore, if the K^* or $\mathcal{E}^*$ defined as above is well-defined, so is K or $\mathcal{E}$ defined as in Section 2 well-defined and *vice versa*. If K^* or $\mathcal{E}^*$ is given, then K or $\mathcal{E}$ is called the ***closed C^*_D-filter base*** or the ***basic closed C^*_D-filter on* Y *induced by*** K^* or $\mathcal{E}^*$ and *vice versa*.

Lemma 3.11 *Let $\mathcal{E}$ be a basic closed $\boldsymbol{C^*_D}$-filter on* **Y** *defined as in* Section **2**. *If $\mathcal{E}$ converges to a point x in* **Y**, *then* (*i*) *$r_f = f(x)$ for all f in D; i.e. $\mathcal{E} = \mathcal{E}_x$,* (*ii*) *$\mathsf{V}_x$ is an open nhood base at x in* **Y** *and* (*iii*)

$$\mathrm{V}^*_{k(x)} = \left\{\cap_{f^* \in H^*} f^{*-1}\left(\left(f(x) - \varepsilon, f(x) + \varepsilon\right)\right) \mid\right.$$
$$\left. H^* \in [D^*]^{<\omega}, H = \{f \mid f^* \in H^*\}, \varepsilon > 0\right\}$$

is an open nhood base at k(*x*) *in Y^w.*

Proof. If $\mathcal{E}$ converges to x in **Y**, (i): for each $f \in D$,

$$x \in f^{-1}\left(\left[r_f - \varepsilon, r_f + \varepsilon\right]\right) \in \mathsf{K} \subset \mathfrak{C}$$

for all $\varepsilon > 0$, thus $f(x) = r_f$; *i.e.*, $\mathcal{E} = \mathcal{E}_x$. (ii): Since $\mathcal{E}$ converges to x in **Y**, for any open nhood U of x, there is

$$E = \cap_{f \in H} f^{-1}\left(\left[f(x) - \delta, f(x) + \delta\right]\right) \in \mathsf{K}_x$$

which is contained in $\mathcal{E}_x = \mathcal{E}$ for some $H \in [D]^{<\omega}, \delta > 0$ such that $\in E \subset U$. Since x is in

$$\mathbf{S} = \cap_{f \in H} f^{-1}\left(\left(f(x) - \delta, f(x) + \delta\right)\right) \subset E \subset U$$

and **S** is in V_x, thus V_x is an open nhood base at x; (iii): For any F^* in τ such that N_x is not in F^*, by **Cor. 3.5** (i), x is not in F, and by (ii) of **Lemma 3.11** above, x is in

$$\mathrm{O} = \cap_{f \in H} f^{-1}\left(\left[f(x) - \delta, f(x) + \delta\right]\right) \subset Y - F$$

for some $H \in [D]^{<\omega}, \delta > 0$. Since

$$x \in P = \cap_{f \in H} f^{-1}\left(\left[f(x)-\delta/2, f(x)+\delta/2\right]\right) \in \mathsf{N}_x,$$

Cor. 3.5 (i), **Lemmas 3.6** (ii) and **3.8** (i) imply that

$$\mathsf{N}_x \in P^* \subset \cap_{f^* \in H^*} f^{*-1}\left(\left(f(x)-\delta, f(x)+\delta\right)\right) = T \in \mathsf{V}^*_{k(x)},$$

where $H^* = \{f^* \mid f \in H\}$. We claim that $T \subset Y^w - F^*$: For any $\mathfrak{C}_s$ in T, if $f^*(\mathfrak{C}_s) = s_f$ for all f in D, then s_f is in $\mathbf{I}_f = (f(x)-\delta, f(x)+\delta)$ for all f in H. Pick a $\rho > 0$ such that $[s_f - \rho, s_f + \rho] \subset \mathbf{I}_f$ for all f in H, then

$$L = \cap_{f \in H} f^{-1}\left(\left[s_f - \rho, s_f + \rho\right]\right) \subset \mathrm{O} \subset Y - F$$

and $L \in \mathsf{K}_s \subset \mathfrak{C}_s$; *i.e.* $\mathfrak{C}_s \in Y^w - F^*$. So

$$k(x) \in T \subset Y^w - F^*.$$

Thus $\mathsf{V}^*_{k(x)}$ is an open nhood base at $k(x)$.

Lemma 3.12 *Let* $\mathcal{E}$ *be a basic* $\mathbf{C}^*_D$*-filter on* $\mathbf{Y}$ *defined as in* Section **2**. *If* $\mathcal{E}$ *does not converge in* $\mathbf{Y}$,

$$V^*_{\mathcal{E}} = \left\{\cap_{f^* \in H^*} f^{*-1}\left(\left(r_f - \varepsilon, r_f + \varepsilon\right)\right) \mid H^* \in [D^*]^{<\omega}, \varepsilon > 0\right\}$$

is an open nhood base at $\mathcal{E}$ *in* Y^w.

Proof. If $\mathcal{E}$ does not converge in $\mathbf{Y}$, then $\mathcal{E}$ is in Y^w. Since $f^*(\mathcal{E}) = r_f$ for all $f^* \in \boldsymbol{D}^*$,

$$\mathcal{E} \in \cap_{f^* \in H^*} f^{*-1}\left(\left(r_f - \varepsilon, r_f + \varepsilon\right)\right)$$

for any $H^* \in [D^*]^{<\varepsilon}, \varepsilon > 0$. For any $F^* \in \tau$ such that $\mathcal{E} \in Y^w - F^*$, by **Cor. 3.5** (iv) there exists a

$$\mathbf{E} = \cap_{f \in H} f^{-1}\left(\left[r_f - \delta, r_f + \delta\right]\right) \in \mathsf{K} \subset \mathcal{E}$$

for some $H \in [D]^{<\omega}, \delta > 0$ such that $\mathbf{E} \subset \mathbf{Y} - \mathbf{F}$. For $H^* = \{f^* \mid f \in H\}$, let

$$U = \cap_{f^* \in H^*} f^{*-1}\left(\left(r_f - \delta, r_f + \delta\right)\right),$$

then $\mathcal{E} \in U \in \mathsf{V}^*$. We claim that $U \subset Y^w - F^*$. For any $\mathcal{E}_t$ in U, let $f^*(\mathcal{E}_t) = t_f$ for each f^* in H^*. Then for each f in H, t_f is in

$$\left(r_f - \delta, r_f + \delta\right) \text{ and } f^{-1}\left(\left[t_f - \gamma, t_f + \gamma\right]\right) \in \mathcal{E}_t$$

for all $\gamma > 0$. Pick a $\sigma > 0$ such that

$$\left[t_f - \sigma, t_f + \sigma\right] \subset \left[r_f - \delta, r_f + \delta\right]$$

for each f in H, then

$$L = \cap_{f \in H} f^{-1}\left(\left[t_f - \sigma, t_f + \sigma\right]\right) \subset E \subset Y - F.$$

Since $L \in \mathsf{K}_t \subset \mathcal{E}_t$, so $\mathcal{E}_t \in Y^w - F^*$. Hence $\mathcal{E}$ is in $U \subset Y^w - F^*$. Thus, $\mathsf{V}^*_{\mathcal{E}}$ is an open nhood base at $\mathcal{E}$.

Proposition 3.13 *For any basic closed* $\mathbf{C}^*_{D^*}$*-filter* $\mathcal{E}^*$ *on* Y^w, $\mathcal{E}^*$ *converges in* Y^w.

Proof. For given $\mathcal{E}^*$, let K and $\mathcal{E}$ be the closed $\boldsymbol{C}^*_D$-filter base and the basic closed $\boldsymbol{C}^*_D$-filter on $\mathbf{Y}$ induced by $\mathcal{E}^*$. ***Case* 1**: If $\mathcal{E}$ converges to an x in $\mathbf{Y}$, then r_f is $f(x)$ for all f in D. For any

$$U = \cap_{f^* \in H^*} f^{*-1}\left(\left(r_f - \delta, r_f + \delta\right)\right)$$

in $\mathsf{V}^*_{k(x)}$, let

$$E = \cap_{f^* \in I^*} f^{*-1}\left(\left[r_f - \delta/2, r_f + \delta/2\right]\right),$$

where $I^* \in [D^*]^{<\omega}$. Then $E \in \mathsf{K}^* \subset \mathcal{E}^*$ and $E \subset U$. Thus, $\mathcal{E}^*$ converges to $k(x) = \mathsf{N}_x$ in Y^w. ***Case 2***: If $\mathcal{E}$ does not converge in $\mathbf{Y}$, then $\mathcal{E}$ is in Y^w. For any

$$U = \cap_{f^* \in I^*} f^{*-1}\left(\left(r_f - \delta, r_f + \delta\right)\right)$$

in $\mathsf{V}^*_{\mathcal{E}}$, let

$$E = \cap_{f^* \in I^*} f^{*-1}\left(\left[r_f - \delta/2, r_f + \delta/2\right]\right),$$

then $E \in \mathsf{K}^* \subset \mathcal{E}^*$ and $E \subset U$. Thus, $\mathcal{E}^*$ converges to $\mathcal{E}$ in Y^w.

Theorem 3.14 (Y^w, k) *is a compactification of* $\mathbf{Y}$.

Proof. First, we show that Y^w is compact. Let G be a sub-collection of τ with the finite intersection property. Let

$$\mathbf{L} = \left\{\cap_{E^* \in H} E^* \mid H \in [G]^{<\omega}\right\},$$

then $\mathbf{L}$ is a filter base on Y^w. Let $\mathcal{F}$ be a closed ultrafilter on Y^w such that $\mathbf{L}$ is contained in $\mathcal{F}$. By **Cor. 2.3**, there is a unique basic closed $\boldsymbol{C}^*_{D^*}$-filter $\mathcal{E}^*$ on Y^w such that $\mathcal{E}^*$ is contained in $\mathcal{F}$. By **Prop. 3.13**, $\mathcal{E}^*$ converges to an $\mathcal{E}_o$ in Y^w. This implies that $\mathcal{F}$ converges to $\mathcal{E}_o$ too. Hence, $\mathcal{E}_o$ is in $\boldsymbol{F}$ for all $\boldsymbol{F}$ in $\mathcal{F}$; *i.e.*, $\mathcal{E}_o \in \cap\{E^* \mid E^* \in G\}$. **Thm. 17.4** in [**1**, p.118], Y^w is compact. Thus, by **Lemma 3.10** (i) and (iii), (Y^w, k) is a compactification of $\mathbf{Y}$.

4. The Hausdorff Compactification (X^w,*k*) of X Induced by a Subset *D* of C^*(X)

Let $\mathbf{X}$ be a Tychonoff space and let D be a subset of $\mathbf{C}^*(\mathbf{X})$ such that D separates points of $\mathbf{X}$ and the topology on $\mathbf{X}$ is the weak topology induced by D. It is clear that D contains a non-constant function. For each x in $\mathbf{X}$, since V_x is an open nhood base at x, it is clear that $\mathcal{E}_x$ converges to x. Let $X^w = X_E \cup X_F$, where $X_E = \{\mathcal{E}_x \mid x \in \mathbf{X}\}$ and $X_E = \{\mathcal{E} \mid \mathcal{E}$ is a basic closed $\boldsymbol{C}^*_D$-filter that does not converge in $\mathbf{X}\}$. Similar to what we have done in Section **3**, we can get the similar definitions, lemmas, propositions and a theorem in the following:

(**4.15.4**) (See **Def. 3.4**) *For a nonempty closed set F in* $\mathbf{X}$, $F^* = \{\mathcal{E} \in X^w \mid F \cap T \neq \phi$ *for all* T *in* $\mathcal{E}\}$.

(**4.15.5**) (See **Cor. 3.5**) *For a nonempty closed set* **F** *in* **X**, (*i*) *x is in* **F** *if* $\mathcal{E}_x$ *is in* **F***; (*ii*) **F** *is* **X** *if* $\mathbf{F}^* = X^w$; (*iii*) *for each* $\mathcal{E}$ *in* X^w, **F** *is in* $\mathcal{E}$ *implying* $\mathcal{E}$ *is in* **F***; (*iv*) $\mathcal{E} \in X^w - F^* \Leftrightarrow$ *there is a* S *in* $\mathcal{E}$ *such that* $S \subset X - F$.

Proof. (i) ($\Leftarrow$) If $\mathcal{E}_x$ is in F^*, then

$$F \cap f^{-1}\left(\left(f(x)-\varepsilon, f(x)+\varepsilon\right)\right) \supset$$
$$F \cap f^{-1}\left(\left[f(x)-\varepsilon/2, f(x)+\varepsilon/2\right]\right) \neq \phi$$

for all *f* in D, $\varepsilon > 0$. Since V_x is a nhood base at x, thus x is a cluster point of **F**, so x is in **F**. (i) implying (ii), (iii) and (iv) are obvious.

(**4.15.6**) (See **Lemma 3.6**) *For any two nonempty sets* E *and* F *in* **X**,

(*i*) $E \subseteq F \Leftrightarrow E^* \subseteq F^*$;

(*ii*) $(E \cap F)^* \subseteq (E^* \cap F^*)$;

(*iii*) $(E \cup F)^* = (E^* \cup F^*)$.

(**4.15.7**) (See **Prop. 3.7**) τ = {**F***|**F** *is a nonempty closed set in* **X**} *is a base for the closed sets of* X^w.

(**4.15.7.1)** (See the definitions for the topology $\Im$ on Y^w and *f** for each *f* in D in Section **3**.)

Equip X^w *with the topology* $\Im$ *induced by* τ. *For each* f *in* D, *define* $\mathrm{f}^*: X^w \to \mathbf{R}$ *by* f*($\mathcal{E}$) = r_f if $f^{-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right) \in \mathcal{E}$ *for all* $\varepsilon > 0$. *Then* f* *is well-defined and* f*($\mathcal{E}_x$) is *f*(*x*) *for all f in* D *and all x in* **X**.

(**4.15.8)** (See **Lemma 3.8**) *For each f in* D, *let* r *be in* $\mathrm{Cl}(\mathrm{f}(\mathbf{X}))$, *then*

(*i*) $\left(f^{-1}\left([r-\delta, r+\delta]\right)\right)^* \subseteq f^{*-1}\left((r-\varepsilon, r+\varepsilon)\right)$

and

(*ii*) $f^{*-1}\left((r-\varepsilon, r+\varepsilon)\right) \subseteq \left(f^{-1}\left([r-\varepsilon, r+\varepsilon]\right)\right)^*$

for any $\varepsilon > \delta > 0$.

(**4.15.9**) (See **Prop. 3.9**) *For each* f in D, *f** *is a bounded real continuous function on* X^w.

(**4.15.10**) (See **Lemma 3.10**) *Let* $k: \mathbf{X} \to X^w$ *be defined by* $k(x) = \mathcal{E}_x$. *Then,* (*i*) k *is an embedding from* **X** *into* X^w; (ii) $f^* \circ k = f$ *for all* f *in* D; *and* (*iii*) $k(\mathbf{X})$ *is dense in* X^w.

(**4.15.11**) (See **Lemmas 3.11** and **3.12**) *For each* $\mathcal{E}$ *in* X^w, *let*

$$\mathsf{K} = \left\{\cap_{f\in H} f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right) \mid \cap_{f\in H} f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right) \neq \phi \text{ for a ny } H \in [D]^{<\omega}, \varepsilon > 0\right\} \subset \mathcal{E}$$

1) *If* $\mathcal{E}$ *converges to x*, *then* $\mathcal{E}$ is $\mathcal{E}_x$ and $\mathsf{V}^*_{k(x)}$ is =

$$\mathsf{V}^*_{\mathcal{E}_x} = \left\{\cap_{f^*\in H^*} f^{*-1}\left(\left(f(x)-\varepsilon, f(x)+\varepsilon\right)\right) \mid H^* \in [D^*]^{<\omega}, \varepsilon > 0\right\}$$

is an open nhood base at $\mathcal{E}_x$. 2) *If* $\mathcal{E}$ *does not converge in* **X**, *then* $\mathcal{E}$ *is in* X^w and

$$\mathsf{V}^*_{\mathcal{E}} = \left\{\cap_{f^*\in H^*} f^{*-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right) \mid \cap_{f^*\in H^*} f^{*-1}\left(\left(r_f-\varepsilon, r_f+\varepsilon\right)\right) \neq \phi \text{ for any } H^* \in [D^*]^{<\omega}, \varepsilon > 0\right\}$$

is an open nhood base at $\mathcal{E}$ *in* X^w.

(**4.15.13**) (See **Prop. 3.13**) *Each basic closed* $\boldsymbol{C}^*_{\boldsymbol{D}^*}$-filter $\mathcal{E}^*$ *on* X^w *converges to* $\mathcal{E}$ *in* X^w.

(**4.15.14**) (See **Theorem 3.14**) (X^w, k) *is a compactification of* **X**.

Lemma 4.16 D^* *separates points of* X^w.

Proof. For $\mathcal{E}_s$, $\mathcal{E}_t$ in X^w, let

$$\mathsf{K}_s = \left\{\cap_{f\in H} f^{-1}\left(\left[s_f-\varepsilon, s_f+\varepsilon\right]\right) \mid \cap_{f\in H} f^{-1}\left(\left[s_f-\varepsilon, s_f+\varepsilon\right]\right) \neq \phi \text{ for any } H \in [D]^{<\omega}, \varepsilon > 0\right\}$$

and similarly for K_t. Since $\mathcal{E}_s$ is not equal to $\mathcal{E}_t$, K_s is not equal to K_t and that D has a g such that $s_g \neq t_g$ are equivalent, where $g^{-1}\left(\left[s_g-\varepsilon, s_g+\varepsilon\right]\right) \in \mathsf{K}_s$ which is contained in $\mathcal{E}_s$ and $g^{-1}\left(\left[s_g-\varepsilon, s_g+\varepsilon\right]\right) \in \mathsf{K}_s$ which is contained in $\mathcal{E}_t$ for all $\varepsilon > 0$, thus by the definition of g*, g*($\mathcal{E}_s$) $= s_g \neq t_g =$ g*($\mathcal{E}_t$).

Theorem 4.17 (X^w, k) *is a Hausdorff compactification of* **X**.

Proof. By **4.15.10** (i) and (iii), **4.15.14** and **Lemma 4.16**, (X^w, k) is a Hausdorff compactification of **X**.

5. The Homeomorphism between (X^w,*k*) and (Z,*h*)

Let $(\mathbf{Z}, h)$ be an arbitrary Hausdorff compactification of **X**, then **X** is a Tychonoff space. Let $^\circ D$ denote $\mathbf{C}(\mathbf{Z})$ which is the family of real continuous functions on **Z**, and let $D = \left\{f \mid f = {}^\circ f \circ h, {}^\circ f \in {}^\circ D\right\}$. Then D is a subset of $\mathbf{C}^*(\mathbf{X})$ such that D separates points of **X**, the topology on **X** is the weak topology induced by D and D contains a non-constant function.

Let (X^w, k) be the Hausdorff compactification of **X** obtained by the process in Section 4 and D is defined as above. For each basic closed $\boldsymbol{C}^*_D$-filter $\mathcal{E}$ in X^w, let $\mathcal{E}$ be generated by

$$\mathsf{K} = \left\{\cap_{f\in H} f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right) \mid \cap_{f\in H} f^{-1}\left(\left[r_f-\varepsilon, r_f+\varepsilon\right]\right) \neq \phi \text{ for any } H \in [D]^{<\omega}, \varepsilon > 0\right\}$$

let $^\circ\mathcal{E}$ be the basic closed $\boldsymbol{C}^*_{\mathfrak{D}}$-filter on **Z** generated by

$$^{\circ}\mathsf{K} = \left\{ \cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \mid \cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \neq \phi \text{ for any } {^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega}, \varepsilon > 0 \right\}$$

and let h^{-1} be the function from h(**X**) to **X** defined by $h^{-1}(h(x)) = x$. Since h and h^{-1} are one-one, $f = {^{\circ}f} \circ h$ and h(**X**) is dense in **Z**, similar to the arguments in the paragraphs prior to **Lemma 3.11**, we have that

$$\cap_{f \in H} f^{-1} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \neq \phi$$

iff

$$\cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \neq \phi$$

for any

$${^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega} \text{ (or any } H \in [D]^{<\omega}),$$

$$H = \left\{ f \mid {^{\circ}f} \in {^{\circ}H} \right\} \text{ (or } {^{\circ}H} = \left\{ {^{\circ}f} \mid f \in H \right\})$$

and all $\varepsilon > 0$. Thus, if K or $\mathcal{E}$ is well-defined, so is $^{\circ}\mathsf{K}$ or $^{\circ}\mathcal{E}$ and *vice versa*. If K or $\mathcal{E}$ is given, $^{\circ}\mathsf{K}$ or $^{\circ}\mathcal{E}$ is called the ***closed C****$_{^{\circ}D}$***-filter base*** or the ***basic closed C****$_{^{\circ}D}$***-filter on Z induced by*** K or $\mathcal{E}$ and *vice versa*. For any z in **Z**,

$$^{\circ}\mathsf{K}_z = \left\{ \cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[{^{\circ}f}(z) - \varepsilon, {^{\circ}f}(z) + \varepsilon \right] \right) \mid {^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega}, \varepsilon > 0 \right\}$$

is the ***closed C****$_{^{\circ}D}$***-filter base at z***. The closed filter $^{\circ}\mathcal{E}_z$ generated by $^{\circ}\mathsf{K}_z$ is the ***basic closed C****$_{^{\circ}D}$***-filter at z***. Since **Z** is compact Hausdorff, each $^{\circ}\mathcal{E}$ on **Z** converges to a unique point z in **Z**. So, we define $\mathrm{T}: X^w \to \mathbf{Z}$ by $\mathrm{T}(\mathcal{E}) = z$, where $\mathcal{E}$ is in X^w and z is the unique point in **Z** such that the basic closed ***C****$_{^{\circ}D}$-filter $^{\circ}\mathcal{E}$ on **Z** induced by $\mathcal{E}$ converges to it. For $\mathcal{E}_s$, $\mathcal{E}_t$ in X^w, let

$$\mathsf{K}_s = \left\{ \cap_{f \in H} f^{-1} \left(\left[s_f - \varepsilon, s_f + \varepsilon \right] \right) \mid \cap_{f \in H} f^{-1} \left(\left[s_f - \varepsilon, s_f + \varepsilon \right] \right) \neq \phi \text{ for any } H \in [D]^{<\omega}, \varepsilon > 0 \right\}$$

and similarly for K_t such that $\mathcal{E}_s$ and $\mathcal{E}_t$ are generated by K_s and K_t, respectively. Assume that $^{\circ}\mathcal{E}_s$ and $^{\circ}\mathcal{E}_t$ converge to z_s and z_t in **Z**, respectively. Then $\mathcal{E}_s$ is not equal to $\mathcal{E}_t$, $^{\circ}\mathcal{E}_s$ is not equal to $^{\circ}\mathcal{E}_t$ and z_s is not equal to z_t are equivalent. Hence T is well-defined and one-one. For each z in **Z**, let $^{\circ}\mathcal{E}_z$ be the basic closed ***C****$_{^{\circ}D}$-filter at z, since **Z** is compact Hausdorff and

$$^{\circ}\mathsf{V}_z = \left\{ \cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[{^{\circ}f}(z) - \varepsilon, {^{\circ}f}(z) + \varepsilon \right] \right) \mid {^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega}, \varepsilon > 0 \right\}$$

is an open nhood base at z, thus $^{\circ}\mathcal{E}_z$ converges to z. Let $\mathcal{E}_z$ be the element in X^w induced by $^{\circ}\mathcal{E}_z$, then, $\mathrm{T}(\mathcal{E}_z) = z$. Hence, T is one-one and onto.

Theorem 5.18 *(*$\left(X^w, k\right)$ *is homeomorphic to* $(\mathbf{Z}, h)$ *under the mapping* T *such that* $\mathrm{T}(k(x)) = h(x)$.

Proof. We show that T^{-1} is continuous. For each $\mathcal{E}$ in **F*** which is in τ, let $^{\circ}\mathcal{E}$ be the basic closed **C***$_{^{\circ}D}$-filter on **Z** induced by $\mathcal{E}$. If $^{\circ}\mathcal{E}$ converges to z in **Z**, ${^{\circ}f}(z) = r_f$ for each f in D and

$$^{\circ}\mathsf{K} = \left\{ \cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \mid \cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \neq \phi \text{ for any } {^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega}, \varepsilon > 0 \right\} \subset {^{\circ}\mathcal{E}}$$

Then (a): $\mathcal{E}$ is in **F*** iff (b):

$$F \cap \left(\cap_{f \in H} f^{-1} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \right) \neq \phi$$

for any $H \in [D]^{<\omega}, \varepsilon > 0$, where

$$\cap_{f \in H} f^{-1} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \in \mathcal{E}.$$

Since h is one-one, $f = {^{\circ}f} \circ h$ for all f in D, so (b) iff (c):

$$h(F) \cap \left[\cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \right] = h\left(F \cap \left[\cap_{f \in H} f^{-1} \left(\left[r_f - \varepsilon, r_f + \varepsilon \right] \right) \right] \right) \neq \phi$$

for any

$$H \in [D]^{<\omega} \text{ (or } {^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega}),$$

$${^{\circ}H} = \left\{ {^{\circ}f} \mid f \in H \right\} \text{ (or } H = \left\{ f \mid {^{\circ}f} \in {^{\circ}H} \right\})$$

and any $\varepsilon > 0$. Since

$${^{\circ}f^{-1}} \left(\left(r_f - \varepsilon, r_f + \varepsilon \right) \right) \supset {^{\circ}f^{-1}} \left(\left[r_f - \varepsilon/2, r_f + \varepsilon/2 \right] \right)$$

for any $^{\circ}$f in $^{\circ}D$, $\varepsilon > 0$, (c) iff (d):

$$h(F) \cap \left[\cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left(r_f - \varepsilon, r_f + \varepsilon \right) \right) \right] \neq \phi$$

for any ${^{\circ}H} \in \left[{^{\circ}D} \right]^{<\omega}, \varepsilon > 0$. Since

$$\cap_{^{\circ}f \in {^{\circ}H}} {^{\circ}f^{-1}} \left(\left(r_f - \varepsilon, r_f + \varepsilon \right) \right)$$

is an arbitrary basic open nhood of z in **Z**. So, (d) iff z is in $\mathrm{Cl}_{\mathbf{Z}}(\mathrm{h}(\mathbf{F}))$; *i.e.*, $\mathcal{E}$ is in **F*** if $\mathrm{T}(\mathcal{E})$ is equal to z which belongs to $\mathrm{Cl}_{\mathbf{Z}}(\mathrm{h}(\mathbf{F}))$. Hence, T(**F***) = Cl$_{\mathbf{Z}}$(h(**F**)) is closed in **Z** for all **F*** in τ. Thus, T^{-1} is continuous. Since T is one-one, onto and both **Z** and X^w are compact Hausdorff, by **Theorem 17.14** in [**1**, p.123], T is a homeomorphism. Finally, from the definitions of k and h, it is clear that $\mathrm{T}(k(x)) = h(x)$ for all x in **X**.

Corollary 5.19 *Let* (β**X**, h) *be the Stone-Čech compactification of a Tychonoff space* **X**,

$$D = \left\{ f \mid f = {^{\circ}f} \circ h, {^{\circ}f} \in C(\beta X) \right\}$$

and $\mathrm{T}_\beta: X^w \to \beta X$ *is defined similarly to* T *as above. Then* (β**X**, h) *is homeomorphic to* $\left(X^w, k\right)$ *such that* $\mathrm{T}_\beta(k(\mathrm{x})) = h(\mathrm{x})$.

Corollary 5.20 *Let* (γ**X**, h) *be the Wallman compactification of a normal* **T**$_1$*-space* **X**,

$$D=\left\{f \mid f={}^{\circ}f\circ h,{}^{\circ}f\in C\left(\gamma X\right)\right\}$$

and $\mathrm{T}_{\gamma}:X^{w}\to\gamma X$ *is defined similarly to* T *as above. Then* (γ**X**, h) *is homeomorphic to* $\left(X^{w},k\right)$ *such that* $\mathrm{T}_{\gamma}\left(k\left(\mathrm{x}\right)\right)=h\left(\mathrm{x}\right)$.

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