*American Journal of Computational Mathematics*, **2013, 3, 211-216**
http://dx.doi.org/10.4236/ajcm.2013.33030 Published Online September 2013 (http://www.scirp.org/journal/ajcm)

# Finite Element Analysis of the Ramberg-Osgood Bar

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Received May 29, 2013; revised June 29, 2013; accepted July 9, 2013



## ABSTRACT

In this work, we present a priori error estimates of finite element approximations of the solution for the equilibrium equation of an axially loaded Ramberg-Osgood bar. The existence and uniqueness of the solution to the associated nonlinear two point boundary value problem is established and used as a foundation for the finite element analysis.

**Keywords:** Nonlinear Two Point Boundary Value Problem; Ramberg-Osgood Axial Bar; Existence and Uniqueness of Solutions; Finite Element Analysis; Convergence and a Priori Error Estimates

## 1. Introduction

The following Ramberg-Osgood stress strain equation

$$\varepsilon(x) = A\sigma(x) + B\left|\sigma(x)\right|^{q-2}\sigma(x), \quad (1.1)$$

is accepted as the model for the material's constitutive equation in the stress analysis for a variety of industrial metals. Numerous data exist in literature that supports the use of (1.1) to represent the stress-strain relationship for aluminum and several other steel alloys exhibiting elastic-plastic behavior (see, for example, [1-4] and the references therein). In Equation (1.1), $\varepsilon(x)$ represents the axial strain, $\sigma(x)$ represents the axial stress, $0 < x < L$, $q \geq 2$ represents the material hardening index (where $q = 2$ describes the linear elastic material), the constants $A$, $B$ and $q$ are determined from the experimental values for the parameters $E$, $\sigma_y$, $\varepsilon_y$, $\varepsilon_u$, and $\varepsilon_u$ by the formula

$$A = \frac{1}{E}, B = 0.002\left(\frac{1}{\sigma_y}\right)^{q-2}, q = 1 + \frac{\ln 20}{\ln\left(\sigma_u/\sigma_y\right)} \quad (1.2)$$

where $E$ is the Young's modulus, $\sigma_y$, $\varepsilon_y$ are the material's yield stress and strain, $\sigma_u$, $\varepsilon_u$ are the ultimate stress and the ultimate strain, and $L > 0$ stands for the length of the solid bar.

We observe that Equation (1.1) splits the strain into two parts: an elastic strain part with coefficient $A$ and a nonlinear part with coefficient $B$. The linear part dominates for $\sigma < \sigma_y$, while the nonlinear part dominates for $\sigma > \sigma_y$. In many industrial applications, e.g., in lightweight ship deck titanium structures, welding-induced plastic zones play important roles in determining the structures' integrity (see [5,6]).

**Figure 1** compares the stress-strain curves for Hooke's law, the double modulus, and Ramberg-Osgood law using material measured data. Among these models, the Ramberg-Osgood model appears to represent the material's behavior the best.

**Table 1** gives experimental values of the material constants for some commonly used metals in industries.

Although (1.1) is widely used in industries for finite element analysis, no solvability and uniqueness or error analysis has been given in literature even for the following one-dimensional boundary value problem:

$$\begin{cases} \dfrac{\mathrm{d}\sigma(x)}{\mathrm{d}x} - c(x)u(x) + f(x) = 0, \ 0 < x < L, \\ \dfrac{\mathrm{d}u(x)}{\mathrm{d}x} = A\sigma(x) + B\left|\sigma(x)\right|^{q-2}\sigma(x), \\ u(0) = 0, u'(L) = \beta \end{cases} \quad (1.3)$$

where $c(x) \geq 0$ satisfies $c \in L^\infty(0,L)$ and $f(x) \in L^q(0,L)$. For simplicity, we consider only one boundary condition. Other Dirichlet type boundary conditions can be treated similarly.

We also consider the case when $c(x)u(x)$ is replaced by $\sum_{i=1}^{N} k_i u(x_i)\delta(x - x_i)$, where $\delta(x - x_i)$ is Dirac impulse functions, and $k_i$ stands for concentrated elastic support constant at $0 \leq x_i \leq L$, for $i = 1, \cdots, N$.

In Section 2, we develop a week formulation of (1.3) subject to the given boundary condition and prove existence and uniqueness of the solution by using the theory

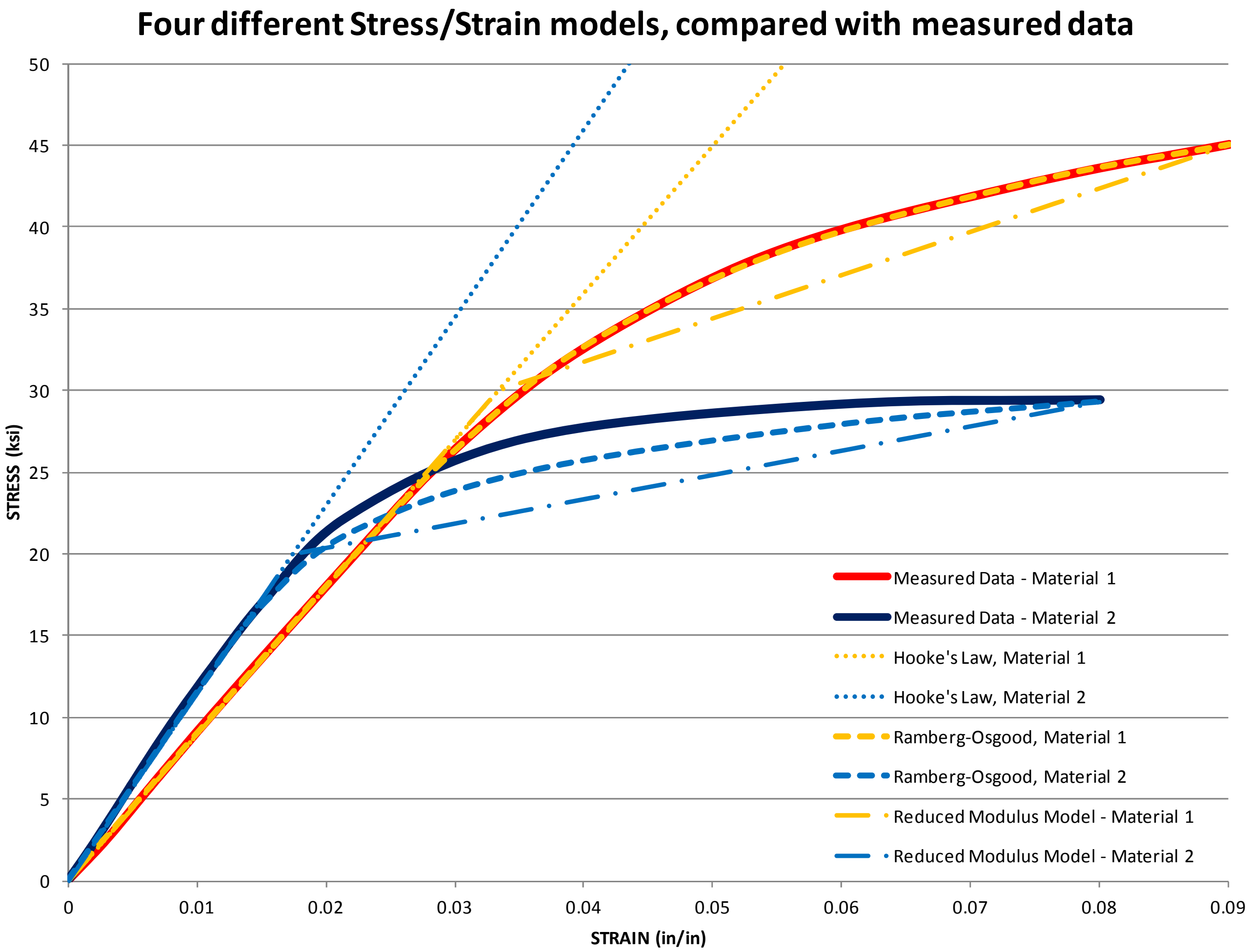


**Figure 1. Ramburg-Osgood curves.**

**Table 1. Constants for Ramburg-Osgood materials.**

| Material | $A$ | $B$ | $q-1$ |
|---|---|---|---|
| Inconel 718 | 3.33e−05 | 4.42e−71 | 32.00 |
| 5083 Aluminum | 9.80e−05 | 2.50e−23 | 13.11 |
| 6061T6 Aluminum | 1.00e−04 | 1.35e−58 | 34.44 |
| 304 Stainless Steel | 3.57e−05 | 3.44e−13 | 6.32 |
| 304 L StainlessSteel | 3.57e−05 | 2.24e−15 | 7.36 |

of perturbed convex variational problems in Sobolev spaces (see [7] for details.) We also prove that the solution is bounded in certain Sobolev norms. In Section 3, we derive an error estimates for the semi-discrete error between the week solution and the Galerkin's finite element solution of (1.3) for the standard conformal finite elements. The results of this section are based on the results in Section 2. We believe that the results established in these sections are novel and preliminary.

## 2. Existence and Uniqueness of Solutions

Let $W^{1,p}(0,L)$ and $W_0^{1,p}(0,L)$ be the standard Sobolev spaces, where $p=\dfrac{q}{q-1}$. Define $\phi(\sigma)=A\sigma+B|\sigma|^{q-2}\sigma$, where $A>0, B>0$, and $q>2$.

Observe that the mapping $\phi(\sigma)$ is one-to-one; however, its inverse cannot be written explicitly.

Since $\varepsilon=u'$, Equation (1.3) can be rewritten as:

$$\begin{cases} -\dfrac{\mathrm{d}\sigma(x)}{\mathrm{d}x}+c(x)u(x)=f(x), & 0<x<L \\ \sigma=\phi^{-1}(u'), & u(0)=0, u'(L)=\beta \end{cases} \tag{2.2}$$

Define the following space of admissible functions as

$$V\equiv\left\{u\in W^{1,p}(0,L)\,\middle|\,u(0)=0,u'(L)=\beta\right\}. \tag{2.3}$$

The weak formulation of (2.2) can then be written as

**Problem I:** Find $u\in V$, such that

$$\int_0^L\phi^{-1}(u')v'\mathrm{d}x+\int_0^L cuv\mathrm{d}x=\int_0^l fv\mathrm{d}x \quad \forall v\in W_0^{1,p}(0,L). \tag{2.4}$$

Let us define the operator:

$$a(u,v) \equiv \int_0^L \phi^{-1}(u')v'\mathrm{d}x + \int_0^L cuv\mathrm{d}x, \tag{2.5}$$

for $u,v \in V$. Then, $a(u,v)$ satisfies the following property:

$$\begin{aligned} a(u,u) &= \int_0^L \phi^{-1}(u')u'\mathrm{d}x + \int_0^L cu^2\mathrm{d}x \\ &= \int_0^L \phi^{-1}(u')\left[A\phi^{-1}(u') + B\left|\phi^{-1}(u')\right|^{q-2}\phi^{-1}(u')\right]\mathrm{d}x \\ &\quad + \int_0^L cu^2\mathrm{d}x \ge \int_0^L\left[A\left|\phi^{-1}(u')\right|^2 + B\left|\phi^{-1}(u')\right|^q\right]\mathrm{d}x \\ &= A\left\|\phi^{-1}(u')\right\|_{L^2}^2 + B\left\|\phi^{-1}(u')\right\|_{L^q}^q, \text{ for } u \in V. \end{aligned} \tag{2.6}$$

Also, by the definition of $\phi$, we have

$$\begin{aligned} \|u'\|_{L^q} &= \left\|A\phi^{-1}(u') + B\left|\phi^{-1}(u')\right|^{q-2}\phi^{-1}(u')\right\|_{L^q} \\ &\le A\left\|\phi^{-1}(u')\right\|_{L^q} + B\left\|\left|\phi^{-1}(u')\right|^{q-1}\right\|_{L^q} \\ &\le A\left\|\phi^{-1}(u')\right\|_{L^q} + \bar{B}\left\|\phi^{-1}(u')\right\|_{L^q}^{q-1} \end{aligned} \tag{2.7}$$

**Lemma 2.1** For given positive constants $A, B, q, L$, there exists a constant $C$ independent of the solutions $u(x) \in V$ of the BVP (1.3) such that $\left\|\phi^{-1}(u')\right\|_{L^2}^2 + \left\|\phi^{-1}(u')\right\|_{L^q}^q \le C$.

**Proof:** For a solution $u(x) \in V$, we can write: $v = u - u_b$, where $u_b \in V$ is a fixed function, so that $v \in W_0^{1,p}(0,L)$, and since: $\int_0^L \phi^{-1}(u')v'\mathrm{d}x + \int_0^L cuv\mathrm{d}x = \int_0^L fv\mathrm{d}x$, we get:

$$\begin{aligned} a(u,u) &= \int_0^L \phi^{-1}(u')u'\mathrm{d}x + \int_0^L cu^2\mathrm{d}x \\ &= \int_0^L fu\mathrm{d}x - \int_0^L fu_b\,\mathrm{d}x + \int_0^L cuu_b\mathrm{d}x \\ &\quad + \int_0^L \phi^{-1}(u')u_b'\,\mathrm{d}x \end{aligned} \tag{2.8}$$

Also, by (2.6) and (2.8), we have:

$$\begin{aligned} &A\left\|\phi^{-1}(u')\right\|_{L^2}^2 + B\left\|\phi^{-1}(u')\right\|_{L^q}^q \\ &\le \|f\|_{L^q}\|u_b\|_{L^p} + \|u\|_{L^p}\left(\|f\|_{L^q} + \|cu_b\|_{L^q}\right) \\ &\quad + \left\|\phi^{-1}(u')\right\|_{L^q}\|u_b'\|_{L^p} \\ &\le C_1 + C_2\|u\|_{L^p} + C_3\left\|\phi^{-1}(u')\right\|_{L^q}, \end{aligned} \tag{2.9}$$

where $C_1 = \|f\|_{L^q}\|u_b\|_{L^p}$, $C_2 = \left(\|f\|_{L^q} + \|cu_b\|_{L^q}\right)$, and $C_3 = \|u_b'\|_{L^p}$.

By the Sobolev inequality, we have (see e.g. [8,9]): $\|u\|_{L^p} \le C\|u'\|_{L^q}$, and therefore:

$$B\left\|\phi^{-1}(u')\right\|_{L^q}^q \le C_1 + C_2\|u'\|_{L^q} + C_3\left\|\phi^{-1}(u')\right\|_{L^q}.$$

Also, since by definition of $\phi$,

$$\begin{aligned} \|u'\|_{L^q} &= \left\|A\phi^{-1}(u') + B\left|\phi^{-1}(u')\right|^{q-2}\phi^{-1}(u')\right\|_{L^q} \\ &\le A\left\|\phi^{-1}(u')\right\|_{L^q} + B\left\|\left|\phi^{-1}(u')\right|^{q-1}\right\|_{L^q} \\ &\le A\left\|\phi^{-1}(u')\right\|_{L^q} + \bar{B}\left\|\phi^{-1}(u')\right\|_{L^q}^{q-1}, \end{aligned}$$

we have

$$\left\|\phi^{-1}(u')\right\|_{L^q}^q \le \bar{C}_1 + \bar{C}_2\left\|\phi^{-1}(u')\right\|_{L^q} + \bar{C}_3\left\|\phi^{-1}(u')\right\|_{L^q}^{q-1} \tag{2.10}$$

where $\bar{C}_i, i = 1,2,3$ are positive constants. From (2.10), we conclude that $\left\|\phi^{-1}(u')\right\|_{L^q}$ is bounded and that there exists a constant $C$ such that $\left\|\phi^{-1}(u')\right\|_{L^q} \le C$, as $u(x)$ varies over the solution set of (1.3) in $V$. Therefore, the result of the lemma is follows.

**Theorem 2.1** For a given $f \in L^q(0,L)$, $q \ge 2$, $A > 0$ & $B > 0$, problem (I) has a unique solution $u \in U$.

**Proof:**

The uniqueness follows from the following argument. Let $u_1$ and $u_2$ be two solutions of (2.4). Then (since $c(x) \ge 0$):

$$0 \ge \int_0^L \left(\phi^{-1}(u_1') - \phi^{-1}(u_2')\right)(u_1' - u_2')\mathrm{d}x,$$

which leads to:

$$\begin{aligned} 0 &\ge \int_0^L (\sigma_1 - \sigma_2)\left(\phi(\sigma_1) - \phi(\sigma_2)\right)\mathrm{d}x \\ &= A\int_0^L (\sigma_1 - \sigma_2)^2\,\mathrm{d}x \\ &\quad + B\int_0^L\left(|\sigma_1|^{q-2}\sigma_1 - |\sigma_2|^{q-2}\sigma_2\right)(\sigma_1 - \sigma_2)\mathrm{d}x \\ &\ge A\int_0^L (\sigma_1 - \sigma_2)^2\mathrm{d}x, \end{aligned}$$

since $\sigma_1 = \phi^{-1}(u_1')$, $\sigma_2 = \phi^{-1}(u_2')$ and

$$\left(|\sigma_1|^{q-2}\sigma_1 - |\sigma_2|^{q-2}\sigma_2\right)(\sigma_1 - \sigma_2) \ge 0,$$

which is well-known [10,11].

Therefore, $\sigma_1 = \sigma_2$ and $u_1 = u_2$, and this establishes the uniqueness of the solution of (2.4).

For existence, we consider the variational formulation of (2.4) and define the total potential energy by:

$$\begin{aligned} J(u) &= \frac{1}{2}\left[\int_0^L \sigma\varepsilon\mathrm{d}x + \int_0^L cu^2\mathrm{d}x\right] - \int_0^L fu\mathrm{d}x \\ &= \frac{1}{2}\left[\int_0^L \phi^{-1}(u')u'\mathrm{d}x + \int_0^L cu^2\mathrm{d}x\right] - \int_0^L fu\mathrm{d}x \end{aligned}$$

Let $\varphi(t) = \frac{1}{2}\phi^{-1}(t)t$, then $J(u)$ can be written as:

$$J(u) = \frac{1}{2}\left[\int_0^L \varphi(u')\mathrm{d}x + \int_0^L cu^2\mathrm{d}x\right] - \int_0^L fu\mathrm{d}x.$$

Also we have: $\varphi'(t) = \frac{1}{2}\phi^{-1}(t) + \frac{1}{2}\left[\phi^{-1}(t)\right]' t$.

Letting $t = \phi(y(t)) = Ay(t) + B|y(t)|^{q-2} y(t)$.

Then $y = \phi^{-1}(t)$ and $1 = Ay'(t) + (q-1)B|y|^{q-2} y'(t)$.

Therefore, we get

$$\left[\phi^{-1}(t)\right]' = y'(t) = \frac{1}{A + (q-1)B|y|^{q-2}}, \text{ and}$$

$$\varphi'(t) = \frac{1}{2}\left[y(t) + \frac{t}{A + (q-1)B|y|^{q-2}}\right] = \frac{1}{2}\left[\phi^{-1}(t) + \phi^{-1}(t)\right] = \phi^{-1}(t).$$

Now the first variation of $J$ can be expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u + \varepsilon v)\bigg|_{\varepsilon=0} = \int_0^L \left[\varphi'(u')v' + cuv\right]\mathrm{d}x - \int_0^L fv\mathrm{d}x = \int_0^L \left[\phi^{-1}(u')v' + cuv\right]\mathrm{d}x - \int_0^L fv\mathrm{d}x.$$

However:

$$\varphi''(t) = \frac{1}{2}\left[\frac{2}{A + (q-1)B|y|^{q-2}} + t\frac{(q-1)(q-2)B|y|^{q-4} yy'}{\left[A + (q-1)B|y|^{q-2}\right]^2}\right]$$

$$= \frac{1}{2}\left\{\frac{2}{\left[A + (q-1)B|y|^{q-2}\right]} + \frac{(q-1)(q-2)B|y|^{q-2}}{\left[A + (q-1)B|y|^{q-2}\right]^2}\right\}$$

$$= \frac{1}{2}\frac{\left[A + q(q-1)B|y|^{q-2}\right]}{\left[A + (q-1)B|y|^{q-2}\right]^2} \geq 0.$$

We rewrite the total energy function as $J(u) = F_1(u) + F_2(u) - F(u)$, where $F_1(u) = \frac{1}{2}\int_0^L \varphi(u')\mathrm{d}x$, $F_2(u) = \frac{1}{2}\int_0^L cu^2\mathrm{d}x$, and $F(u) = \int_0^L f(x)u(x)\mathrm{d}x$. Then weak formulation (2.4) is equivalent to $\underset{u\in V}{\text{Min}}\ J(u)$.

Since $\varphi''(t) \geq 0$, $F_1 : V \to R$ is convex , and since $c \in L^\infty(0,L)$, $F_2 : V \to R$ is weekly sequentially continuous (since $\{u_n\}$ converges weekly in $V$ implies that $\{u_n\}$ converges strongly in $L^q(0,L)$.) Also (2.6) and (2.7) imply the coercivity of $J(u)$, see, e.g., [9-11 ]. Therefore, $J(u)$ satisfies the conditions of the theorem of 42.7, pp. 225-226, in [9], and the existence of a weak solution follows.

We now consider the second case when the term $c(x)u(x)$ is replaced by $\sum_{i=1}^N k_i u(x_i)\delta(x - x_i)$.

In this case, $F_2(u) = \frac{1}{2}\sum_{i=1}^N k_i\left[u(x_i)\right]^2$ and we only need show that it is weakly sequentially continuous. Suppose that $\{u_n\}$ converges weekly in $V$, then for a $v \in W^{-1,q'}(0,L)$,

$$\lim_{k\to\infty}\int_0^L \left(v'u_k' + vu_k\right)\mathrm{d}x = \int_0^L \left(v'u' + vu\right)\mathrm{d}x$$

and $\lim_{k\to\infty}\int_0^L vu_k\mathrm{d}x = \int_0^L vu\mathrm{d}x$. Therefore, since

$$u_k(x_i) = \int_0^{x_i} u_k'(x)\mathrm{d}x = \int_0^L v'u_k'(x)\mathrm{d}x,$$

where

$$v(x) = \begin{cases} x, & 0 \leq x \leq x_i \\ x_i, & x_i < x \leq L \end{cases}$$

We have

$$\lim_{k\to\infty} u_k(x_i) = \lim_{k\to\infty}\int_0^L v'u_k'(x)\mathrm{d}x = \int_0^L v'u'(x)\mathrm{d}x = \int_0^{x_i} u'(x)\mathrm{d}x = u(x_i)$$

$$\lim_{k\to\infty} F_2(u_k) = \lim_{k\to\infty}\sum_{i=1}^N k_i\left[u_k(x_i)\right]^2 = \sum_{i=1}^N k_i\left[u(x_i)\right]^2 = F_2(u).$$

Therefore, Theorem 2.1 holds with the same conditions for the case when $c(x)u(x)$ is replaced by $\sum_{i=1}^N k_i u(x_i)\delta(x - x_i)$.

## 3. Finite Element Error Estimates

Let $V_h \equiv S_h^k(0,L) \subset W^{1,q}(0,L)$ be a standard conformal finite element space of order $k$ (See [12-15]) satisfying the interpolation property:

$$\|v - \Pi_h v\|_{1,p} \leq C(v)h^k, \forall v \in W^{1,p}(0,L), \tag{3.1}$$

where $C$ is a positive constant depending only on $v$ and $L$, $\Pi_h v$ is the finite element interpolation of $v$, $k$ is the polynomial degree for the interpolation shape functions, and $h$ the mesh size, $\|v - \Pi_h v\|_{1,p}$ the $W^{1,p}(0, L)$ norm.

The corresponding finite element Galerkin's finite element approximation problem for (2.1) is:

**Problem II:**

Find $u_h \in V_h \equiv \left\{ _h \in S_h^k(0,L) \middle| v_h(0) = A, v_h'(L) = B \right\}$, such that

$$\int_0^L \phi^{-1}(u_h')v_h'\mathrm{d}x + \int_0^L u_h v_h dx v_h'\mathrm{d}x = \int_0^L f v_h \mathrm{d}x, \tag{3.2}$$

$$\forall v_h \in V_0 \equiv \left\{ v_h \in S_h^k(0,L) \middle| v_h(0) = 0, v_h'(L) = 0 \right\}.$$

**Theorem 3.1** Problem II has a unique solution.

**Proof:** The proof is similar to the proof of Theorem 2.1.

**Lemma 3.1**

For given positive constants $A, B, q, L$, there exists a constant $C$ independent of the solutions $u_h \in V_h$ of Problem II such that $\left\| \phi^{-1}(u_h') \right\|_{L^q} \le C$.

**Proof:**

The proof is similar to that of **Lemma 2.1**.

To derive finite element error estimates, let $u$ denotes the exact solution of Problem I and $u_h$ the finite element solution of Problem II.

Then

$$\begin{aligned}
& a(u, u-u_h) - a(u_h, u-u_h) \\
& = a(u, u-\Pi_h u) - a(u_h, u-\Pi_h u) \\
& = \int_0^L \left( \phi^{-1}(u') - \phi^{-1}(u_h') \right)(u' - \Pi_h u')\mathrm{d}x \\
& \quad + \int_0^L c(u-u_h)(u-\Pi_h u)\,\mathrm{d}x \\
& \le \left\| \phi^{-1}(u') - \phi^{-1}(u_h') \right\|_{L^q} \left\| u' - \Pi_h u' \right\|_{L^p} \\
& \quad + \left\| c(u-u_h) \right\|_{L^q} \left\| u - \Pi_h u \right\|_{L^p}
\end{aligned} \tag{3.3}$$

Let $\sigma \equiv \phi^{-1}(u')$, and $\sigma \equiv \phi^{-1}(u')$, . Also

$$\begin{aligned}
& a(u, u-u_h) - a(u_h, u-u_h) \\
& = \int_0^L \left( \phi^{-1}(u') - \phi^{-1}(u_h') \right)(u' - u_h')\mathrm{d}x + \int_0^L c(u-u_h)^2\,\mathrm{d}x \\
& = \int_0^L (\sigma - \sigma_h)\left( \phi(\sigma) - \phi(\sigma_h) \right)\mathrm{d}x + \int_0^L c(u-u_h)^2\,\mathrm{d}x \\
& = A\int_0^L (\sigma - \sigma_h)^2\,\mathrm{d}x + B\int_0^L \left( |\sigma|^{q-2}\sigma - |\sigma_h|^{q-2}\sigma_h \right)(\sigma - \sigma_h)\mathrm{d}x \\
& \quad + \int_0^L c(u-u_h)^2\,\mathrm{d}x \\
& \ge A\int_0^L |\sigma - \sigma_h|^2\,\mathrm{d}x + \int_0^L c(u-u_h)^2\,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

As a result of (3.3) and (3.4), we get:

$$\begin{aligned}
& \left\| c(u-u_h) \right\|_{L^2}^2 + \left\| \sigma - \sigma_h \right\|_{L^2}^2 \\
& \le \frac{1}{A}\Bigg[ \left\| \phi^{-1}(u') - \phi^{-1}(u_h') \right\|_{L^q} \left\| u' - (\Pi_h u)' \right\|_{L^p} \\
& \quad + \left\| c(u-u_h) \right\|_{L^q} \left\| u - \Pi_h u \right\|_{L^p} \Bigg]
\end{aligned} \tag{3.5}$$

By **Lemma 2.1**, **Lemma 3.1**, and (3.4), we get the following error estimates:

$$\begin{aligned}
& \left\| c(u-u_h) \right\|_{L^2}^2 + \left\| \sigma - \sigma_h \right\|_{L^2}^2 \\
& \le C\left( \left\| u' - \Pi_h u' \right\|_{L^p} + \left\| u - \Pi_h u \right\|_{L^p} \right) \\
& \le Ch^k
\end{aligned} \tag{3.6}$$

Therefore, by (3.6), we have established the following convergence and error estimate result.

**Theorem 2.3** For $c(x) = \sum_{i=1}^N k_i \delta(x - x_i)$, $k_i > 0$, or any $c(x) \ge 0$, let $u$ and $u_h$ be the unique solutions of Problems I and II, respectively, then

$$\left\| \sigma - \sigma_h \right\|_{L^2} \le Ch^{k/2}, \text{ and } \lim_{h \to 0} \left\| u' - u_h' \right\|_{L^p} = 0,$$

and if $c(x) \ge c_0$ for some $c_0 > 0$, or

$c(x) = \sum_{i=1}^N k_i \delta(x - x_i), k_i > 0,$ then

$$\left\| u - u_h \right\|_{L^2}^2 + \left\| \sigma - \sigma_h \right\|_{L^2} \le Ch^{k/2}, \text{ and } \lim_{h \to 0} \left\| u' - u_h' \right\|_{1,p} = 0,$$

in which $\sigma \equiv \phi^{-1}(u')$ and $\sigma_h \equiv \phi^{-1}(u_h')$ stand for the stresses.

Note that $\sigma$ stands for the stress corresponding to the strain $\varepsilon = u'$.

## 4. Conclusion

In this work, we establish existence and uniqueness of the solution $u$ of (2.4) in the Sobolev space $U$ and its finite element solution $u_h$ in a general finite element space $S_0^h(0,L)$ with elastic support for a class of load functions *f*. We derive convergence and error estimates for the semi-discrete error $e_h(x) \equiv u(x) - u_h(x)$.


## 5. Acknowledgements

The research in this paper is a part of a research project funded by the Research office, Texas A & M University at Qatar.


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