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On the Harmonic Index of Triangle-Free Graphs*

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ABSTRACT

The harmonic index of a graph G is defined as $H(G)=\sum_{uv\in E(G)}\frac{2}{d(u)+d(v)}$, where $d(u)$ denotes the degree of a vertex u in G. In this work, we give another expression for the Harmonic index. Using this expression, we give the minimum value of the harmonic index for any triangle-free graphs with order n and minimum degree $\delta\geq k$ for $k\leq n/2$ and show the corresponding extremal graph is the complete graph $K_{k,n-k}$.

Keywords: Harmonic Index; Minimum Degree; Triangle-Free

1. Introduction

All graphs considered in the following will be simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of graph G are the number of its vertices and number of its edges, respectively. For undefined terminology and notations, we refer the reader to [1].

For a graph G, the harmonic index $H(G)$ is defined as

$$H(G)=\sum_{uv\in E(G)}\frac{2}{d(u)+d(v)}.$$

It has been found that the harmonic index, which is a special case of general sum-connectivity index, correlates well with the Randić index [2,3] and the π-electronic energy of benzenoid hydrocarbons [4,5]. In [6], Favaron *et al.* considered the relation between harmonic index and the eigenvalues of graphs. Zhong [7] found the minimum and maximum values of the harmonic index for connected graphs and trees, and characterized the corresponding extremal graphs. Recently, Wu *et al.* [8] give a best possible lower bound for the harmonic index of a graph (a triangle-free graph, respectively) with order n and minimum degree at least two and characterize the extremal graphs. In this work, we will give a best possible lower bound for the harmonic index of a triangle-free graph with order n and minimum degree at least k. We show the corresponding extremal graph is the complete bipartite graph $K_{k,n-k}$.

2. Another Expression for the Harmonic Index

Before we go forwards to investigate the relationship between the Harmonic index and the minimum degree $\delta(G)$ of triangle-free graphs, we will give another expression for the Harmonic index in this section, which is vital in sequel.

Let G be a graphs with order n and minimum degree $\delta(G)\geq k$. Denote by $x_{i,j}$ $(x_{i,j}\geq 0)$, the number of edges joining the vertices of degrees i and j. Denote by n_i the number of vertices of degree of i. Then

$$H(G)=\sum_{k\leq i\leq j\leq n-1}\frac{2}{i+j}x_{i,j} \tag{1}$$

$$n_k+n_{k+1}+\cdots+n_{n-1}=n. \tag{2}$$

By counting the edges that incident to a vertex of degree i, $i=k,\cdots,n-1$, one obtains

$$\sum_{\substack{j=k\\ j\neq i}}^{n-1}x_{i,j}+2x_{i,i}=in_i,\quad i.e.\ \ n_i=\frac{1}{i}\left(\sum_{\substack{j=k\\ j\neq i}}^{n-1}x_{i,j}+2x_{i,i}\right). \tag{3}$$

Substituting Equation (3) back into Equation (2) and performing appropriate rearrangements, we get

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$$\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}\right)x_{i,j}+2\sum_{i=k}^{n-1}\frac{x_{i,i}}{i}=n. \tag{4}$$

Now, rewriting Equation (1) as

$$H=\sum_{k\le i<j\le n-1}\frac{2}{i+j}x_{i,j}+\sum_{i=k}^{n-1}\frac{x_{i,i}}{i} \tag{5}$$

and combining Equations (4) and (5) so as to eliminate the term $\sum(x_{ii}/i)$, we arrive at

$$n-2H=\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}=\sum_{k\le i<j\le n-1}\frac{(i-j)^2}{ij(i+j)}x_{i,j}$$

i.e.,

$$H=\frac{n}{2}-\frac{1}{2}\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j} \tag{6}$$

$$=\frac{n}{2}-\frac{1}{2}\sum_{k\le i<j\le n-1}\frac{(i-j)^2}{ij(i+j)}x_{i,j} \tag{7}$$

Remark 2.1 *From* (7), *we see that* $H(G)\le\frac{n}{2}$ *for* n*-vertex graph* G *and the equality holds if and only if* G *is regular.*

3. Main Results

First, we give a lower bound for any triangle-free graph with order n and size m.

Lemma 3.1 *For any triangle-free graph* G *with order* n *and size* m, *then*

$$H(G)\ge\frac{2m}{n},$$

where equality holds if and only if m is of the form $m=p(n-p)$ for some natural numbers p, and $G\cong K_{p,n-p}$.

Proof. For any edge uv of G, we have $d(u)+d(v)\le n$, or it would contain triangle(s). By (1), we have

$$H(G)=\sum_{uv\in E(G)}\frac{2}{d(u)+d(v)}\ge\sum_{uv\in E(G)}\frac{2}{n}=\frac{2m}{n},$$

equality holds if and only if $d(u)+d(v)=n$ for every $uv\in E(G)$. Thus, if we denote $d(u)=p$ for an edge uv, then each of the p neighbors, including v, of u should has degree $n-p$. Similarly, each of the $n-p$ neighbors of v has degree p. Therefore, $m=p(n-p)$ and $G\cong K_{k,n-k}$.

Lemma 3.2 *Let* $w(i,j)=\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}$, *for* $0<i<j$, *then* $w(i,j)$ *is decreasing in* i *and increasing in* j.

Proof. We have

$$\frac{\partial w(i,j)}{\partial i}=-\frac{1}{i^2}+\frac{4}{(i+j)^2}=\frac{(3i+j)(i-j)}{i^2(i+j)^2}<0 \text{ and}$$

$$\frac{\partial w(i,j)}{\partial j}=-\frac{1}{j^2}+\frac{4}{(i+j)^2}=\frac{(i+3j)(j-i)}{j^2(i+j)^2}>0 \text{ for}$$

$i<j$.

Theorem 3.3 *Let* G *be a triangle-free graph with order* n *and the minimum degree* $\delta(G)\ge k$ $(k\le n/2)$. *Then*

$$H(G)\ge\frac{2k(n-k)}{n},$$

where equality holds if and only if $G\cong K_{k,n-k}$.

Proof. Assume m is the size of graph G. We divide the proof into the following two cases.

Case 1. $m>k(n-k)$. The result follows by Lemma 3.1.

Case 2. $m\le k(n-k)$. Note that the maximum degree $\Delta(G)\le n-k$ for graph G, or it would contain triangle(s). By (6) and Lemma 3.2, we have

$$\begin{aligned}
H(G)&=\frac{n}{2}-\frac{1}{2}\sum_{k\le i<j\le n-1}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}\\
&=\frac{n}{2}-\frac{1}{2}\sum_{k\le i<j\le n-k}\left(\frac{1}{i}+\frac{1}{j}-\frac{4}{i+j}\right)x_{i,j}\\
&\ge\frac{n}{2}-\frac{1}{2}\sum_{k\le i<j\le n-k}\left(\frac{1}{k}+\frac{1}{n-k}-\frac{4}{n}\right)x_{i,j}\\
&=\frac{n}{2}-\frac{1}{2}\left(\frac{1}{k}+\frac{1}{n-k}-\frac{4}{n}\right)\sum_{k\le i<j\le n-k}x_{i,j}\\
&\ge\frac{n}{2}-\frac{m}{2}\left(\frac{1}{k}+\frac{1}{n-k}-\frac{4}{n}\right)\\
&\ge\frac{n}{2}-\frac{k(n-k)}{2}\left(\frac{1}{k}+\frac{1}{n-k}-\frac{4}{n}\right)\\
&=\frac{2k(n-k)}{n}.
\end{aligned}$$

For equalities to hold above, we must have $m=k(n-k)=x_{k,n-k}$, which means that $G\cong K_{k,n-k}$.

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