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Best Simultaneous Approximation of Finite Set in Inner Product Space

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ABSTRACT

In this paper, we find a way to give best simultaneous approximation of *n* arbitrary points in convex sets. First, we introduce a special hyperplane which is based on those *n* points. Then by using this hyperplane, we define best approximation of each point and achieve our purpose.

Keywords: Best Approximation; Hyperplane; Best Simultaneous Approximation

1. Introduction

As we known, best approximation theory has many applications. One of the best results is best simultaneous approximation of a bounded set but this target cannot be achieved easily. Frank Deutsch in [1] defined hyperplanes and gave the best approximation of a point in convex sets.

In [3,4] we can see that a hyperplane of an *n*-dimensional space is a flat subset with dimension $n-1$.

In this paper we try to find best simultaneous approximation of *n* arbitrary points in convex sets. We say theorems of best approximation of a point in convex sets.

Then we give the method of finding best simultaneous approximation of *n* points in convex set.

2. Preliminary Notes

In this paper, we consider that *X* is a real inner product space. For a nonempty subset *W* of *X* and $x \in X$, define

$$d(x,W) = \inf_{w\in W} \|x-w\|.$$

Recall that a point $w_0 \in W$ is a best approximation of $x \in X$ if $\|x-w_0\| = d(x,W)$.

If each $x \in X$ has at least one best approximation $w_0 \in W$, then *W* is called proximinal.

We denote by $\mathbf{P}_W(x)$, the set of all best approximations of *x* in *W*. Therefore

$$\mathbf{P}_W(x) := \{w : w \in W, \|x-w\| = d(x,W)\}$$

It is well-known that $\mathbf{P}_W(x)$ is a closed and bounded subset of *X*. If $x \in X$, then $\mathbf{P}_W(x)$ is located in the boundary of *W*.

In 2.4 lemma of [1] we can see that if *K* be a convex subset of *X*. Then each $x \in X$ has at most one best approximation in *K*.

In particular, every closed convex subset *K* of a Hilbert space *X* has a unique best approximation in *K*.

Also in 4.1 lemma of [1] if *K* be a convex set and $x \in X$, $y_0 \in K$. Then $y_0 = P_K(x)$ if and only if

$$\langle x-y_0, y-y_0\rangle \le 0 \text{ for all } y \in K$$

For a nonempty subset *W* of *X* and a nonempty bounded set *S* in *X*, define

$$d(S,W) = \inf_{w\in W} \sup_{s\in S} \|s-w\|$$

and

$$\mathbf{P}_W(S) = \left\{w \in W : \sup_{s\in S}\|s-w\| = d(S,W)\right\}$$

Each element in $\mathbf{P}_W(S)$ (If $\mathbf{P}_W(S) \ne \emptyset$) is called a best simultaneous approximation to *S* from *W* (see [2] Preliminary Notes).

For $f \in X^* \setminus (0)$ and $c \in R$ hyperplane *H* in *X* defined by

$$H = \{y \in X; f(y) = c\}$$

and we denote *H* by $H = \langle f, c\rangle$.

The Kernel of a functional *f* is the set

$$\ker f := \langle f, 0\rangle$$

and for

$$H = \langle f, c\rangle,$$

we say that $x \in X$ is in the below of hyperplane H, if $f(x) \le c$.

3. Best simultaneous Approximation in Convex Sets

In this section,we consider

$$S = \{x-1, x-2, \cdots, x-n\}$$

and

$$i = 1, \cdots, n,\ k = 1, 2, \cdots, n(k \ne i)$$

Define

$$W_i := \{w \in W; \max_{x_j \in S} d(w, x_j) = d(w, x_i)\}$$

$$f_{ik}(y) := \langle y, x_i - x_k \rangle, \forall y \in X$$

$$c_{ik} := \frac{\|x_i\|^2 - \|y_i\|^2}{2} \tag{1.1}$$

$$V_{ik} := \{v \in W; f_{ik}(v) \le c_{ik}\}$$

$$H_{ik} := \{v \in W; f_{ik}(v) = c_{ik}\}$$

Lemma 3.1. Let $x_i, x_k \in S$ consider the hyperplane $H = \{y \in X\ ;\ f_{ik}(y) = c_{ik}\}$ then

$$d(x_i, H) = d(x_k, H)$$

Proof. Give $y \in H$ so we have

$$f_{ik}(y) = \langle y, x_i - x_k \rangle = c_{ik}$$

$$\langle y, x_i \rangle - \langle y, x_k \rangle = \frac{\|x_i\|^2 - \|x_k\|^2}{2}$$

$$\|x_k\|^2 - 2\langle y, x_k \rangle = \|x_i\|^2 - 2\langle y, x_i \rangle$$

So by adding $\|y\|^2$ with equation of above, we have

$$\|y\|^2 + \|x_k\|^2 - 2\langle y, x_k \rangle = \|y\|^2 + \|x_i\|^2 - 2\langle y, x_i \rangle$$

$$\langle x_k - y, x_k - y \rangle = \langle x_i - y, x_i - y \rangle$$

Therefore have

$$\|x_k - y\|^2 = \|x_i - y\|^2$$

$$d(x_k, y) = d(x_i, y) \qquad ■$$

Note 3.2. It is obvious that $\bigcup_i W_i \subseteq W$. Now let $w \in W$, so there exist $i \in \{1, 2, \cdots, n\}$ such that $d(w, x_i) \ge d(w, x_j)$ for all $j \in \{1, 2, \cdots, n\}$.

Thus $d(w, x_i) = \max_{x_j \in S} d(w, x_j)$, therefore w will be in W_i, that we conclude

$$W = \bigcup_i W_i.$$

Theorem 3.3. Let $i = 1, 2, \cdots, n$ then:

1) $W_i = \bigcap_{k=1,\ k\ne i}^{n} V_{ik}$

2) If W be a convex subset of X, then W_i is a convex set.

3) If W be a closed set, then W_i is a closed set.

Proof. 1) Let $v \in \bigcap_{k=1,\ k\ne i}^{n} V_{ik}$ therefore $v \in V_{ik} \forall k = 1, \cdots, n(k \ne i)$ so $f_{ik}(v) \le c_{ik}$ then we have

$$\langle v, x_i - x_k \rangle \le \frac{\|x_i\|^2 - \|x_k\|^2}{2}$$

$$2\langle v, x_i \rangle - 2\langle y, x_k \rangle \le \|x_i\|^2 - \|x_k\|^2$$

$$\|x_k\|^2 - 2\langle y, x_k \rangle \le \|x_i\|^2 - 2\langle v, x_i \rangle$$

so by adding $\|v\|^2$ with equation of above, we have

$$\|v\|^2 + \|x_k\|^2 - 2\langle v, x \rangle_k \le \|v\|^2 + \|x_i\|^2 - 2\langle v, x_i \rangle$$

$$\langle x_k - v, x_k - v \rangle \le \langle x_i - v, x_i - v \rangle$$

therefore we have

$$\|v\|^2 + \|x_k\|^2 - 2\langle v, x \rangle_k \le \|v\|^2 + \|x_i\|^2 - 2\langle v, x_i \rangle$$

$$\langle x_k - v, x_k - v \rangle \le \langle x_i - v, x_i - v \rangle.$$

Thus we have

$$\|x_k - v\|^2 \le \|x_i - v\|^2$$

$$d(x_k, v) \le d(x_i, v) \forall k = 1, \cdots, n(k \ne i).$$

Therefore $v \in W_i$.

Since all previous steps will be reversible, so for any $w \in W_i$ in a fixed i, we have $\sup_{x_j \in S} d(w, x_j) = d(w, x_i)$ that consider

$$\|x_k - w\|^2 \le \|x_i - w\|^2\ \forall k = 1, \cdots, n$$

thus we have

$$f_{ik}(w) \le c_{ik}\ \ \forall k = 1, \cdots, n(k \ne i)$$

so

$$w \in V_{ik}\ \ \forall k = 1, \cdots, n(k \ne i)$$

therefore

$$w \in \bigcap_{k=1,\ k\ne i}^{n} V_{ik}$$

and finally

$$W_i = \bigcap_{k=1, k\ne i}^{n} V_{ik}.$$

2) First we proof V_{ik}, for all $i, k(k \ne i)$ is convex set.

Give $y_1, y_2 \in V_{ik}$ and $0 \le \lambda \le 1$, set

$$y := \lambda y_1 + (1-\lambda) y_2$$

thus we have

$$\begin{aligned} f(y) &= \lambda f(y_1) + (1-\lambda) f(y_2) \\ &\le \lambda c_{ik} + (1-\lambda) c_{ik} = c_{ik} \end{aligned}$$

So $y \in V_{ik}$. Thus V_{ik} is convex set and since intersection of any convex set is convex, therefore W_i is convex set.

3) It is obviously that f is continuous function and we know

$$V_{ik} = f^{-1}\left[c_{ik}, +\infty\right) \cap W \,.$$

So, V_{ik} is closed set, this implies W_i is closed set. ■

4. Algorithm

The following theorem states that to find best simultaneous approximation of a bounded set S of W, it is enough to obtain the best approximation to any

$$x_i \text{ in } W_i \left(i.e.\, P_{w_i}\left(x_j\right)\right).$$

Thus $P_{w_i}\left(x_j\right)$ would be the best simultaneous approximation of S from W if $d\left(x_j, P_{W_j}\left(x_j\right)\right)$ is minimal.

Theorem 4.1. If W be a convex subset of X and there exist $P_{w_i}\left(x_j\right)$ for all $i = 1, 2, \cdots, n$, then

$$d\left(S, W\right) = \inf_i d\left(\mathbf{P}_{W_i}\left(x_i\right), x_i\right) = \inf_i d\left(W_i, x_i\right)$$

Proof. With attention of best simultaneous approximation and (3.2) notation, we have

$$\begin{aligned} d\left(S, W\right) &= \inf_{w \in W} \sup_{x_j \in S} \left\| x_j - w \right\| \\ &= \inf_i \inf_{w \in W_i} \sup_{x_j \in S} \left\| x_j - w \right\| \end{aligned}$$

but according to the definition of W_i we have

$$\sup_{x_j \in S} \left\| x_j - w \right\| = \left\| x_i - w \right\|$$

thus the above equation can be written as follows

$$d\left(S, W\right) = \inf_i \inf_{w \in W_i} \left\| x_i - w \right\| = \inf_i d\left(W_i, x_i\right)$$

and since exist

$$\mathbf{P}_{W_i}\left(x_i\right) \in W_i$$

so we have

$$d\left(S, W\right) = \inf_i d\left(\mathbf{P}_{W_i}\left(x_i\right), x_i\right).$$ ■

Corollary 4.2. With the assumptions of the previous theorem there exist i, such that $\mathbf{P}_{W_i}\left(x_i\right)$ is best simultaneous approximation of S in W.

Proof. With attention previous theorem, there exist $i \in \{1, 2, \cdots, n\}$ such that

$$d\left(S, W\right) = d\left(\mathbf{P}_{W_i}\left(x_i\right), x_i\right)$$

and by the definition of W_i we have

$$d\left(S, W\right) = d\left(\mathbf{P}_{W_i}\left(x_i\right), x_i\right) = \sup_{x_j \in S} d\left(\mathbf{P}_{W_i}\left(x_i\right), x_j\right)$$

after according to the above equation and define the best simultaneous approximation of the relationship will

$$\mathbf{P}_{W_i}\left(x_i\right) \in \mathbf{P}_W\left(S\right)$$

However, the algorithm with assumes a convex set W and $S = \{x_1, x_2, \cdots, x_n\}$ introduce the following.

With attention **3.1** lemma for points x_1, x_2 the hyperplane $H_{12} = \{y \in X;\ f_{12}(y) = c_{12}\}$ are possible to obtain, by **3.4** definition the points W in below H_{12} are V_{12} called.

Also for points x_1, x_3 the hyperplane

$$H_{13} = \{y \in X;\ f_{13}(y) = c_{13}\}$$

are formed and the points of W in below H_{13} are V_{13} called and so we do order to the points x_1, x_n.

By taking subscribe of any V_{1n}, find W_1 that this set is convex (by Theorem 3.3, 2).

Therefore, if best approximation x_1 exists in this set, it is called $\mathbf{P}_{W_1}\left(x_1\right)$. Thus obtain $\mathbf{P}_{W_i}\left(x_i\right)$ for any $i = \{1, 2, \cdots, n\}$.

Finally, the point $\mathbf{P}_{W_j}\left(x_j\right)$ which has minimal distance to x_i, is the best simultaneous approximation of S in W.

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