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Two-Sided First Exit Problem for Jump Diffusion Processes Having Jumps with a Mixture of Erlang Distribution*

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ABSTRACT

In this paper, we consider the two-sided first exit problem for jump diffusion processes having jumps with rational Laplace transforms. We investigate the probabilistic property of conditional memorylessness, and drive the joint distribution of the first exit time from an interval and the overshoot over the boundary at the exit time.

Keywords: First Exit Time; Two-Sided Jumps; Jump Diffusion Process; Overshoot

1. Introduction

Consider the following jump diffusion process

$$X_t = u + ct + \sigma W_t + \sum_{k=1}^{N_t} Y_k,\ t \ge 0, \tag{1.1}$$

where the constant u is the starting point of $\{X_t\}_{t\ge0}$, c and σ represent the drift and the volatility of the diffusion part, respectively, $\{W_t\}_{t\ge0}$ is a standard Brownian motion with $W_0 = 0$, $\{N_t\}_{t\ge0}$ is a Poisson process with rate λ, and the jumps sizes $\{Y_1, Y_2, \cdots\}$ are assumed to be i.i.d. real valued random variables with common density $p(x)$. Moreover, it is assumed that the random processes $\{W_t\}_{t\ge0}$, $\{N_t\}_{t\ge0}$ and random variables $\{Y_1, Y_2, \cdots\}$ are mutually independent. In this paper we are interested in the density p of following type

$$p(x) = \sum_{j=1}^{J}\sum_{i=1}^{m_j} p_{ij} \frac{\rho_j^i x^{i-1}}{(i-1)!} e^{-\rho_j x} I_{\{x>0\}} + \sum_{j=1}^{\hat{J}}\sum_{i=1}^{\hat{m}_j} \hat{p}_{ij} \frac{\hat{\rho}_j^i |x|^{i-1}}{(i-1)!} e^{\hat{\rho}_j x} I_{\{x<0\}}, \tag{1.2}$$

where $J, \hat{J}, m_j, \hat{m}_j \in \mathbb{N}$, $p_{ij}, \hat{p}_{ij} \in \mathbb{R}^+$, $\mathrm{Re}(\rho_j) > 0$, $\mathrm{Re}(\hat{\rho}_j) > 0$ and that $\rho_i \ne \rho_j$, $\hat{\rho}_i \ne \hat{\rho}_j$ for all $i \ne j$. Moreover,

$$\sum_{j=1}^{J}\sum_{i=1}^{m_j} p_{ij} + \sum_{j=1}^{\hat{J}}\sum_{i=1}^{\hat{m}_j} \hat{p}_{ij} = 1.$$

Define τ to be the first exit time of X_t to two flat barriers a and b $(a < b)$, *i.e.*

$$\tau = \inf\{t \ge 0 : X_t \ge b \text{ or } X_t \le a\}.$$

Recently, one-sided and two-sided first exit problems for processes with two-sided jumps have attracted a lot of attentions in applied probability (see [1-7]). For example, Perry and Stadje [1] studied two-sided first exit time for processes with two-sided exponential jumps; Kou and Wang [2] studied the one-sided first passage times for a jump diffusion process with exponential positive and negative jumps. Cai [3] investigated the first passage time of a hyper-exponential jump diffusion process. Cai *et al.* [4] discussed the first passage time to two barriers of a hyper-exponential jump diffusion process. Closed form expressions are obtained in Kadankova and Veraverbeke [5] for the integral transforms of the joint distribution of the first exit time from an interval and the value of the overshoot through boundaries at the exit time for the Poisson process with an exponential component. For some related works, see Perry *et al.* [8], Cai and Kou [9], Lewis and Mordecki [10] and the references therein.

Motivated by works mentioned above, the main objective of this paper is to study the first exit time of the process (1.1) with jump density (1.2) from an interval

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and the overshoot over the boundary at the exit time. In Section 2, we study the roots of the generalized Lundberg equation and conditional memory lessness. The main results of this paper are given in Section 3.

2. Preliminary Results

It is easy to see that the infinitesimal generator of $\{X_t\}_{t\ge 0}$ is given by

$$(L\varphi)(x)=\frac{1}{2}\sigma^2\varphi''(x)+c\varphi'(x)+\lambda\int_{-\infty}^{\infty}\left[u(x+y)-u(x)\right]p(y)\mathrm{d}y$$

for any twice continuously differentiable function $\varphi(x)$ and the Lévy exponent of $\{X_t\}_{t\ge 0}$ is given by

$$g(z)=\frac{1}{t}\ln E\mathrm{e}^{zX_t}=\frac{1}{2}z^2\sigma^2+zc+\lambda\left(\sum_{j=1}^{J}\sum_{i=1}^{m_j}\frac{p_{ij}(\rho_j)^i}{(\rho_j-z)^i}+\sum_{j=1}^{\hat{J}}\sum_{i=1}^{\hat{m}_j}\frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j+z)^i}-1\right).$$

By analytic continuation, the function $g(z)$ can be extended to the complex plane except at finitely many poles. In the following, we consider the resulting extension $G(z)$ of $g(z)$, *i.e.*,

$$G(z)=\frac{1}{2}z^2\sigma^2+zc+\lambda\left(\sum_{j=1}^{J}\sum_{i=1}^{m_j}\frac{p_{ij}(\rho_j)^i}{(\rho_j-z)^i}+\sum_{j=1}^{\hat{J}}\sum_{i=1}^{\hat{m}_j}\frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j+z)^i}-1\right), z\in C.$$

Let us denote $M=\sum_{j=1}^{J}m_j$ and $\hat{M}=\sum_{j=1}^{\hat{J}}\hat{m}_j$.

In [11], Kuznetsov has studied the roots of the equation $G(z)=\theta$. However, for this particular Lévy process X, we will give another simple proof for the roots of this equation.

Lemma 2.1. For fix $\theta>0$, the generalized Cramér-Lundberg equation

$$G(z)=\theta$$

has $M+\hat{M}+2$ complex roots $\gamma_1(\theta),\gamma_2(\theta),\cdots,\gamma_{M+1}(\theta)$ with $\mathrm{Re}(\gamma_i(\theta))>0$ for $i=1,2,\cdots,M+1$ and $\hat{\gamma}_1(\theta),\hat{\gamma}_2(\theta),\cdots,\hat{\gamma}_{\hat{M}+1}(\theta)$ with $\mathrm{Re}(\hat{\gamma}_i(\theta))<0$ for $i=1,2,\cdots,\hat{M}+1$.

Proof. Let

$$G_1(z)=\frac{1}{2}z^2\sigma^2+zc-\lambda-\theta, z\in\mathbb{C},$$

$$G_2(z)=\lambda\left(\sum_{j=1}^{J}\sum_{i=1}^{m_j}\frac{p_{ij}(\rho_j)^i}{(\rho_j-z)^i}+\sum_{j=1}^{\hat{J}}\sum_{i=1}^{\hat{m}_j}\frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j+z)^i}\right), z\in\mathbb{C}.$$

Firstly, we prove that for given $\theta>0$, $G(z)=\theta$ has $\hat{M}+1$ roots with negative real parts. Set $C_r^-=\{z:|z|=r,z\in C^-\}$ with $r>\varepsilon+\max_{1\le j\le\hat{J}}\{|\hat{\rho}_j|\}$, where ε is an arbitrary positive constant. Applying Rouchés theorem on the semi-circle C_r^-, consisting of the imaginary axis running from $-ir$ to ir and with radius r running clockwise from ir to $-ir$. We let $r\to\infty$ and denote by C^- the limiting semi-circle. It is known that both $\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_1(z)$ and $\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_2(z)$ are analytic in C^-. We want to show that

$$\left|\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_1(z)\right|>\left|\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_2(z)\right|, z\in C^-.$$

Notice that $|G_1(z)|\to\infty$ for $\mathrm{Re}(z)\to-\infty$, and $|G_2(z)|\le\lambda\sum_{j=1}^{J}\sum_{i=1}^{m_j}\frac{|p_{ij}||\rho_j|^i}{\varepsilon^i}+\lambda\sum_{j=1}^{\hat{J}}\sum_{i=1}^{\hat{m}_j}\frac{|\hat{p}_{ij}||\hat{\rho}_j|^i}{\varepsilon^i}$ is bounded for $\mathrm{Re}(z)\to-\infty$. Hence, for $\mathrm{Re}(z)\to-\infty$,

$$\left|\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_1(z)\right|>\left|\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_2(z)\right|$$

on the boundary of the half circle in C^-. For $a\in R$, we have $|G_2(ia)|<\lambda$ (see Lewis and Mordecki [10]). On the other hand,

$$|G_1(\mathrm{i}a)|\ge-\mathrm{Re}G_1(\mathrm{i}a)=\frac{1}{2}\sigma^2a^2+\lambda+\theta>\lambda.$$

Thus we have $|G_1(\mathrm{i}a)|>|G_2(\mathrm{i}a)|$. Since $\left(\prod_{j=1}^{\hat{J}}(\hat{\rho}_j+z)^{\hat{m}_j}\right)G_1(z)$ has $\hat{M}+1$ roots with negative real parts, so equation $G(z)=\theta$ has $\hat{M}+1$ roots with negative real parts. Similarly, we can prove $G(z)=\theta,\theta>0$ has $M+1$ roots with positive real parts.

In the rest of this paper, we assume all the roots of equation $G(z)=\theta$ are distinct and denote $\gamma_i(\theta)=\gamma_i\ (i=1,2,\cdots,M+1)$, $\hat{\gamma}_i(\theta)=\hat{\gamma}_i\ (i=1,2,\cdots,\hat{M}+1)$ for notational simplicity, and denote E^u (or P^u in the sequel) representing the expectation (or probability) when X_t starts from u. We denote a sequence of events

$$K_0=\{\omega:X_\tau=b\},\quad G_0=\{\omega:X_\tau=a\}$$

$K_{jil}=\{\omega:\ X_{(\cdot)}$ crosses b at time τ by the l th phase of i th positive jump whose parameter is $\rho_j\}$,

$G_{j'i'l'}$ = { ω : $X_{(\cdot)}$ crosses a at time τ by the l'th phase of i'th negative jump whose parameter is $\hat{\rho}_{j'}$ }
for $j=1,2,\cdots,J$, $j'=1,2,\cdots,\hat{J}$, $i=1,\cdots,m_j$, $i'=1,\cdots,\hat{m}_{j'}$, $l=1,\cdots,i$ and $l'=1,\cdots,i'$.

Theorem 2.2. For any $x>0$, we have

$$P^u\left(X_\tau-b\ge x\middle|K_{jil}\right)=\sum_{h=0}^{i-l}\frac{\left(\rho_j x\right)^h}{h!}\mathrm{e}^{-\rho_j x}, \tag{2.1}$$

$$P^u\left(X_\tau-a\le -x\middle|G_{j'i'l'}\right)=\sum_{h=0}^{i'-l'}\frac{\left(\hat{\rho}_{j'} x\right)^h}{h!}\mathrm{e}^{-\hat{\rho}_{j'} x}. \tag{2.2}$$

Furthermore, conditional on $\left\{\omega:\omega\in K_{jil}\right\}$ $\left(\left\{\omega:\omega\in G_{j'i'l'}\right\}\right)$, the stopping time τ is independent of the overshoot $X_\tau-b$ (the undershoot $X_\tau-a$). More precisely, for any $x>0$, we have

$$\begin{aligned}&P^u\left(\tau\le t, X_\tau-b\ge x\middle|K_{jil}\right)\\&=P^u\left(\tau\le t,\middle|K_{jil}\right)\times P^u\left(X_\tau-b\ge x\middle|K_{jil}\right),\end{aligned} \tag{2.3}$$

$$\begin{aligned}&P^u\left(\tau\le t, X_\tau-a\le -x\middle|G_{j'i'l'}\right)\\&=P^u\left(\tau\le t,\middle|G_{j'i'l'}\right)\times P^u\left(X_\tau-a\le -x\middle|G_{j'i'l'}\right).\end{aligned} \tag{2.4}$$

Proof. Firstly, we prove (2.1) and (2.3). It suffices to show

$$\begin{aligned}&P^u\left(\tau\le t, X_\tau-b\ge x, K_{jil}\right)\\&=\sum_{h=0}^{i-l}\frac{\left(\rho_j x\right)^h}{h!}\mathrm{e}^{-\rho_j x}P^u\left(\tau\le t, K_{jil}\right),\end{aligned} \tag{2.5}$$

since (2.1) can be obtained by letting $t\to\infty$ in (2.5) and then dividing both sides of the resulting equation by $P^u\left(K_{jil}\right)$. It is known that an Erlang(n) random variable can be expressed as an independent sum of n exponential random variables with same parameters. Let $V_i,\left(i=1,2,\cdots,n\right)$ the n independent exponentially distributed random variables with parameter ρ_i. Denote by $T_1,T_2,\cdots$ the arrival times of the Poisson process N, and let $\mathfrak{F}_{t-}=\sigma\left(\left\{X_s,0\le s<t\right\}\right)$ be the field generated by process X_s, $0\le s<t$. It follows that

$$\begin{aligned}&P^u\left(\tau\le t, X_\tau-b\ge x, I_{K_{jil}}\right)\\&=\sum_{n=1}^{\infty}P^u\left(T_n=\tau\le t, X_\tau-b\ge x, I_{K_{jil}}\right)=\sum_{n=1}^{\infty}P_n.\end{aligned}$$

With $U_u\left(t\right)=u+ct+\sigma W\left(t\right)$, we have

$$\begin{aligned}P_n&=P^u\left(T_n=\tau\le t, X_\tau-b\ge x, I_{K_{jil}}\right)\\
&=P^u\left(\max_{0\le s<T_n}X_s<b, X_{T_n}-b\ge x, T_n=\tau\le t, I_{K_{jil}}\right)\\
&=E\left[P^u\left(X_{T_n}-b\ge x, I_{K_{jil}}\middle|\mathfrak{F}_{T_n-},T_n\right)I_{\left(\max\limits_{0\le s<T_n}X_s<b,T_n=\tau\le t\right)}\right]\\
&=E\left[P^u\left(U_u\left(T_n\right)+Y_1+Y_2+\cdots+Y_n-b\ge x, I_{K_{jil}}\middle|\mathfrak{F}_{T_n-},T_n\right)I_{\left(\max\limits_{0\le s<T_n}X_s<b,T_n=\tau\le t\right)}\right]\\
&=E\left[P^u\left(Y_n\ge b+x-\left(U_u\left(T_n\right)+\sum_{k=1}^{n-1}Y_k\right), I_{K_{jil}}\middle|\mathfrak{F}_{T_n-},T_n\right)I_{\left(\max\limits_{0\le s<T_n}X_s<b,T_n=\tau\le t\right)}\right]\\
&=p_{ij}E\left[P^u\left(\sum_{k=l}^{i}V_k\ge b+x-U_u\left(T_n\right)-\sum_{k=1}^{n-1}Y_k-\sum_{k=1}^{l-1}V_k, V_l\ge b-U_u\left(T_n\right)-\sum_{k=1}^{n-1}Y_j-\sum_{k=1}^{l-1}V_k\middle|\mathfrak{F}_{T_n-},T_n\right)I_{\left(\max\limits_{0\le s<T_n}X_s<b,T_n=\tau\le t\right)}\right]\\
&=\left(\sum_{h=0}^{i-l}\frac{\left(\rho_j x\right)^h}{h!}\mathrm{e}^{-\rho_j x}\right)P^u\left(T_n=\tau\le t, I_{K_{jil}}\right).\end{aligned}$$

Thus we have

$$\begin{aligned}&P^u\left(\tau\le t, X_\tau-b\ge x, I_{K_{jil}}\right)\\&=\left(\sum_{h=0}^{i-l}\frac{\left(\rho_j x\right)^h}{h!}\mathrm{e}^{-\rho_j x}\right)P^u\left(\tau\le t, I_{K_{jil}}\right).\end{aligned}$$

(2.2) and (2.4) can be obtained similarly. This completes the proof.

The following results are immediate consequences of Theorem 2.2.

Corollary 2.3. For $j=1,2,\cdots,J$, $j'=1,2,\cdots,\hat{J}$, $i=1,\cdots,m_j$, $i'=1,\cdots,\hat{m}_{j'}$, $l=1,\cdots,i$, $l'=1,\cdots,i'$, we have

$$E^u\left[e^{\alpha(X_\tau-b)}I(K_{jil})\right]=\left(\frac{\rho_j}{\rho_j-\alpha}\right)^{i+1-l},\ \alpha\le 0,$$

$$E^u\left[e^{\alpha(X_\tau-a)}I(G_{j'i'l'})\right]=\left(\frac{\hat\rho_j}{\hat\rho_j+\alpha}\right)^{i'+1-l'},\ \alpha\ge 0.$$

Corollary 2.4. For any $x>0$, we have

$$P^u\left(X_\tau-b\ge x\middle|K_{jk}\right)=\sum_{h=0}^{k-1}\frac{(\rho_j x)^h}{h!}e^{-\rho_j x},$$

$$P^u\left(X_\tau-a\le -x\middle|G_{j'k'}\right)=\sum_{h=0}^{k'-1}\frac{(\hat\rho_{j'} x)^h}{h!}e^{-\hat\rho_j x},$$

$$P^u\left(\tau\le t, X_\tau-b\ge x\middle|K_{jk}\right)$$
$$=P^u\left(\tau\le t,\middle|K_{jk}\right)\times P^u\left(X_\tau-b\ge x\middle|K_{jk}\right),$$

$$P^u\left(\tau\le t, X_\tau-a\le -x\middle|G_{j'k'}\right)$$
$$=P^u\left(\tau\le t,\middle|G_{j'k'}\right)\times P^u\left(X_\tau-a\le -x\middle|G_{j'k'}\right)$$

where

$$K_{jk}=K_{jk1}\cup K_{j(k+1)2}\cup\cdots\cup K_{jm_j(m_j-k+1)},$$

$$G_{j'k'}=G_{j'k'1}\cup G_{j'(k'+1)2}\cup\cdots\cup G_{j'm'_j(m'_j-k'+1)}$$

for $j=1,2,\cdots,J$, $k=1,2,\cdots,m_j$, $j'=1,2,\cdots,\hat J$, $k'=1,2,\cdots,\hat m_{j'}$.

Corollary 2.5. For $j=1,2,\cdots,J$, $k=1,2,\cdots,m_j$, $j'=1,2,\cdots,\hat J$, $k'=1,2,\cdots,\hat m_{j'}$, we have

$$E^u\left[e^{\alpha(X_\tau-b)}I(K_{jk})\right]=\left(\frac{\rho_j}{\rho_j-\alpha}\right)^{k},\ \alpha\le 0,$$

$$E^u\left[e^{\alpha(X_\tau-a)}I(G_{j'k'})\right]=\left(\frac{\hat\rho_j}{\hat\rho_j+\alpha}\right)^{k'},\ \alpha\ge 0.$$

Remark 2.6. When $m_i=1$, $n_j=1$, (2.1) and (2.3) reduce to Equations (8) and (9) of Cai [3], respectively.

3. Main Results

In this section, we study the distribution of the first exit problem to two barriers. We first define three vectors:

$$f=\left(f(b),f_{11}^u,\cdots,f_{jk}^u,\cdots,f_{Jm_J}^u,f(a),f_{11}^d,\cdots,f_{j'k'}^d,\cdots,f_{\hat J\hat m_{\hat J}}^d\right)^{\mathrm T},$$

$$\pi=\left(E^u\left[e^{-\theta\tau}I_{K_0}\right],\cdots,E^u\left[e^{-\theta\tau}I_{K_{Jm_J}}\right],E^u\left[e^{-\theta\tau}I_{G_0}\right],\cdots,E^u\left[e^{-\theta\tau}I_{G_{\hat J\hat m_{\hat J}}}\right]\right),$$

$$\varpi(u)=\left(e^{\gamma_1(u-b)},e^{\gamma_2(u-b)},\cdots,e^{\gamma_{M+1}(u-b)},e^{\hat\gamma_1(u-a)},e^{\hat\gamma_2(u-a)},\cdots,e^{\hat\gamma_{\hat M+1}(u-a)}\right).$$

where

$$f_{jk}^u=\int_0^\infty f(y+b)\frac{(\rho_j y)^{k-1}\rho_j e^{-\rho_j y}}{(k-1)!}dy,\ j=1,\cdots,J,k=1,\cdots,m_j,$$

$$f_{j'k'}^d=\int_{-\infty}^0 f(y+a)\frac{(-\hat\rho_j y)^{k'-1}\hat\rho_j e^{\hat\rho_j y}}{(k'-1)!}dy,\ j'=1,\cdots,\hat J,k'=1,\cdots,\hat m_{j'}.$$

Let

$$C_1=\left(1,1,\cdots,1,e^{\hat\gamma_1(b-a)},e^{\hat\gamma_2(b-a)},\cdots,e^{\hat\gamma_{\hat M+1}(b-a)}\right),\qquad C_2=\left(e^{\gamma_1(a-b)},e^{\gamma_2(a-b)},\cdots,e^{\gamma_{M+1}(a-b)},1,1,\cdots,1\right),$$

$$A_i=\begin{pmatrix}\frac{\rho_i}{\rho_i-\gamma_1} & \cdots & \frac{\rho_i}{\rho_i-\gamma_{M+1}} & \frac{e^{\hat r_1(b-a)}\rho_i}{\rho_i-\hat\gamma_1} & \cdots & \frac{e^{\hat r_{\hat M+1}(b-a)}\rho_i}{\rho_i-\hat\gamma_{\hat M+1}}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \left(\frac{\rho_i}{\rho_i-\gamma_1}\right)^{m_i} & \cdots & \left(\frac{\rho_i}{\rho_i-\gamma_{M+1}}\right)^{m_i} & \frac{e^{\hat r_1(b-a)}\rho_i^{m_i}}{(\rho_i-\hat\gamma_1)^{m_i}} & \cdots & \frac{e^{\hat r_{\hat M+1}(b-a)}\rho_i^{m_i}}{(\rho_i-\hat\gamma_{\hat M+1})^{m_i}}\end{pmatrix},i=1,2,\cdots,J,$$

$$B_i = \begin{pmatrix} \frac{e^{\gamma_1(a-b)}\hat{\rho}_i}{\hat{\rho}_i+\gamma_1} & \cdots & \frac{e^{\gamma_{M+1}(a-b)}\hat{\rho}_i}{\hat{\rho}_i+\gamma_{M+1}} & \frac{\hat{\rho}_i}{\hat{\rho}_i+\hat{\gamma}_1} & \cdots & \frac{\hat{\rho}_i}{\hat{\rho}_i+\hat{\gamma}_{\hat{M}+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\gamma_1(a-b)}\hat{\rho}_i^{\hat{m}_i}}{(\hat{\rho}_i+\gamma_1)^{\hat{m}_i}} & \cdots & \frac{e^{\gamma_{M+1}(a-b)}\hat{\rho}_i^{\hat{m}_i}}{(\hat{\rho}_i+\gamma_{M+1})^{\hat{m}_i}} & \left(\frac{\hat{\rho}_i}{\hat{\rho}_i+\hat{\gamma}_1}\right)^{\hat{m}_i} & \cdots & \left(\frac{\hat{\rho}_i}{\hat{\rho}_i+\hat{\gamma}_{\hat{M}+1}}\right)^{\hat{m}_i} \end{pmatrix}, i = 1,2,\cdots,\hat{J}.$$

Define a matrix

$$D = \begin{pmatrix} C_1^T & A_1^T & \cdots & A_J^T & C_2^T & B_1^T & \cdots & B_{\hat{J}}^T \end{pmatrix}^{\mathrm{T}}.$$

Theorem 3.1. Consider any nonnegative measurable function f such that $f_{jk}^u < \infty$ and $f_{j'k'}^d < \infty$ for $j = 1,2,\cdots,J$, $k = 1,\cdots,m_j$, $j' = 1,\cdots,\hat{J}$, $k' = 1,\cdots,\hat{m}_{j'}$. For any $\theta > 0$ and $u \in (a,b)$, we have

$$E^u\left[e^{-\theta\tau} f(X_\tau)\right] = \pi\mathbf{f}, \tag{3.1}$$

where π satisfies

$$\pi D = \varpi. \tag{3.2}$$

Moreover, when D is a non-singular matrix, π is the unique solution of (3.2), *i.e.*,

$$\pi = \varpi \cdot D^{-1}. \tag{3.3}$$

Proof. By the law of total probability, we have

$$\begin{aligned} E^u\left[e^{-\theta\tau} f(X_\tau)\right] &= E^u\left[e^{-\theta\tau} f(X_\tau) I_{K_0}\right] + \sum_{j=1}^{J}\sum_{i=1}^{m_j}\sum_{l=1}^{i} E^u\left[e^{-\theta\tau} f(X_\tau) I_{K_{jil}}\right] \\ &\quad + E^u\left[e^{-\theta\tau} f(X_\tau) I_{G_0}\right] + \sum_{j'=1}^{\hat{J}}\sum_{i'=1}^{\hat{m}_{j'}}\sum_{l'=1}^{i'} E^u\left[e^{-\theta\tau} f(X_\tau) I_{G_{j'i'l'}}\right] \\ &= E^u\left[e^{-\theta\tau} f(X_\tau) I_{K_0}\right] + \sum_{j=1}^{J}\sum_{k=1}^{m_j} E^u\left[e^{-\theta\tau} f(X_\tau) I_{K_{jk}}\right] \\ &\quad + E^u\left[e^{-\theta\tau} f(X_\tau) I_{G_0}\right] + \sum_{j'=1}^{\hat{J}}\sum_{k'=1}^{\hat{m}_{j'}} E^u\left[e^{-\theta\tau} f(X_\tau) I_{G_{j'k'}}\right]. \end{aligned}$$

It follows from Corollary 2.4, for $j = 1,2,\cdots,J$, $k = 1,\cdots,m_j$, $j' = 1,\cdots,\hat{J}$, $k' = 1,\cdots,\hat{m}_{j'}$, we have

$$E^u\left[e^{-\theta\tau} f(X_\tau) I_{K_0}\right] = E^u\left[e^{-\theta\tau} I_{K_0}\right] f(b),$$

$$E^u\left[e^{-\theta\tau} f(X_\tau) I_{G_0}\right] = E^u\left[e^{-\theta\tau} I_{G_0}\right] f(a),$$

$$E^u\left[e^{-\theta\tau} f(X_\tau) I_{K_{jk}}\right] = E^u\left[e^{-\theta\tau} I_{K_{jk}}\right] \int_0^\infty f(y+b) \frac{(\rho_j y)^{k-1} \rho_j e^{-\rho_j y}}{(k-1)!} dy,$$

$$E^u\left[e^{-\theta\tau} f(X_\tau) I_{G_{j'k'}}\right] = E^u\left[e^{-\theta\tau} I_{G_{j'k'}}\right] \int_{-\infty}^0 f(y+a) \frac{(-\hat{\rho}_j y)^{k'-1} \hat{\rho}_j e^{\hat{\rho}_j y}}{(k'-1)!} dy.$$

Combining these equations, we get

$$\begin{aligned} &E^u\left[e^{-\theta\tau} f(X_\tau)\right] \\ &= E^u\left[e^{-\theta\tau} I_{K_0}\right] f(b) + \sum_{j=1}^{J}\sum_{k=1}^{m_j} E^u\left[e^{-\theta\tau} I_{K_{jk}}\right] f_{jk}^u \\ &\quad + E^u\left[e^{-\theta\tau} I_{G_0}\right] f(a) + \sum_{j'=1}^{\hat{J}}\sum_{k'=1}^{\hat{m}_{j'}} E^u\left[e^{-\theta\tau} I_{G_{j'k'}}\right] f_{j'k'}^d. \end{aligned}$$

The expressions for $E^u\left[e^{-\theta\tau} I_{K_0}\right]$, $E^u\left[e^{-\theta\tau} I_{K_{jk}}\right]$, $E^u\left[e^{-\theta\tau} I_{G_0}\right]$ and $E^u\left[e^{-\theta\tau} I_{G_{j'k'}}\right]$ can be determined as follows. Let Δ denote the set of functions $g: R \to R$ such that $g(u)$ is twice continuously differentiable and bounded for $a < u < b$ with $g'(u)$ and $g''(u)$ bounded for $a < u < b$. By applying Itô formula to the process X_t, we have that for $t \ge 0$ and $g \in \Delta$,

$$g(X_{t\wedge\tau}) = g(u) + \int_0^{t\wedge\tau} Lg(X_s) ds + M_t,$$

where M_t is a martingale with $M_0 = 0$. Note that we have $a < X_s < b$ as $s \le \tau$.

For any $\theta > 0$, we can easily obtain from the above equation that

$$\mathrm{e}^{-\theta(t\wedge\tau)} g\left(X_{t\wedge\tau}\right)$$

$$= g(u) + \int_0^{t\wedge\tau} \mathrm{e}^{-\theta s}\left(Lg\left(X_s\right) - \theta g\left(X_s\right)\right)\mathrm{d}s + \int_0^{t\wedge\tau} \mathrm{e}^{-\theta s}\mathrm{d}M_t,$$

where the last term of the above equation is a mean-0 martingale. This implies that

$$E^u\left[\mathrm{e}^{-\theta(t\wedge\tau)} g\left(X_{t\wedge\tau}\right)\right]$$

$$= g(u) + E^u\left[\int_0^{t\wedge\tau} \mathrm{e}^{-\theta s}\left(Lg\left(X_s\right) - \theta g\left(X_s\right)\right)\mathrm{d}s\right], t \ge 0. \tag{3.4}$$

By simple calculation, the function $g(x) = \mathrm{e}^{\beta x}$ with $G(\beta) = \theta$ and $\beta \in C$ satisfies $Lg\left(X_s\right) - \theta g\left(X_s\right) = 0$ for $a < x < b$. It follows from (3.4) that the process $\left\{\mathrm{e}^{-\theta(t\wedge\tau)+\beta X_{t\wedge\tau}} : t \ge 0\right\}$ is a martingale. Then

$$\begin{aligned}
\mathrm{e}^{\beta u} &= E^u\left[\mathrm{e}^{-\theta\tau+\beta X_\tau}\right] = E^u\left[\mathrm{e}^{-\theta\tau} I_{K_0}\right]\mathrm{e}^{\beta b} + \sum_{j=1}^{J}\sum_{i=1}^{m_j}\sum_{l=1}^{i} E^u\left[\mathrm{e}^{-\theta\tau} I_{K_{jil}}\right]\left(\frac{\rho_j}{\rho_j - \beta}\right)^{i+1-l}\mathrm{e}^{\beta b} \\
&\quad + E^u\left[\mathrm{e}^{-\theta\tau} I_{G_0}\right]\mathrm{e}^{\beta a} + \sum_{j'=1}^{\hat{j}}\sum_{i'=1}^{\hat{m}_{j'}}\sum_{l'=1}^{i'} E^u\left[\mathrm{e}^{-\hat{\rho}\tau} I_{G_{j'i'l'}}\right]\left(\frac{\hat{\rho}_{j'}}{\hat{\rho}_{j'} + \beta}\right)^{i'+1-l'}\mathrm{e}^{\beta a} \\
&= E^u\left[\mathrm{e}^{-\theta\tau} I_{K_0}\right]\mathrm{e}^{\beta b} + \sum_{j=1}^{J}\sum_{k=1}^{m_j} E^u\left[\mathrm{e}^{-\theta\tau} I_{K_{jk}}\right]\left(\frac{\rho_j}{\rho_j - \beta}\right)^{k}\mathrm{e}^{\beta b} \\
&\quad + E^u\left[\mathrm{e}^{-\theta\tau} I_{G_0}\right]\mathrm{e}^{\beta a} + \sum_{j'=1}^{\hat{j}}\sum_{k'=1}^{\hat{m}_{j'}} E^u\left[\mathrm{e}^{-\hat{\rho}\tau} I_{G_{j'k'}}\right]\left(\frac{\hat{\rho}_{j'}}{\hat{\rho}_{j'} + \beta}\right)^{k'}\mathrm{e}^{\beta a}.
\end{aligned} \tag{3.5}$$

Setting $\beta = \gamma_i$ for $i = 1, 2, \cdots, M+1$ and $\beta = \hat{\gamma}_i$ for $i = 1, 2, \cdots, \hat{M}+1$ in (3.5), we have the following linear equations:

$$\begin{aligned}
\mathrm{e}^{\gamma_i u} &= E^u\left[\mathrm{e}^{-\theta\tau+\gamma_i X_\tau}\right] = E^u\left[\mathrm{e}^{-\theta\tau} I_{K_0}\right]\mathrm{e}^{\gamma_i b} + \sum_{j=1}^{J}\sum_{k=1}^{m_j} E^u\left[\mathrm{e}^{-\theta\tau} I_{K_{jk}}\right]\left(\frac{\rho_j}{\rho_j - \gamma_i}\right)^{k}\mathrm{e}^{\gamma_i b} \\
&\quad + E^u\left[\mathrm{e}^{-\theta\tau} I_{G_0}\right]\mathrm{e}^{\gamma_i a} + \sum_{j'=1}^{\hat{j}}\sum_{k'=1}^{\hat{m}_{j'}} E^u\left[\mathrm{e}^{-\theta\tau} I_{G_{j'k'}}\right]\left(\frac{\hat{\rho}_{j'}}{\hat{\rho}_{j'} + \gamma_i}\right)^{k'}\mathrm{e}^{\gamma_i a},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{e}^{\hat{\gamma}_i u} &= E^u\left[\mathrm{e}^{-\theta\tau+\hat{\gamma}_i X_\tau}\right] = E^u\left[\mathrm{e}^{-\theta\tau} I_{K_0}\right]\mathrm{e}^{\hat{\gamma}_i b} + \sum_{j=1}^{J}\sum_{k=1}^{m_j} E^u\left[\mathrm{e}^{-\theta\tau} I_{K_{jk}}\right]\left(\frac{\rho_j}{\rho_j - \hat{\gamma}_i}\right)^{k}\mathrm{e}^{\hat{\gamma}_i b} \\
&\quad + E^u\left[\mathrm{e}^{-\theta\tau} I_{G_0}\right]\mathrm{e}^{\hat{\gamma}_i a} + \sum_{j'=1}^{\hat{j}}\sum_{k'=1}^{\hat{m}_{j'}} E^u\left[\mathrm{e}^{-\hat{\rho}\tau} I_{G_{j'k'}}\right]\left(\frac{\hat{\rho}_{j'}}{\hat{\rho}_{j'} + \hat{\gamma}_i}\right)^{k'}\mathrm{e}^{\hat{\gamma}_i a}.
\end{aligned}$$

Then the vector π satisfies $\pi \cdot D = \varpi(u)$ and we have (3.1). If D is non-singular, we have $\pi = \varpi(u) D^{-1}$. This completes the proof.

Corollary 3.2. For any $\delta \in \left(-\min\left(\left|\hat{\rho}_1\right|, \left|\hat{\rho}_2\right|, \cdots, \left|\hat{\rho}_J\right|\right), \min\left(\left|\rho_1\right|, \left|\rho_2\right|, \cdots, \left|\rho_{\hat{J}}\right|\right)\right)$, we have

$$E^u\left[\mathrm{e}^{-\theta\tau+\delta X_\tau}\right] = e^{\delta b}\left(\sum_{i=1}^{M+1}\omega_i \mathrm{e}^{\gamma_i(u-b)} + \sum_{j=1}^{\hat{M}+1}\nu_j \mathrm{e}^{\hat{\gamma}_j(u-a)}\right), \tag{3.6}$$

where

$$D\left(\omega_1, \omega_2, \cdots, \omega_{M+1}, \nu_1, \nu_2, \cdots, \nu_{\hat{M}+1}\right)^{\mathrm{T}} = J(\delta),$$

and

$$J(\delta) = \left(1, \left(\frac{\rho_1}{\rho_1 - \delta}\right)^{m_1}, \cdots, \frac{\rho_1}{\rho_1 - \delta}, \cdots, \left(\frac{\rho_J}{\rho_J - \delta}\right)^{m_J}, \cdots, \frac{\rho_J}{\rho_J - \delta}, \mathrm{e}^{\delta(a-b)}, \left(\frac{\hat{\rho}_1}{\hat{\rho}_1 + \delta}\right)^{\hat{m}_1}\mathrm{e}^{\delta(a-b)}, \cdots, \frac{\hat{\rho}_{\hat{J}}}{\hat{\rho}_{\hat{J}} + \delta}\mathrm{e}^{\delta(a-b)}\right)^{\mathrm{T}}.$$

Remark 3.3. When $m_i = 1$, $n_j = 1$, (3.1) and (3.6) reduce to equation (6) and (15) of [4], respectively.

From Theorem 3.1, choosing $f(X_\tau)$ to be $I_{(X_\tau - b \geq 0)}$, $I_{(X_\tau - a \leq 0)}$, $I_{(X_\tau - b > y)}$, $I_{(X_\tau - a < -y)}$, $I_{(X_\tau = b)}$, $I_{(X_\tau = a)}$ and $e^{\delta X_\tau}$ respectively, we can obtain the following corollaries.

Corollary 3.4. 1) For any $\theta > 0$, we have

$$E^u\left[e^{-\theta\tau} I_{(X_\tau - b \geq 0)}\right] = \begin{cases} 0 & u \leq a \\ \sum_{i=1}^{M+1} \hat{\omega}_i e^{-\gamma_i(b-u)} + \sum_{j=1}^{\hat{M}+1} \hat{\nu}_j e^{-\hat{\gamma}_j(a-u)} & a < u < b, \\ 1 & u \geq b \end{cases} \tag{3.7}$$

where

$$\hat{\omega}^{\mathrm{T}} := \left(\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_{M+1}, \hat{\nu}_1, \hat{\nu}_2, \cdots, \hat{\nu}_{\hat{M}+1}\right)$$

is determined by the linear system $D\hat{\omega} = I_1$. Here

$$I_1^{\mathrm{T}} \equiv (1, 1, \cdots, 1, 0, 0, \cdots, 0)_{1\times(M+\hat{M}+2)}.$$

2) For any $\theta > 0$, we have

$$E^u\left[e^{-\theta\tau} I_{(X_\tau - a \leq 0)}\right] = \begin{cases} 1 & u \leq a \\ \sum_{i=1}^{M+1} \hat{\hat{\omega}}_i e^{-\gamma_i(b-u)} + \sum_{j=1}^{\hat{M}+1} \hat{\hat{\nu}}_j e^{-\hat{\gamma}_j(a-u)} & b > u > a, \\ 0 & u \geq b \end{cases} \tag{3.8}$$

where

$$\hat{\hat{\omega}}^T := \left(\hat{\hat{\omega}}_1, \hat{\hat{\omega}}_2, \cdots, \hat{\hat{\omega}}_{M+1}, \hat{\hat{\nu}}_1, \hat{\hat{\nu}}_2, \cdots, \hat{\hat{\nu}}_{\hat{M}+1}\right)$$

is determined by the linear system $D\hat{\hat{\omega}} = I_2$. Here

$$I_2^{\mathrm{T}} \equiv (0, 0, \cdots, 0, 1, 1, \cdots, 1)_{1\times(M+\hat{M}+2)}.$$

Corollary 3.5. 1) For $f(X_\tau) = I_{(X_\tau - a > y)}$ and any $\theta > 0$, $y > 0$, we have

$$E^u\left[e^{-\theta\tau} I_{(X_\tau - b > y)}\right] = \begin{cases} 0 & u \in [b, b+y] \text{ or } u \leq a \\ \sum_{i=1}^{M+1} \breve{\omega}_i e^{-\gamma_i(b-u)} + \sum_{j=1}^{\hat{M}+1} \breve{\nu}_j e^{-\hat{\gamma}_j(a-u)} & a < u < b \\ 1 & u > b+y \end{cases}, \tag{3.9}$$

where

$$\breve{\omega}^{\mathrm{T}} = \left(\breve{\omega}_1, \breve{\omega}_2, \cdots, \breve{\omega}_{M+1}, \breve{\nu}_1, \breve{\nu}_2, \cdots, \breve{\nu}_{\hat{M}+1}\right)$$

is determined by the linear system $D\breve{\omega} = \breve{I}_1$. Here

$$\breve{I}_1^{\mathrm{T}} \equiv \left(0, e^{-\rho_1 y}, \cdots, \sum_{t=0}^{m_1-1} \frac{(\rho_1 y)^t}{t!} e^{-\rho_1 y}, \cdots, e^{-\rho_J y}, \cdots, \sum_{t=0}^{m_J-1} \frac{(\rho_J y)^t}{t!} e^{-\rho_J y}, 0, \cdots, 0\right)_{1\times(M+\hat{M}+2)};$$

2) For $f(X_\tau) = I_{(X_\tau - a < -y)}$ and any $\theta > 0$, $y > 0$, we have

$$E^u\left[e^{-\theta\tau} I_{(X_\tau - a < -y)}\right] = \begin{cases} 1 & u < a-y \\ \sum_{i=1}^{M+1} \breve{\breve{\omega}}_i e^{-\gamma_i(b-u)} + \sum_{j=1}^{\hat{M}+1} \breve{\breve{\nu}}_j e^{-\hat{\gamma}_j(a-u)} & b > u > a \\ 0 & u \in [a-y, a] \text{ or } u \geq b \end{cases}, \tag{3.10}$$

where

$$\breve{\breve{\omega}}^{\mathrm{T}} = \left(\breve{\breve{\omega}}_1, \breve{\breve{\omega}}_2, \cdots, \breve{\breve{\omega}}_{M+1}, \breve{\breve{\nu}}_1, \breve{\breve{\nu}}_2, \cdots, \breve{\breve{\nu}}_{\hat{M}+1}\right)$$

is determined by the linear system $D\breve{\breve{\omega}} = \breve{I}_2$. Here

$$\breve{I}_2^{\mathrm{T}} \equiv \left(0,\cdots,0,0,\mathrm{e}^{-\hat{\rho}_1 y},\cdots,\sum_{t=0}^{\hat{m}_1-1}\frac{(\hat{\rho}_1 y)^t}{t!}\mathrm{e}^{-\hat{\rho}_1 y},\cdots,\mathrm{e}^{-\hat{\rho}_{\hat{j}} y},\cdots,\sum_{t=0}^{\hat{m}_{\hat{j}}-1}\frac{(\hat{\rho}_{\hat{j}} y)^t}{t!}\mathrm{e}^{-\hat{\rho}_{\hat{j}} y}\right)_{1\times(M+\hat{M}+2)}.$$

Note that the difference of $E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau-b\geq 0)}\right]$ $\left(E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau-a\leq 0)}\right]\right)$ and $E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau-b>0)}\right]$ $\left(E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau-a<0)}\right]\right)$ is exactly $E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau=b)}\right]$ $\left(E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau=a)}\right]\right)$. Thus we obtain the following results.

Corollary 3.6. 1) For $f(X_\tau)=I_{(X_\tau=b)}$, and for any $\theta>0$, we have

$$E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau=b)}\right]=\begin{cases}0 & u>b \text{ or } u\leq a\\ \sum_{i=1}^{M+1}\tilde{\omega}_i\mathrm{e}^{-\gamma_i(b-u)}+\sum_{j=1}^{\hat{M}+1}\tilde{\nu}_j\mathrm{e}^{-\hat{\gamma}_j(a-u)} & a<u<b\\ 1 & u=b\end{cases}, \tag{3.11}$$

where

$$\tilde{\omega}^{\mathrm{T}}=\left(\tilde{\omega}_1,\tilde{\omega}_2,\cdots,\tilde{\omega}_{M+1},\tilde{\nu}_1,\tilde{\nu}_2,\cdots,\tilde{\nu}_{\hat{M}+1}\right)$$

is determined by the linear system $D\tilde{\omega}=\tilde{I}_1$. Here

$$\tilde{I}_1^{\mathrm{T}}\equiv(1,0,\cdots,0,0,\cdots,0)_{1\times(M+\hat{M}+2)}.$$

2) For $f(X_\tau)=I_{(X_\tau=a)}$ and any $\theta>0$, $y>0$, we have

$$E^u\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau=a)}\right]=\begin{cases}0 & u<a \text{ or } u\geq b\\ \sum_{i=1}^{M+1}\bar{\omega}_i\mathrm{e}^{-\gamma_i(b-u)}+\sum_{j=1}^{\hat{M}+1}\bar{\nu}_j\mathrm{e}^{-\hat{\gamma}_{\hat{M}+1}(a-u)} & b>u>a\\ 1 & u=a\end{cases}, \tag{3.12}$$

where

$$\bar{\nu}^{\mathrm{T}}=\left(\bar{\nu}_1,\bar{\nu}_2,\cdots,\bar{\nu}_{M+1},\bar{\nu}_1,\bar{\nu}_2,\cdots,\bar{\nu}_{\hat{M}+1}\right)$$

is determined by the linear system $D\bar{\nu}=\bar{I}_2$. Here

$$\bar{I}_2^{\mathrm{T}}\equiv(0,0,\cdots,0,1,0,\cdots,0)_{1\times(M+\hat{M}+2)}.$$

To end the paper, we give an example.

Example 3.7. When $J=m_1=\hat{J}=\hat{m}_1=1$, $p(x)=p_{11}\rho_1\mathrm{e}^{-\rho_1 x}I_{(x>0)}+\hat{p}_{11}\hat{\rho}_1\mathrm{e}^{\hat{\rho}_1 x}I_{(x<0)}$ and $p_{11}+\hat{p}_{11}=1$, the equation $G(z)=\theta$ has 4 real roots: γ_1, γ_2, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ $(-\infty<\hat{\gamma}_2<-\hat{\rho}_1<\hat{\gamma}_1<0<\gamma_1<\rho_1<\gamma_2<\infty)$. Let

$$D=\begin{pmatrix}1 & 1 & \mathrm{e}^{\hat{\gamma}_1(b-a)} & \mathrm{e}^{\hat{\gamma}_2(b-a)}\\ \dfrac{\rho_1}{\rho_1-\gamma_1} & \dfrac{\rho_1}{\rho_1-\gamma_2} & \dfrac{\rho_1\mathrm{e}^{\hat{\gamma}_1(b-a)}}{\rho_1-\hat{\gamma}_1} & \dfrac{\rho_1\mathrm{e}^{\hat{\gamma}_2(b-a)}}{\rho_1-\hat{\gamma}_2}\\ \mathrm{e}^{\gamma_1(a-b)} & \mathrm{e}^{\gamma_2(a-b)} & 1 & 1\\ \dfrac{\hat{\rho}_1\mathrm{e}^{\gamma_1(a-b)}}{\hat{\rho}_1+\gamma_1} & \dfrac{\hat{\rho}_1\mathrm{e}^{\gamma_2(a-b)}}{\hat{\rho}_1+\gamma_2} & \dfrac{\hat{\rho}_1}{\hat{\rho}_1+\hat{\gamma}_1} & \dfrac{\hat{\rho}_1}{\hat{\rho}_1+\hat{\gamma}_1}\end{pmatrix}.$$

Denote D^{-1} by

$$D^{-1}=\frac{1}{|D|}\begin{pmatrix}d_{11} & -d_{21} & d_{31} & -d_{41}\\ -d_{12} & d_{22} & -d_{32} & d_{42}\\ d_{13} & -d_{23} & d_{33} & -d_{43}\\ -d_{14} & d_{24} & -d_{34} & d_{44}\end{pmatrix}.$$

Then we have

$$E^u\left[\mathrm{e}^{-\theta\tau}f(X_\tau)\right]$$
$$=\omega_1\mathrm{e}^{\gamma_1(u-b)}+\omega_2\mathrm{e}^{\gamma_2(u-b)}+\omega_3\mathrm{e}^{\hat{\gamma}_1(u-a)}+\omega_4\mathrm{e}^{\hat{\gamma}_2(u-a)},$$

where

$$\omega_1=\frac{d_{11}f(b)-d_{21}f_{11}^u+d_{31}f(a)-d_{41}f_{11}^d}{|D|},$$

$$\omega_2=\frac{-d_{12}f(b)+d_{22}f_{11}^u-d_{32}f(a)+d_{42}f_{11}^d}{|D|},$$

$$\omega_3=\frac{d_{13}f(b)-d_{23}f_{11}^u+d_{33}f(a)-d_{43}f_{11}^d}{|D|},$$

$$\omega_4=\frac{-d_{14}f(b)+d_{24}f_{11}^u-d_{34}f(a)+d_{44}f_{11}^d}{|D|},$$

$$f_{11}^{u}=\int_{0}^{\infty}f\left(y+b\right)\rho_{1}\mathrm{e}^{-\rho_{1}y}\mathrm{d}y,\qquad f_{11}^{d}=\int_{-\infty}^{0}f\left(y+a\right)\hat{\rho}_{1}\mathrm{e}^{\hat{\rho}_{1}y}\mathrm{d}y,$$

$$\begin{aligned}|D|=&\rho_1\hat{\rho}_1\frac{(\gamma_2-\gamma_1)(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}\\&+\left(\frac{(\gamma_1-\hat{\gamma}_2)(\hat{\gamma}_1-\gamma_2)}{(\hat{\rho}_1+\hat{\gamma}_2)(\rho_1-\hat{\gamma}_1)}\mathrm{e}^{\hat{\gamma}_1(b-a)}-\frac{(\gamma_1-\hat{\gamma}_1)(\gamma_2-\hat{\gamma}_2)}{(\hat{\rho}_1+\hat{\gamma}_1)(\rho_1-\hat{\gamma}_2)}\mathrm{e}^{\hat{\gamma}_2(b-a)}\right)\frac{\rho_1\hat{\rho}_1\mathrm{e}^{\gamma_1(a-b)}}{(\hat{\rho}_1+\gamma_1)(\rho_1-\gamma_2)}\\&+\left(\frac{(\hat{\gamma}_2-\gamma_2)(\hat{\gamma}_1-\gamma_1)}{(\hat{\rho}_1+\hat{\gamma}_2)(\rho_1-\hat{\gamma}_1)}\mathrm{e}^{\hat{\gamma}_1(b-a)}-\frac{(\hat{\gamma}_1-\gamma_2)(\hat{\gamma}_2-\gamma_1)}{(\hat{\rho}_1+\hat{\gamma}_1)(\rho_1-\hat{\gamma}_2)}\mathrm{e}^{\hat{\gamma}_2(b-a)}\right)\frac{\rho_1\hat{\rho}_1\mathrm{e}^{\gamma_2(a-b)}}{(\hat{\rho}_1+\gamma_2)(\rho_1-\gamma_1)}\\&+\frac{(\gamma_1-\gamma_2)(\hat{\gamma}_2-\hat{\gamma}_1)}{(\rho_1-\hat{\gamma}_1)(\rho_1-\hat{\gamma}_2)(\hat{\rho}_1+\gamma_1)(\hat{\rho}_1+\gamma_2)}\rho_1\hat{\rho}_1\mathrm{e}^{(\hat{\gamma}_1+\hat{\gamma}_2-\gamma_1-\gamma_2)(b-a)},\end{aligned}$$

$$d_{11}=\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}+\left(\frac{(\gamma_2-\hat{\gamma}_1)\mathrm{e}^{\hat{\gamma}_2(b-a)}}{(\rho_1-\hat{\gamma}_2)(\hat{\rho}_1+\hat{\gamma}_1)}-\frac{(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\hat{\gamma}_1(b-a)}}{(\rho_1-\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}\right)\frac{\rho_1\hat{\rho}_1\mathrm{e}^{\gamma_2(a-b)}}{\hat{\rho}_1+\gamma_2},$$

$$d_{13}=\frac{\rho_1\hat{\rho}_1(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\gamma_2(a-b)}}{(\rho_1-\gamma_1)(\hat{\rho}_1+\gamma_2)(\hat{\rho}_1+\hat{\gamma}_2)}-\frac{\rho_1\hat{\rho}_1(\gamma_1-\hat{\gamma}_2)\mathrm{e}^{\gamma_1(a-b)}}{(\rho_1-\gamma_2)(\hat{\rho}_1+\gamma_1)(\hat{\rho}_1+\hat{\gamma}_2)}+\frac{\rho_1\hat{\rho}_1(\gamma_1-\gamma_2)\mathrm{e}^{(\gamma_1+\gamma_2-\hat{\gamma}_2)(a-b)}}{(\rho_1-\hat{\gamma}_2)(\hat{\rho}_1+\gamma_1)(\hat{\rho}_1+\gamma_2)},$$

$$d_{21}=\frac{\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}+\left(\frac{(\gamma_2-\hat{\gamma}_1)\mathrm{e}^{\hat{\gamma}_2(b-a)}}{\hat{\rho}_1+\hat{\gamma}_1}-\frac{(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\hat{\gamma}_1(b-a)}}{\hat{\rho}_1+\hat{\gamma}_2}\right)\frac{\hat{\rho}_1\mathrm{e}^{\gamma_2(a-b)}}{\hat{\rho}_1+\gamma_2},$$

$$d_{23}=\frac{\hat{\rho}_1(\hat{\gamma}_2-\gamma_1)\mathrm{e}^{\gamma_1(a-b)}}{(\hat{\rho}_1+\gamma_1)(\hat{\rho}_1+\hat{\gamma}_2)}+\frac{\hat{\rho}_1(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\gamma_2(a-b)}}{(\hat{\rho}_1+\gamma_2)(\hat{\rho}_1+\hat{\gamma}_2)}+\frac{\hat{\rho}_1(\gamma_1-\gamma_2)\mathrm{e}^{(\gamma_1+\gamma_2-\hat{\gamma}_2)(a-b)}}{(\hat{\rho}_1+\gamma_1)(\hat{\rho}_1+\gamma_2)},$$

$$d_{31}=\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_1-\gamma_2)\mathrm{e}^{\hat{\gamma}_1(b-a)}}{(\rho_1-\gamma_2)(\rho_1-\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}-\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_2-\gamma_2)\mathrm{e}^{\hat{\gamma}_2(b-a)}}{(\rho_1-\gamma_2)(\rho_1-\hat{\gamma}_2)(\hat{\rho}_1+\hat{\gamma}_1)}+\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_2-\hat{\gamma}_1)\mathrm{e}^{(\hat{\gamma}_1+\hat{\gamma}_2-\gamma_2)(b-a)}}{(\rho_1-\hat{\gamma}_1)(\rho_1-\hat{\gamma}_2)(\hat{\rho}_1+\gamma_2)},$$

$$d_{33}=\frac{\rho_1\hat{\rho}_1(\gamma_2-\gamma_1)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_2)}+\left(\frac{(\gamma_1-\hat{\gamma}_2)\mathrm{e}^{\gamma_2(a-b)}}{(\rho_1-\gamma_1)(\hat{\rho}_1+\gamma_2)}-\frac{(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\gamma_1(a-b)}}{(\rho_1-\gamma_2)(\hat{\rho}_1+\gamma_1)}\right)\frac{\rho_1\hat{\rho}_1\mathrm{e}^{\hat{\gamma}_2(b-a)}}{\rho_1-\hat{\gamma}_2},$$

$$d_{41}=\frac{\rho_1(\hat{\gamma}_1-\gamma_2)\mathrm{e}^{\hat{\gamma}_1(b-a)}}{(\rho_1-\gamma_2)(\rho_1-\hat{\gamma}_1)}+\frac{\rho_1(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\hat{\gamma}_2(b-a)}}{(\rho_1-\gamma_2)(\rho_1-\hat{\gamma}_2)}+\frac{\rho_1(\hat{\gamma}_2-\hat{\gamma}_1)\mathrm{e}^{(\hat{\gamma}_1+\hat{\gamma}_2-\gamma_2)(b-a)}}{(\rho_1-\hat{\gamma}_1)(\rho_1-\hat{\gamma}_2)},$$

$$d_{43}=\frac{\rho_1(\gamma_2-\gamma_1)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)}+\left(\frac{(\gamma_1-\hat{\gamma}_2)\mathrm{e}^{\gamma_2(a-b)}}{\rho_1-\gamma_1}-\frac{(\gamma_2-\hat{\gamma}_2)\mathrm{e}^{\gamma_1(a-b)}}{\rho_1-\gamma_2}\right)\frac{\rho_1\mathrm{e}^{\hat{\gamma}_2(b-a)}}{\rho_1-\hat{\gamma}_2}.$$

We define d_{12} (d_{22}, d_{32}, d_{42}) and d_{14} (d_{24}, d_{34}, d_{44}) as follows: let d_{12} (d_{22}, d_{32}, d_{42}) be obtained from d_{11} (d_{21}, d_{31}, d_{41}) by changing γ_2 to γ_1 in d_{11} (d_{21} d_{31}, d_{41}); let d_{14} (d_{24}, d_{34}, d_{44}) be obtained from d_{13} (d_{23}, d_{33}, d_{43}) by changing $\hat{\gamma}_2$ to $\hat{\gamma}_1$ in d_{13} (d_{23}, d_{33}, d_{43}).

- If $f(X_\tau)=\mathrm{e}^{\delta X_\tau}$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau+\delta X_\tau}\right]=\mathrm{e}^{\delta b}\left(\omega_1'\mathrm{e}^{\gamma_1(u-b)}+\omega_2'\mathrm{e}^{\gamma_2(u-b)}+\omega_3'\mathrm{e}^{\hat{\gamma}_1(u-a)}+\omega_4'\mathrm{e}^{\hat{\gamma}_2(u-a)}\right),\quad a<u<b,$$

where

$$f(a)=\mathrm{e}^{\delta a},\ f(b)=\mathrm{e}^{\delta b},\ f_{11}^u=\mathrm{e}^{\delta b}\frac{\rho_1}{\rho_1-\delta},\ f_{11}^d=\mathrm{e}^{\delta a}\frac{\hat{\rho}_1}{\hat{\rho}_1+\delta},$$

$$\omega_1'=\frac{d_{11}-d_{21}\frac{\rho_1}{\rho_1-\delta}+d_{31}\mathrm{e}^{\delta(a-b)}-d_{41}\mathrm{e}^{\delta(a-b)}\frac{\hat{\rho}_1}{\hat{\rho}_1+\delta}}{|D|}$$

$$\omega_2'=\frac{-d_{12}+d_{22}\frac{\rho_1}{\rho_1-\delta}-d_{32}\mathrm{e}^{\delta(a-b)}+d_{42}\mathrm{e}^{\delta(a-b)}\frac{\hat{\rho}_1}{\hat{\rho}_1+\delta}}{|D|}$$

$$\omega_3' = \frac{d_{13} - d_{23}\dfrac{\rho_1}{\rho_1 - \delta} + d_{33}e^{\delta(a-b)} - d_{43}e^{\delta(a-b)}\dfrac{\hat{\rho}_1}{\hat{\rho}_1 + \delta}}{|D|}$$

$$\omega_4' = \frac{-d_{14} + d_{24}\dfrac{\rho_1}{\rho_1 - \delta} - d_{34}\mathrm{e}^{\delta(a-b)} + d_{44}\mathrm{e}^{\delta(a-b)}\dfrac{\hat{\rho}_1}{\hat{\rho}_1 + \delta}}{|D|}$$

- If $f(X_\tau) = I_{(X_\tau - b \geq 0)}$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau} I_{(X_\tau - b \geq 0)}\right] = \hat{\omega}_1 \mathrm{e}^{\gamma_1(u-b)} + \hat{\omega}_2 \mathrm{e}^{\gamma_2(u-b)} + \hat{\omega}_3 \mathrm{e}^{\hat{\gamma}_1(u-a)} + \hat{\omega}_4 \mathrm{e}^{\hat{\gamma}_2(u-a)}, \quad a < u < b,$$

where

$$f(a) = 0, \quad f(b) = 1, \quad f_{11}^u = 1, \quad f_{11}^d = 0,$$

$$\hat{\omega}_1 = \frac{d_{11} - d_{21}}{|D|}, \hat{\omega}_2 = \frac{-d_{12} + d_{22}}{|D|},$$

$$\hat{\omega}_3 = \frac{d_{13} - d_{23}}{|D|}, \hat{\omega}_4 = \frac{-d_{14} + d_{24}}{|D|};$$

- If $f(X_\tau) = I_{(X_\tau - a \leq 0)}$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau} I_{(X_\tau - a \leq 0)}\right] = \hat{\hat{\omega}}_1 \mathrm{e}^{\gamma_1(u-b)} + \hat{\hat{\omega}}_2 \mathrm{e}^{\gamma_2(u-b)} + \hat{\hat{\omega}}_3 \mathrm{e}^{\hat{\gamma}_1(u-a)} + \hat{\hat{\omega}}_4 \mathrm{e}^{\hat{\gamma}_2(u-a)}, \quad a < u < b,$$

where

$$f(a) = 1, \quad f(b) = 0, \quad f_{11}^u = 0, \quad f_{11}^d = 1,$$

$$\hat{\hat{\omega}}_1 = \frac{d_{31} - d_{41}}{|D|}, \hat{\hat{\omega}}_2 = \frac{-d_{32} + d_{42}}{|D|},$$

$$\hat{\hat{\omega}}_3 = \frac{d_{33} - d_{43}}{|D|}, \hat{\hat{\omega}}_4 = \frac{-d_{34} + d_{44}}{|D|};$$

- If $f(X_\tau) = I_{(X_\tau - b > y)}$, $y > 0$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau} I_{(X_\tau - b > y)}\right] = \breve{\omega}_1 \mathrm{e}^{\gamma_1(u-b)} + \breve{\omega}_2 \mathrm{e}^{\gamma_2(u-b)} + \breve{\omega}_3 \mathrm{e}^{\hat{\gamma}_1(u-a)} + \breve{\omega}_4 \mathrm{e}^{\hat{\gamma}_2(u-a)}, \quad a < u < b,$$

$$f(a) = 0, \quad f(b) = 0, \quad f_{11}^u = \mathrm{e}^{-\rho_1 y}, \quad f_{11}^d = 0,$$

$$\breve{\omega}_1 = \frac{-d_{21}\mathrm{e}^{-\rho_1 y}}{|D|}, \quad \breve{\omega}_2 = \frac{d_{22}\mathrm{e}^{-\rho_1 y}}{|D|},$$

$$\breve{\omega}_3 = \frac{-d_{23}\mathrm{e}^{-\rho_1 y}}{|D|}, \quad \breve{\omega}_4 = \frac{d_{24}\mathrm{e}^{-\rho_1 y}}{|D|};$$

- If $f(X_\tau) = I_{(X_\tau - a < -y)}$, $y > 0$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau} I_{(X_\tau - a < -y)}\right] = \breve{\breve{\omega}}_1 \mathrm{e}^{\gamma_1(u-b)} + \breve{\breve{\omega}}_2 \mathrm{e}^{\gamma_2(u-b)} + \breve{\breve{\omega}}_3 \mathrm{e}^{\hat{\gamma}_1(u-a)} + \breve{\breve{\omega}}_4 \mathrm{e}^{\hat{\gamma}_2(u-a)}, \quad a < u < b,$$

where

$$f(a) = 0, \quad f(b) = 0, \quad f_{11}^u = 0, \quad f_{11}^d = \mathrm{e}^{-\hat{\rho}_1 y},$$

$$\omega_1 = \frac{-d_{41}\mathrm{e}^{-\hat{\rho}_1 y}}{|D|}, \quad \omega_2 = \frac{d_{42}\mathrm{e}^{-\hat{\rho}_1 y}}{|D|},$$

$$\omega_3 = \frac{-d_{43}\mathrm{e}^{-\hat{\rho}_1 y}}{|D|}, \quad \omega_4 = \frac{d_{44}\mathrm{e}^{-\hat{\rho}_1 y}}{|D|};$$

- If $f(X_\tau) = I_{(X_\tau = b)}$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau} I_{(X_\tau = b)}\right] = \tilde{\omega}_1 \mathrm{e}^{\gamma_1(u-b)} + \tilde{\omega}_2 \mathrm{e}^{\gamma_2(u-b)} + \tilde{\omega}_3 \mathrm{e}^{\hat{\gamma}_1(u-a)} + \tilde{\omega}_4 \mathrm{e}^{\hat{\gamma}_2(u-a)}, \quad a < u < b,$$

where

$$f(a) = 0, \quad f(b) = 1, \quad f_{11}^u = 0, \quad f_{11}^d = 0,$$

$$\tilde{\omega}_1 = \frac{d_{11}}{|D|}, \quad \tilde{\omega}_2 = \frac{-d_{12}}{|D|}, \quad \tilde{\omega}_3 = \frac{d_{13}}{|D|}, \quad \tilde{\omega}_4 = \frac{-d_{14}}{|D|};$$

- If $f(X_\tau) = I_{(X_\tau = a)}$, then we have

$$E^u\left[\mathrm{e}^{-\theta\tau} I_{(X_\tau = a)}\right] = \bar{\omega}_1 \mathrm{e}^{\gamma_1(u-b)} + \bar{\omega}_2 \mathrm{e}^{\gamma_2(u-b)} + \bar{\omega}_3 \mathrm{e}^{\hat{\gamma}_1(u-a)} + \bar{\omega}_4 \mathrm{e}^{\hat{\gamma}_2(u-a)}, \quad a < u < b,$$

where

$$f(a) = 1, \quad f(b) = 0, \quad f_{11}^u = 0, \quad f_{11}^d = 0,$$

$$\bar{\omega}_1 = \frac{d_{31}}{|D|}, \quad \bar{\omega}_2 = \frac{-d_{32}}{|D|}, \quad \bar{\omega}_3 = \frac{d_{33}}{|D|}, \quad \bar{\omega}_4 = \frac{-d_{34}}{|D|}.$$

When $a \to -\infty$, we have

$$\hat{\omega}_1 = \frac{d_{11} - d_{21}}{|D|} = \frac{\dfrac{\rho_1\hat{\rho}_1(\hat{\gamma}_1 - \hat{\gamma}_2)}{(\rho_1 - \gamma_2)(\hat{\rho}_1 + \hat{\gamma}_1)(\hat{\rho}_1 + \hat{\gamma}_2)} - \dfrac{\hat{\rho}_1(\hat{\gamma}_1 - \hat{\gamma}_2)}{(\hat{\rho}_1 + \hat{\gamma}_1)(\hat{\rho}_1 + \hat{\gamma}_2)}}{\rho_1\hat{\rho}_1\dfrac{(\gamma_2 - \gamma_1)(\hat{\gamma}_1 - \hat{\gamma}_2)}{(\rho_1 - \gamma_1)(\rho_1 - \gamma_2)(\hat{\rho}_1 + \hat{\gamma}_1)(\hat{\rho}_1 + \hat{\gamma}_2)}} = \frac{\gamma_2(\rho_1 - \gamma_1)}{\rho_1(\gamma_2 - \gamma_1)},$$

$$\hat{\omega}_2 = \frac{-d_{12}+d_{22}}{|D|} = \frac{\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)} - \frac{\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}}{\rho_1\hat{\rho}_1\frac{(\gamma_2-\gamma_1)(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}} = -\frac{\gamma_1(\rho_1-\gamma_2)}{\rho_1(\gamma_2-\gamma_1)},$$

$$\breve{\omega}_1 = \frac{-d_{21}\mathrm{e}^{-\rho_1 y}}{|D|} = \frac{-\frac{\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}\mathrm{e}^{-\rho_1 y}}{\rho_1\hat{\rho}_1\frac{(\gamma_2-\gamma_1)(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}} = -\frac{(\rho_1-\gamma_1)(\rho_1-\gamma_2)}{\rho_1(\gamma_2-\gamma_1)}\mathrm{e}^{-\rho_1 y},$$

$$\breve{\omega}_2 = \frac{d_{22}\mathrm{e}^{-\rho_1 y}}{|D|} = \frac{\frac{\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}\mathrm{e}^{-\rho_1 y}}{\rho_1\hat{\rho}_1\frac{(\gamma_2-\gamma_1)(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}} = \frac{(\rho_1-\gamma_1)(\rho_1-\gamma_2)}{\rho_1(\gamma_2-\gamma_1)}\mathrm{e}^{-\rho_1 y},$$

$$\tilde{\omega}_1 = \frac{d_{11}}{|D|} = \frac{\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}}{\rho_1\hat{\rho}_1\frac{(\gamma_2-\gamma_1)(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}} = \frac{\rho_1-\gamma_1}{\gamma_2-\gamma_1},$$

$$\tilde{\omega}_2 = \frac{-d_{12}}{|D|} = -\frac{\frac{\rho_1\hat{\rho}_1(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}}{\rho_1\hat{\rho}_1\frac{(\gamma_2-\gamma_1)(\hat{\gamma}_1-\hat{\gamma}_2)}{(\rho_1-\gamma_1)(\rho_1-\gamma_2)(\hat{\rho}_1+\hat{\gamma}_1)(\hat{\rho}_1+\hat{\gamma}_2)}} = -\frac{\rho_1-\gamma_2}{\gamma_2-\gamma_1},$$

$$\hat{\omega}_3 = \hat{\omega}_4 = \breve{\omega}_3 = \breve{\omega}_4 = \tilde{\omega}_3 = \tilde{\omega}_4 = 0.$$

Therefore, we have

$$E^0\left[\mathrm{e}^{-\theta\tau}\right] = \frac{\gamma_2(\rho_1-\gamma_1)}{\rho_1(\gamma_2-\gamma_1)}\mathrm{e}^{-\gamma_1 b} - \frac{\gamma_1(\rho_1-\gamma_2)}{\rho_1(\gamma_2-\gamma_1)}\mathrm{e}^{-\gamma_2 b},$$

$$E^0\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau-b>y)}\right] = \frac{(\rho_1-\gamma_1)(\rho_1-\gamma_2)}{\rho_1(\gamma_2-\gamma_1)}\left(\mathrm{e}^{-\gamma_2 b}-\mathrm{e}^{-\gamma_1 b}\right)\mathrm{e}^{-\rho_1 y},$$

$$E^0\left[\mathrm{e}^{-\theta\tau}I_{(X_\tau=b)}\right] = \frac{\rho_1-\gamma_1}{\gamma_2-\gamma_1}\mathrm{e}^{-\gamma_1 b} - \frac{\rho_1-\gamma_2}{\gamma_2-\gamma_1}\mathrm{e}^{-\gamma_2 b}.$$

These results are all consistent with that of Theorem 3.1 of Kou and Wang [2] for the one-sided exit problem of the doubly exponential jump diffusion process.

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