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Classifying Traveling Wave Solutions to the Zhiber-Shabat Equation

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ABSTRACT

By the complete discrimination system for polynomials, we classify exact traveling wave solutions to the Zhiber-Shabat equation, and compute some new traveling wave solutions.

Keywords: Traveling Wave Solution; Complete Discrimination System for Polynomials; The Zhiber-Shabat Equation

1. Introduction

The study of exact solutions to nonlinear partial differential equations is an important component of integrable systems [1]. Many methods, such as the transformed rational function method [2], the multiple exp-function algorithm [3] and the factorization method [4], have been proposed to find exact traveling wave solutions to nonlinear partial differential equations. At the same time, Ma has obtained complexiton solutions, a kind of multi-wave solutions, to some nonlinear partial differential equations [5,6]. Liu [7] introduced a simple and efficient method to give the classification of exact traveling wave solutions to some nonlinear equations [8].

In this paper, we focus on the Zhiber-Shabat equation to classify its traveling wave solutions. A. M. Wazwaz [9] and A. G. Davodi *et al.* [10] have got some traveling wave solutions to the Zhiber-Shabat equation. By Liu's method, we'll classify exact traveling wave solutions to the Zhiber-Shabat equation, and compute some new traveling wave solutions to the equation.

2. Exact Traveling Wave Solutions

The Zhiber-Shabat equation reads as:

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0, \tag{1}$$

where $p \neq 0$, q, r are three constants. Take the traveling wave transformation

$$u = u(\xi), \xi = kx + \omega t, \tag{2}$$

the corresponding reduced ordinray differential equation is given by

$$k\omega u'' + pe^{u} + qe^{-u} + re^{-2u} = 0. \tag{3}$$

Furthermore, we take $u' = z$, where z is a function of u, and so, we have $u'' = z\mathrm{d}z/\mathrm{d}u$. Substituting these terms into Equation (2) yields

$$k\omega z\frac{\mathrm{d}z}{\mathrm{d}u} + pe^{u} + qe^{-u} + re^{-2u} = 0. \tag{4}$$

Using the method of the variation of constants, the general solution of Equation (3) is given by

$$z = u' = \pm\sqrt{\frac{2q}{k\omega}e^{-u} + \frac{r}{k\omega}e^{-2u} - \frac{2p}{k\omega}e^{u} + \frac{2c}{k\omega}}, \tag{5}$$

where c is an arbitrary constant. Thus the general solution of Equation (4) is

$$\pm(\xi - \xi_0) = \int\frac{\mathrm{d}u}{\sqrt{\frac{2q}{k\omega}e^{-u} + \frac{r}{k\omega}e^{-2u} - \frac{2p}{k\omega}e^{u} + \frac{2c}{k\omega}}}. \tag{6}$$

We take the transformation $v = e^{u}$, the corresponding integral becomes

$$\pm(\xi - \xi_0) = \int\frac{\mathrm{d}v}{\sqrt{\frac{-2p}{k\omega}v^3 + \frac{c}{k\omega}v^2 + \frac{2q}{k\omega}v + \frac{r}{k\omega}}}. \tag{7}$$

Denote

$$F(w) = w^3 + d_2w^2 + d_1w + d_0, \tag{8}$$

where

$$w = \left(\frac{-2p}{k\omega}\right)^{1/3}v, d_2 = \frac{c}{k\omega}\left(\frac{-2p}{k\omega}\right)^{-2/3}, \quad d_1 = \frac{2q}{k\omega}\left(\frac{-2p}{k\omega}\right)^{-1/3}, d_0 = \frac{r}{k\omega}. \tag{9}$$

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The complete discrimination system for $F(w)$ is given by

$$\Delta = -27\left(\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_3}{3}\right)^2 - 4\left(d_1 - d_2^2\right)^3,$$
$$D_1 = d_1 - \frac{d_2^2}{3}. \tag{10}$$

Case 1. $\Delta = 0, D_1 < 0$:

We have $F(w) = (w-\alpha)^2 (w-\beta), \alpha \neq \beta$. When $\omega > \beta$, we have

$$\pm(\xi - \xi_0) = \frac{1}{\sqrt{\alpha-\beta}} \ln\left|\frac{\sqrt{w-\beta}-\sqrt{\alpha-\beta}}{\sqrt{w-\beta}+\sqrt{\alpha-\beta}}\right|, \alpha > \beta, \tag{11}$$

$$\pm(\xi - \xi_0) = \frac{2}{\sqrt{\beta-\alpha}} \arctan\sqrt{\frac{w-\beta}{\beta-\alpha}}, \alpha < \beta. \tag{12}$$

The corresponding solutions are

$$u_1 = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3} \times[(\alpha-\beta)\tanh^2 \frac{1}{2}\sqrt{\alpha-\beta}(\xi-\xi_0)+\beta]\right\}, \tag{13}$$

$$u_2 = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3} \times\left[(\alpha-\beta)\coth^2 \frac{1}{2}\sqrt{\alpha-\beta}(\xi-\xi_0)+\beta\right]\right\}, \tag{14}$$

$$u_3 = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3} \times\left[(\beta-\alpha)\sec^2 \frac{1}{2}\sqrt{\beta-\alpha}(\xi-\xi_0)+\alpha\right]\right\}. \tag{15}$$

Case 2. $\Delta = 0, D_1 = 0$:

Then $F(w) = (w-\alpha)^3$. The solution is given by

$$u = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3}\left[\frac{4}{(\xi-\xi_0)^2}+\alpha\right]\right\}. \tag{16}$$

Case 3. $\Delta > 0, D_1 < 0$:

Then $F(w) = (w-\alpha)(w-\beta)(w-\gamma)$ with $\alpha < \beta < \gamma$. Therefore, we have

$$\pm(\xi-\xi_0) = \int \frac{\mathrm{d}w}{\sqrt{(w-\alpha)(w-\beta)(w-\gamma)}}. \tag{17}$$

When $\alpha < w < \beta$, we obtain a new traveling wave solution

$$u = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3}\alpha + \left(\frac{-2p}{k\omega}\right)^{-1/3} \times(\beta-\alpha)\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}}{2}(\xi-\xi_0), k\right)\right\}. \tag{18}$$

When $w > \gamma$, we obtain another new traveling wave solution

$$u = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3} \times\frac{\left(-\beta\,\mathrm{sn}\left(\frac{\sqrt{\gamma-\alpha}}{2}(\xi-\xi_0), k\right)+\gamma\right)}{\mathrm{cn}\left(\frac{\sqrt{\gamma-\alpha}}{2}(\xi-\xi_0), k\right)}\right\}, \tag{19}$$

where

$$k^2 = \frac{\beta-\alpha}{\gamma-\alpha}.$$

Case 4. $\Delta < 0$:

Then $F(w) = (w-\alpha)(w^2+pw+q), p^2-4q<0$. The corresponding integral becomes

$$\pm(\xi-\xi_0) = \int \frac{\mathrm{d}w}{\sqrt{(w-\alpha)(w^2+pw+q)}}. \tag{20}$$

When $w > \alpha$, we obtain the following new traveling wave solution

$$u = \ln\left\{\left(\frac{-2p}{k\omega}\right)^{-1/3}\left(\alpha + \frac{2\sqrt{\alpha^2+p\alpha+q}}{1+\mathrm{cn}\left[(\alpha^2+p\alpha+q)^{1/4}(\xi-\xi_0), k\right]} - \sqrt{\alpha^2+p\alpha+q}\right)\right\}, \tag{21}$$

where

$$k^2 = \frac{1}{2}\left(1 - \frac{\alpha+\frac{p}{2}}{\sqrt{\alpha^2+p\alpha+q}}\right).$$

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