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ε-Optimality in Multivalued Optimization

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ABSTRACT

In this paper we apply the directional derivative technique to characterize D-multifunction, quasi D-multifunction and use them to obtain ε-optimality for set valued vector optimization problem with multivalued maps. We introduce the notions of local and partial-ε-minimum (weak) point and study ε-optimality, ε-Lagrangian multiplier theorem and ε-duality results.

Keywords: D-Multifunction; Partial-ε-Minimum Point; ε-Optimality; ε-Duality

1. Introduction

The theory of efficiency plays an important role in various knowledge fields. It is proposed as a new frontier in mathematical physics and engineering in context of priorities concerning the alternative energies, the climate exchange and education. Pareto efficiency or Pareto optimality is a central theory in economics with broad applications in game theory, social sciences, management sciences, various industries etc. In set valued vector optimization problems, it is important to know when the set of efficient points is nonempty to establish its main properties (existence, connectedness and compactness) and to extend the concepts to set valued vector optimization in infinite dimensional ordered vector spaces. The notion of proper efficiency was first introduced by Kuhn and Tucker [1] in their well known paper on nonlinear programming and many other notions have been proposed since then. Some of the well known notions are Geoffrion proper efficiency [2], Borwein proper efficiency [3], Benson proper efficiency [4] and super efficiency [5]. Chinaie and Zafarani [6] introduced the concepts of feeble multifunction minimum (weak) point, multifunction minimum (weak) point and obtained optimality conditions for set valued vector optimization problem having multivalued objective and constraints.

While it is theoretically possible to identify the complete set of solutions, finding an exact description of this set often turns out to be practically impossible or computationally too expensive. In practical situations we often stop the calculations at values that are sufficiently close to the optimal solutions, that is, we use algorithms that find approximate of the Pareto optimal set. Stability aspect in set valued vector optimization deals with the study of behaviour of the solution set under perturbations of the data. One of the approaches in this regard is the convergence of sequence of ε-solutions to a solution of the original problem. These facts justify the need of study of approximate efficiency which is equivalent to ε-optimality for set valued vector optimization problems. Some of the researchers who contributed in this area are Hamel [7], Rong and Wu [8].

Chinaie and Zafarani [9] introduced the concepts of ε-feeble multifunction minimum (weak) point and obtained optimality conditions for set valued vector optimization problem having multivalued objective and constraints. In this paper, we have given the notions of (local) partial-ε-minimum point and (local) partial-ε-weak minimum point, for set valued vector optimization problem and used them to study ε-optimality, ε-Lagrangian multiplier theorem and ε-duality results.

This paper is organized as follows: In Section 2 we have given the preliminaries and results related to quasi D-multifunction. In Section 3 we apply the directional derivative technique used by Yang [10] to characterize ε-optimality conditions for set valued vector optimization problem in terms of ε-feeble multifunction minimum point given by Chinaie and Zafarani [9]. In Section 4, we introduce (local) partial-ε-minimum point and (local) partial-ε-weak minimum point, and show that it is different from ε-feeble mutifunction minimum point. Also, we prove that every local partial-ε-minimum (weak) point is a partial-ε-minimum (weak) point if the objective function of set valued vector optimization problem is strict quasi D-multifunction and constraint function is quasi D-multifunction and show that this result is not true in the case of local ε-feeble multifunction minimum point. In Section 5, we obtain ε-Lagrangian multiplier theorem in terms of partial-ε-weak minimum point and in Section 6, we establish ε-weak duality and ε-strong duality for dual problem of set valued vector optimization problem.

2. Preliminaries and Definitions

Let X be locally convex topological vector space, Y, Z be real locally convex Hausdorff topological vector spaces; let $D \subset Y, E \subset Z$ be pointed closed convex cones with

$$\operatorname{int} D \neq \phi \text{ and } \operatorname{int} E \neq \phi.$$

Let Y^* be the dual space of Y, the positive dual cone D^+ of D is given by

$$D^+ = \left\{ f \in Y^* : f(y) \geq 0, \text{ for all } y \in D \right\}.$$

The set of strictly positive functions in D^+ is denoted by D^{+i}, that is

$$D^{+i} = \left\{ fY^* : f(y) > 0, \text{ for all } y \in D \setminus \{0\} \right\}.$$

For a set $A \subset Y$, we write

$$\operatorname{cone} A = \left\{ \lambda a : \lambda \geq 0, a \in A \right\}.$$

If D is convex cone in Y, then $\operatorname{int} D^+ \subset D^{+i}$ and equality holds if $\operatorname{int} D^+ \neq \phi$ [1].

A partial order $\leq_D$ in Y is defined by $y_1 \leq_D y_2$ iff, $y_2 - y_1 \in D$, for all $y_1, y_2 \in Y$.

Through out this paper, we denote $D^o := \operatorname{int} D$ and

$$D_o := D \setminus \{0\}.$$

Let $F : U \rightrightarrows Y$, be a multifunction defined on a non empty subset U of X with values in Y, which is partially ordered by cone D.

Now, for a multifunction $F : U \rightrightarrows Y$, denote by dom$F$ and im$F$ the domain and the image of F, respectively. In other words

$$\operatorname{dom} F = \left\{ x \in X : F(x) \neq \phi \right\},$$

$$\operatorname{im} F = F(X) = \bigcup_{x \in X} F(x).$$

The set

$$\operatorname{gr} F := \left\{ (x, y) : x \in \operatorname{dom} F, y \in F(x) \right\}$$
$$= \bigcup_{x \in X} \left[\{x\} \times F(x) \right]$$

is called the graph of F.

Definition 2.1: [10,11] Let U be convex subset of X. Let $F : U \rightrightarrows Y$ be a multifunction:

1) F is said to be a D-multifunction on U if, for all $x_1, x_2 \in U$ and $t \in [0,1]$, we have

$$tF(x_1) + (1-t) F(x_2) \subseteq F(tx_1 + (1-t) x_2) + D;$$

2) F is said to be a quasi D-multifunction on U if, for all $x_1, x_2 \in U$ and $t \in [0,1]$, we have:

$$\left(F(x_1) + D \right) \cap \left(F(x_2) + D \right) \subseteq F(tx_1 + (1-t) x_2) + D;$$

3) F is said to be a strictly quasi D-multifunction on U iff, for all $x_1, x_2 \in U$, $x_1 \neq x_2$ and $t \in (0,1)$, we have:

$$\left(F(x_1) + D \right) \cap \left(F(x_2) + D \right) \subseteq F(tx_1 + (1-t) x_2) + D^o.$$

Yang [10] gave the following definitions:

Definition 2.2: A function $f : X \to Y$ is said to be a continuous selection of F if f is continuous and $f(x) \in F(x)$, for all $x \in X$. Denote by CS (F) the set of all continuous selections of F.

Definition 2.3: Let

$$S(x_0, V) := \left\{ v \in X : \text{there exists } \delta > 0, x_0 + tv \in V, 0 < t < \delta \right\}$$

be the cone of feasible directions. Then the limit set of F at x_0 in the direction

$v \in S(x_0, F)$ is $Y_F^{y_o}(x_0, v)$:

$$= \left\{ z : z = \lim_{(t_n, u_n) \to (0^+, v)} \frac{f(x_0 + t_n u_n) - y_0}{t_n}, u_n \in S(x_0, V) \right\},$$

where $f \in CS(F)$ with $f(x_0) = y_0$.

The union of all limit sets of F at x_0 in all directions $v \in S(x_0, V)$ is denoted by

$$Y_F^{y_o}\left(x_0, S(x_0, V) \right).$$

We need the following assumption:

Assumption 2.1: Let $x, y \in X$. If

$$z \in F(tx + (1-t) y),$$

for all $t \in (0,1), y \in F(x)$, then there exists a continuous selection $f \in CS(F)$ such that

$$z \in f(tx + (1-t) y),$$

for all $t \in (0,1)$.

Theorem 2.1: Let U be convex subset of X and $F : U \rightrightarrows Y$. If assumption 2.1 holds and F is D-multifunction, then for any

$$x, x^* \in U \text{ and } y^* \in F(x^*),$$

$$F(x) - y^* \subset Y_F^{y^*}(x^*, x - x^*) + D.$$

***Proof*:** Since F is D-multifunction therefore, for $t \in (0,1), x, x^* \in U$, we have

$$tF(x) + (1-t) F(x^*) F(tx + (1-t) x^*) + D;$$

which gives that

$$tF(x) + (1-t) y^* F(tx + (1-t) x^*) + D.$$

If $w \in F(x)$, by assumption 2.1, there exist

$f \in CS(F)$ such that

$$tw+(1-t)y^* \in f\left(tx+(1-t)x^*\right)+D.$$

That is,

$$w-y^* \in \frac{f\left(tx+(1-t)x^*\right)-y^*}{t}+D$$

Thus,

$$w-y^* \in Y_F^{y^*}\left(x^*, x-x^*\right)+D.$$

Hence,

$$F(x)-y^*Y_F^{y^*}\left(x^*, x-x^*\right)+D.$$

Theorem 2.2: Let U be convex subset of X, $F:U \rightrightarrows Y$ be quasi D-multifunction on U and assumption 2.1 hold then, for any

$$x, x^* \in U, y^* \in F\left(x^*\right), \left(y^*-F(x)\right) \cap D \neq \phi$$

implies that,

$$-Y_F^{y^*}\left(x^*, x-x^*\right) \cap D \neq \phi..$$

Proof: For $x, x^* \in U, y^* \in F\left(x^*\right)$, let

$$\left(y^*-F(x)\right) \cap D \neq \phi$$

That is, $y^* \in F(x)+D$ and $y^* \in F\left(x^*\right)+D.$

Since F is quasi D-multifunction, therefore for $t \in (0,1)$,

$$\left(F(x)+D\right) \cap \left(F\left(x^*\right)+D\right) \subseteq F\left(tx+(1-t)x^*\right)+D,$$

which implies that $y^* \in F\left(tx+(1-t)x^*\right)+D.$

Then, by assumption 2.1 there is $f \in CS(F)$ such that

$$y^* \in f\left(tx+(1-t)x^*\right)+D, \text{for all } t \in (0,1),$$

which gives that

$$-\left(\frac{f\left(t\,x+(1-t)x^*\right)-y^*}{t}\right) \in D, \text{ for all } t \in (0,1).$$

That is, $-Y_F^{y^*}\left(x^*, x-x^*\right) \cap D \neq \phi.$

3. ε-Optimality in Terms of Directional Derivatives

In this section, we obtain ε-optimality conditions for set valued vector optimization problem in terms of directional derivatives given by Yang [10] for ε-feeble multifunction minimum point given by Chinaie and Zafarani [9]. We consider the following set valued vector optimization problem:

$$(\text{VP}) \ D-\min_{x \in U} F(x)$$
$$\text{s.t. } G(x) \cap (E) \neq \phi$$

where $U \subseteq X$ is non empty set, $F:U \rightrightarrows Y$, $G:U \rightrightarrows Z$, are multifunctions with nonempty values. The set of feasible solutions of (VP) is denoted by V, that is

$$V=\left\{x \in U : G(x) \cap (-E) \neq \phi\right\}.$$

Chinaie and Zafarani [9] gave the following definitions.

Definition 3.1: Let $\bar{x} \in V, \varepsilon \in D_o$.

1) $\bar{x}$ is called a ε-feeble multifunction minimum point (ε-f. m. m. p.) of problem (VP), if there exists, $\bar{y} \in F(\bar{x})$, such that

$$F\left(V \setminus \{\bar{x}\}\right) \cap \left(\bar{y}-\varepsilon-D\right)=\phi; \tag{3.1}$$

2) $\bar{x}$ is called a ε-feeble multifunction weak minimum point (ε-f. m. w. m. p.) of problem (VP), if there exists, $\bar{y} \in F(\bar{x})$, such that

$$F\left(V \setminus \{\bar{x}\}\right) \cap \left(\bar{y}-\varepsilon-D^o\right)=\phi; \tag{3.2}$$

The set of $\bar{x} \in V$ which satisfies (3.1) or (3.2) is denoted by $\varepsilon-\hat{S}(F,D)$ and $\varepsilon-\widehat{WS}(F,D)$ respectively.

When $V \setminus \{\bar{x}\}$ is replaced by $V \setminus \{\bar{x}\} \cap N(\bar{x})$ in (3.1) and (3.2), $N(\bar{x})$ being neighbourhood of $\bar{x}$, then we have local ε-f. m. m. p. and local ε-f. m. w. m. p. of problem (VP).

We now give the necessary optimal conditions for local ε-feeble weak minimum point [9] of (VP).

Theorem 3.1: Let $x_0 \in V$ and $y_0 \in F(x_0)$ be local ε-feeble multifunction weak minimum point of problem (VP). Then,

$$-Y_F^{y_o}\left(x_0, S(x_0, V)\right) \cap \left(D^o+\varepsilon\right)=\phi, \varepsilon \in D_o.$$

Proof Suppose $x_0 \in V$ is local ε-feeble multifunction weak minimum point of problem (VP) and f is any continuous selection of F such that $y_0=f(x_0)$.

Then,

$$F\left(V \setminus \{x_0\} \cap N(x_0)\right) \cap \left(y_0-\varepsilon-D^o\right)=\phi,$$

where $N(x_0)$ is neighbourhood of x_0.

If

$$z \in Y_F^{y_o}\left(x_0, S(x_0, V)\right),$$

then there exists v, $u_n \in S(x_0, V)$, $u_n \to v$, $t_n \to 0^+$ such that

$$z=\lim_{(t_n, u_n) \to (0^+, v)} \frac{f(x_o+t_n u_n)-y_o}{t_n},$$

for some $f \in CS(F)$.

Let $x_n=x_0+t_n u_n$; then there exist $n_0 \in N$ such that

$$x_n \in V \setminus \{x_0\} \cap N(x_0), \text{ for all } n \geq n_0;$$

$$-\left(F(x_n)-y_0\right) \cap \left(D^o+\varepsilon\right)=\phi, \text{ for all } n \geq n_0.$$

Since f is any continuous selection of F such that

$y_0 = f(x_0)$, therefore:

$$-\left(f(x_n) - f(x_0)\right) \cap \left(D^o + \varepsilon\right) = \phi, \text{ for all } n \ge n_0;$$

$$-\left(\frac{f(x_n) - f(x_0)}{t_n}\right) \cap \left(D^o + \varepsilon\right) = \phi, \text{ for all } n \ge n_0.$$

It follows that $-z \notin D^o + \varepsilon$.

Now, we give the sufficient conditions for ε-feeble multifunction minimum point of problem (VP).

Theorem 3.2: Let $x_0 \in V$,

$$y_0 \in F(x_0), \quad \varepsilon \in D_o, F: U \rightrightarrows Y$$

be a D-mulifunction and $G: U \rightrightarrows Z$ be quasi D-multifunction and f be continuous selection of F such that

$$y_0 = f(x_0).$$

If

$$-Y_F^{y_o}(x_0, x - x_0) \cap (D + \varepsilon) = \phi, \text{for all } x \in S(x_0, V),$$

then x_0 is ε-feeble multifunction minimum point of problem (VP).

Proof: Let x_0 be not ε-feeble multifunction minimum point of problem (VP), then there exists $y_0 \in F(x_0)$ such that

$$F(V \setminus \{x_o\}) \cap (y_0 - \varepsilon - D) \ne \phi,$$

which implies that, there exists $x \ne x_0 \in V, y \in F(x)$, such that $y_0 - \varepsilon - y \in D.$

Since G is quasi D-multifunction, therefore feasible set V is convex,

$$x_0 + t(x - x_0) \in V \text{ for } 0 < t < 1.$$

Thus, $x \in S(x_0, V)$ which implies that

$$-Y_F^{y_o}(x_0, x - x_0) \cap (D + \varepsilon) = \phi \tag{3.3}$$

Since F is D-multifunction therefore,

$$F(x) - y_0 \in Y_F^{y_o}(x_0, x - x_0) + D,$$

which gives that

$$y_0 - y \in -Y_F^{y_o}(x_0, x - x_0) - D.$$

Thus, there exists

$$k \in -Y_F^{y_o}(x_0, x - x_0)$$

such that

$$y_0 - y \in k - D,$$

that is

$$k - y_0 + y \in D.$$

Also,

$$y_0 - \varepsilon - y \in D$$

which implies that $k \in D + \varepsilon.$ Also,

$$k \in -Y_F^{y_o}(x_0, x - x_0)$$

Hence,

$$-Y_F^{y_o}(x_0, x - x_0) \cap (D + \varepsilon) \ne \phi,$$

which is contradiction to given condition (3.3).

4. Partial-ε-Minimum (Weak) Point

In this section we introduce the notion of partial-ε-minimum point, and partial-ε-weak minimum point.

Definition 4.1: Let $\bar{x} \in V, \varepsilon \in D$.

1) $\bar{x}$ is called a partial-ε-minimum point (p.-ε-m. p.) of problem (VP), if there exists, $\bar{y} \in F(\bar{x})$, such that

$$F(V \setminus \{\bar{x}\}) \cap (\bar{y} + \varepsilon - D_o) = \phi; \tag{4.1}$$

2) $\bar{x}$ is called a partial-ε-weak minimum point (p.-ε-w. m. p.) of problem (VP), if there exists, $\bar{y} \in F(\bar{x})$, such that

$$F(V \setminus \{\bar{x}\}) \cap (\bar{y} + \varepsilon - D^o) = \phi; \tag{4.2}$$

The set of $\bar{x} \in V$ which satisfies (4.1) or (4.2) is denoted by $\varepsilon - \hat{P}(F, D)$ and $\varepsilon - \widehat{WP}(F, D)$ respectively.

When $V \setminus \{\bar{x}\}$ is replaced by $V \setminus \{\bar{x}\} \cap N(\bar{x})$ in (4.1) and (4.2), $N(\bar{x})$ being neighbourhood of $\bar{x}$, then we have local p.-ε-.m. p. and local p.-ε-w. m. p. of problem (VP). If $(\bar{x}, \bar{y}) \in \text{gr}F$ satisfies (4.1) then it is called partial-ε-minimizer of (VP) and if satisfies (4.2) then it is called partial-ε-weak minimizer of (VP).

Now we show that partial-ε-minimum point is different from ε-feeble multifunction minimum point.

The following example illustrates that

$$\varepsilon - \hat{S}(F, D) \not\subset \varepsilon - \hat{P}(F, D).$$

Example 4.1: Let $U = X = R$, $Y = R^2$, $Z = R^2$,

$$\varepsilon = \left(\frac{1}{2}, \frac{1}{2}\right), \quad D = R_+^2, \quad E = R_+^2$$

and $G: U \rightrightarrows Z$ be defined by

$$G(x) = \begin{cases} [0, x] \times [0, x] & \text{if } x \ge 0 \\ (x, 0) & \text{if } x < 0 \end{cases}$$

and $F: U \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(0,0), (x,0)\} & \text{if } x < 0 \\ [0, x] \times [0, x] & \text{if } x \ge 0 \end{cases}$$

then, $V = \{x \in R : x \le 0\}.$

Let

$$\bar{x} = 0 \in V, \bar{y} = (0,0) \in F(\bar{x}).$$

Then,

$$F(V \setminus \{\bar{x}\}) \cap (\bar{y} - \varepsilon - D) = \phi.$$

Thus, $\bar{x} \in \varepsilon - \hat{S}(F,D)$.
But

$$(0,0) \in F(V \setminus \{\bar{x}\}) \cap (\bar{y} + \varepsilon - D_o).$$

Thus,

$$\bar{x} \notin \varepsilon - \hat{P}(F,D).$$

The following example illustrates that

$$\varepsilon - \hat{P}(F,D) \not\subset \varepsilon - \hat{S}(F,D).$$

Example 4.2: Let $U = X = R$, $Y = R^2$,

$$Z = R^2, \quad \varepsilon = \left(\frac{5}{2}, 3\right),$$

$$D = \{(x,y) : x \le y, -x \le y,\ y \ge 0\},$$

$$E = \{(x,y) : y \ge 0, x \ge 0\},$$

and $G : U \rightrightarrows Z$ be defined by

$$G(x) = \begin{cases} [0,x] \times [0,x] & \text{if } x \ge 0 \\ (x,0) & \text{if } x < 0 \end{cases}$$

and $F : U \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(0,0), (-2.5,-3.1), (x,0)\} & \text{if } x \le 0 \\ [1,x] \times [1,x] & \text{if } x > 0 \end{cases}$$

then, $V = \{x \in R : x \le 0\}$.
Let

$$\bar{x} = 0 \in V, \ \bar{y} = (0,0) \in F(\bar{x}).$$

Then,

$$F(V \setminus \{\bar{x}\}) \cap (\bar{y} + \varepsilon - D_o) = \phi.$$

Thus,

$$\bar{x} \in \varepsilon - \hat{P}(F,D).$$

But

$$(-2.5,-3.1) \in F(V \setminus \{\bar{x}\}) \cap (\bar{y} + \varepsilon - D).$$

Thus,

$$\bar{x} \notin \varepsilon - \hat{S}(F,D).$$

The following lemma can be proved as in [9].

Lemma 4.1: Let $F : U \rightrightarrows Y$ be a strictly quasi D-multifunction and $G : U \rightrightarrows Z$ be a quasi D-multifunction. Then,

$$\varepsilon - \hat{P}(F,D) = \varepsilon - \widehat{W}P(F,D).$$

Now, we show that every local partial-ε-minimum (weak) point is a partial-(weak) point if F is srictly quasi D-multifunction and G is quasi D-multifunction and prove local ε-feeble multifunction minimum point is not ε-feeble multifunction minimum point of problem (VP) in above conditions.

Theorem 4.1: Let F be strictly quasi-D-multifunction and G be a quasi D-multifunction. Then, any local partial-ε-minimum point of problem (VP) is a partial-ε-minimum point of problem (VP).

***Proof*:** Let $\bar{x}$ be local partial-ε-minimum point of problem (VP), then there exists a neighbourhood $N(\bar{x})$ of $\bar{x}$ and $\bar{y} \in F(\bar{x})$ such that

$$F\left(V \cap \left(N(\bar{x}) \setminus \{\bar{x}\}\right)\right) \cap (\bar{y} + \varepsilon - D_o) = \phi. \tag{4.3}$$

Let if possible, $\bar{x}$ be not partial-ε-minimum point of problem (VP). Then, there exist $\bar{y} \in F(\bar{x})$ such that

$$F(V \setminus \{\bar{x}\}) \cap (\bar{y} + \varepsilon - D_o) = \phi.$$

Thus, there exists $x \in V \setminus \{\bar{x}\}$ and $y \in F(x)$, such that $y \in \bar{y} + \varepsilon - D_o$, which gives that

$$\bar{y} + \varepsilon \in y + D_o \subset y + D.$$

That is,

$$\bar{y} + \varepsilon \in (F(x) + D) \cap (F(\bar{x}) + D).$$

Since F is a strict quasi D-multifunction, therefore for each $t \in (0,1)$. We have

$$(F(x) + D) \cap (F(\bar{x}) + D) \subseteq F(tx + (1-t)\bar{x}) + D^o,$$

which implies that

$$\bar{y} + \varepsilon \in F(\bar{x} + t(x - \bar{x})) + D^o.$$

Let $x_t = \bar{x} + t(x - \bar{x})$. Then, for each $t \in (0,1)$,

$$\bar{y} + \varepsilon \in F(x_t) + D_o$$

and consequently there exists $y_t \in F(x_t)$ such that, $y_t \in \bar{y} + \varepsilon - D_o$.

On the other hand for $t \in (0,\delta)$ with $\delta > 0$ small enough, $\bar{x} + t(x - \bar{x}) \in N(\bar{x})$. Since G is quasi D-multifunction, therefore, feasible set is convex and we have

$$x_t = \bar{x} + t(x - \bar{x}) \in V \cap \left(N(\bar{x}) \setminus \{\bar{x}\}\right), \text{for } t \in (0,\delta).$$

Thus, we deduce that

$$y_t \in F\left(V \cap \left(N(\bar{x}) \setminus \{\bar{x}\}\right)\right) \cap (\bar{y} + \varepsilon - D_o),$$

which contradicts (4.3).

The following example illustrates that above result is not true for ε-feeble multifunction minimum point of (VP).

Example 4.3: Let $U = X = R$, $Y = R^2$, $Z = R^2$, $D = R_+^2$, $\varepsilon = (3.5, 3.5)$, $E = R_+^2$ and $G : U \rightrightarrows Z$ defined by

$$G(x) = \begin{cases} [0,x] \times [0,x] & \text{if } x \ge 0 \\ (x,0) & \text{if } x < 0 \end{cases}$$

and $F: U \rightrightarrows Y$ defined by

$$F(x)=\begin{cases}\left\{(1,1),(x,x),\left(\frac{1}{2},\frac{1}{2}\right)\right\} & \text{if } x<-1\\ \left\{(1,1),\left(x^2,x^2\right)\right\} & \text{if } -1\le x\le 0\\ \left\{(x,0),\left(x^2,x^2\right)\right\} & \text{if } x>0\end{cases}$$

Here G is quasi D-multifunction and F is strict quasi D-multifunction.

Then, Let $\bar{x}=0\in V$, $\bar{y}=(1,1)\in F(\bar{x})$.

Then

$$F\left(V\setminus\{\bar{x}\}\cap N(\bar{x})\right)\cap\left(\bar{y}-\varepsilon-D\right)=\phi.$$

Thus, $\bar{x}$ is local ε-feeble multifunction minimum point.

$$\bar{y}-\varepsilon-D=\left(-5/2,-5/2\right)-D,$$

$$(-3,-3)\in F\left(V\setminus\{\bar{x}\}\right)\cap\left(\bar{y}-\varepsilon-D\right)=\phi.$$

Thus, $\bar{x}$ is not ε-feeble multifunction minimum point of problem (VP).

5. ε-Lagrangian Multiplier Theorem

In this section, let $L(Z, Y)$ be the set of continuous linear operators from Z to Y, and let

$$L_+(Z,Y)=\left\{T\in L(Z,Y):T(E)\subset D\right\}$$

Denote by (F, G) the multivalued map from X to $Y\times Z$ defined by

$$(F,G)(x)=F(x)\times G(x),$$

for all $x\in X$.

If $h\in Y^*$, $T\in L(Z,Y)$, we define $hF: X\to R$ and $F+TG: X\rightrightarrows Y$ as $(hF)(x)=h(F(x))$ and

$$(F+TG)(x)=F(x)+T(G(x)),$$

respectively.

Lemma 5.1: [14]. Let $F: X\rightrightarrows Y$ be D-multifunction on X. Then, one and only one of the following statements is true:

1) there exists $x\in X$ such that $F(x)\cap\left(-D^o\right)\neq\phi.$

2) there exists $\xi\in D^+\setminus\{0\}$ such that $\xi(y)\ge 0$ for all $y\in F(X)$.

Theorem 5.1: Let $D^o\neq\phi$, $G(V)\cap\left(E^o\right)\neq\phi$, $\bar{y}\in F(\bar{x})$ and let $(F-\bar{y}-\varepsilon,G)$ be D-multifunction on V. If $\bar{x}$ is partial-ε-weak minimum point of problem (VP), then there exists $T\in L_+(Z,Y)$ such that $\bar{x}$ is partial-ε-weak minimum point of following problem:

$$(\mathrm{VP})_{\mathrm{T}}\ \min_{x\in V}\left(F(x)+T(G(x))\right)$$

and

$$-T\left(G(\bar{x})\cap(-E)\right)\subset\left(D^o\cup\{0\}\right)$$

Proof: Since $\bar{x}$ is partial-ε-weak minimum point of problem (VP), therefore there exists, $\bar{y}\in F(\bar{x})$, such that

$$F\left(V\setminus\{\bar{x}\}\right)\cap\left(\bar{y}+\varepsilon-D^o\right)=\phi \tag{5.1}$$

Hence,

$$\left(F\left(V\setminus\{\bar{x}\}\right)-\bar{y}-\varepsilon,G(V)\right)\cap\left(-D^o,-E^o\right)=\phi$$

Since $(F-\bar{y}-\varepsilon,G)$ is D-multifunction on V, therefore by Lemma 5.1, there exists

$$(h,p)\in\left(D^+,E^+\right)\setminus\{(0,0)\}$$

such that

$$\begin{aligned}&h(y-\bar{y}-\varepsilon)+p(s)\ge 0,\ \text{for all}\\ &x\in V\setminus\{\bar{x}\},y\in F(x),s\in G(x).\end{aligned} \tag{5.2}$$

We claim that $h\neq 0$. In fact, if h = 0, then $p\neq 0$ and $p(s)\ge 0$, for all $s\in G(x)$ (5.3). Since

$$G(V)\cap\left(E^o\right)\neq\phi,$$

there exists $x_1\in V$ and

$$s_1\in G(x_1)\cap\left(-E^o\right).$$

Hence, $p(s_1)<0$, which contradicts (5.3).

Therefore, $h\neq 0$. Fix $d\in D^o$ with $h(d)=1$ and define $T: Z\to Y$ as $T(z)=p(z)d$, for all $z\in Z$ (5.4).

Clearly, $T\in L_+(Z,Y)$.

Using (5.2) and (5.4), we get

$$h\left(y-\bar{y}-\varepsilon+T(s)\right)\ge 0. \tag{5.5}$$

Since $\bar{x}\in V$, therefore $G(\bar{x})\cap(-E)\neq\phi$.

Let $\bar{s}\in G(\bar{x})\cap(-E)$, then $\bar{s}\in G(\bar{x})$ and $\bar{s}\in -E$.

This gives that $0\ge p(\bar{s})$ (5.6).

Therefore we get that,

$$-T(\bar{s})=-p(\bar{s})d\in\left(D^o\cup\{0\}\right).$$

Thus, we have

$$-T\left(G(\bar{x})\cap(-E)\right)\left(D^o\cup\{0\}\right).$$

Suppose that $\bar{x}$ is not partial-ε-weak minimum point of problem $(\mathrm{VP})_T$, which gives that

$$F\left(V\setminus\{\bar{x}\}\right)+T\left(G\left(V\setminus\{\bar{x}\}\right)\right)\cap\left(\bar{y}+\varepsilon-D^o\right)=\phi$$

Then there exist $x_0\in V\setminus\{\bar{x}\}$, $y_0\in F(x_0)$ and $s_0\in G(x_0)$ such that

$$\bar{y}-\left(y_0+T(s_0)\right)+\varepsilon\in D^o.$$

Since $h\in D^+\setminus\{0\}$, we get:

$$h\left(\bar{y}-\left(y_0+T(s_0)\right)+\varepsilon\right)>0$$

From (5.4), we get

$$h(y_0-\bar{y}-\varepsilon)+p(s_0)<0,$$

which contradicts (5.2).

Hence $\bar{x}$ is partial-ε-weak minimum point of problem $(\mathrm{VP})_T$.

6. ε-Duality

Let us define a multivalued mapping $\psi: L_+(Z,Y)\rightrightarrows Y$ by $\psi(T)$ = {y: there exists $x\in V$, $y\in F(x)$ such that x is partial-ε-weak minimum point of problem $(\mathrm{VP})_T$}.

Consider the following maximum problem: (VD) $\max\psi(T)$ subject to $T\in L_+(Z,Y)$.

Definition 6.1: A point $T\in L_+(Z,Y)$ is said to be a feasible point of problem (VD) if $\psi(T)\neq\phi$. We say that (T_0,y_0) is partial-ε-weak maximizer of (VD) if there exists no feasible point $T\in L_+(Z,Y)$ such that:

$$(\psi(T))\cap(y_0-\varepsilon+D^o)\neq\phi.$$

We now establish the following ε-duality results.

Theorem 6.1 (ε-Weak duality): If $\bar{x}\in V$ and $T\in L_+(Z,Y)$ is a feasible point of problem (VD), then

$$F(\bar{x})\cap(\psi(T)+\varepsilon-D^o)=\phi.$$

***Proof*:** Since $\psi(T)\neq\phi$, for any $y\in\psi(T)$, there exists $\bar{x}\in V, y\in F(\bar{x})$ such that $\bar{x}$ is partial-ε-weak minimum point of (VD) corresponding to T.

It follows that,

$$\left[(F+T(G))(V\setminus\{\bar{x}\})-y-\varepsilon\right]\cap(-D^o)=\phi. \tag{6.1}$$

Now, we show that

$$F(\bar{x})\cap(y+\varepsilon-D^o)=\phi.$$

On contrary, suppose that

$$F(\bar{x})\cap(y+\varepsilon-D^o)\neq\phi.$$

Then there exists $\bar{y}\in F(\bar{x})$, such that

$$\bar{y}\in(y+\varepsilon-D^o),$$

which implies that $(\bar{y}-y-\varepsilon)\in -D^o$.

Since $\bar{x}$ is a feasible point of problem $(\mathrm{VP})_T$, there exist $\bar{z}\in G(\bar{x})\cap(-E)$.

It is given that $T\in L_+(Z,Y)$, therefore $T\bar{z}\in -D$, which implies that

$$\bar{y}+T\bar{z}-y-\varepsilon\in T\bar{z}-D^o\subset -D-D^o\subset -D^o.$$

Thus,

$$\left[(F+T(G))(V\setminus\{\bar{x}\})-y-\varepsilon\right]\cap(-D^o)\neq\phi,$$

which contradicts (6.1).

Therefore, we have

$$F(\bar{x})\cap(\psi(T)+\varepsilon-D^o)=\phi.$$

Theorem 6.2: (ε-Strong duality): Let $(F-y_0-\varepsilon,G)$ be D-multifunction on V. If $x_0\in V, y_0\in F(x_0), x_0$ is partial-ε-weak minimum point of problem (VP) and $G(V)\cap(E^o)=\phi$, then there exists $T_0\in L_+(Z,Y)$ such that (T_0,y_0) is partial-ε-weak maximizer of problem (VD).

***Proof*:** Suppose x_0 is partial-ε-weak minimum point of problem (VP) and $G(V)\cap(E^o)\neq\phi$.

Then, by Theorem 5.1 there exists $T_0\in L_+(Z,Y)$ such that x_0 is partial-ε-weak minimum point of problem $(\mathrm{VP})_{T_0}$, corresponding to T_0.

It follows that, $\psi(T_0)\neq\phi$.

Thus, T_0 is feasible point of (VD) and $y_0\in\psi(T_0)$.

By ε-weak duality, we obtain

$$(y_0-\psi(T_0)-\varepsilon)\cap(-D^o)=\phi.$$

Thus, (T_0,y_0) is partial-ε-weak maximizer of problem (VD).

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