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A Series Solution for the Ginzburg-Landau Equation with a Time-Periodic Coefficient

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Abstract

The solution of the real Ginzburg-Landau (GL) equation with a time-periodic coefficient is obtained in the form of a series, with assured convergence, using the computer-assisted 'Homotopy Analysis Method' (HAM) propounded by Liao [1]. The formulation has been kept quite general to keep open the possibility of obtaining the solution of the GL equation for different continua as limiting cases of the present study. New ideas have been added and clear explanations are provided in the paper to the existing concepts in HAM. The method can easily be extended to solve complex GL equation, system of GL equations or even the GL equations with a diffusion term, each having a time-periodic coefficient. The necessary code in Mathematica that implements the HAM for the current problem is appended to the paper for use by the readers.

1. Introduction

GL equations arise as a solvability condition in a wide variety of problems in continuum mechanics while dealing with a weakly nonlinear stability of systems, e.g., one comes across the GL equation with constant and real coefficients in the case of Rayleigh-Bénard convection in fluids wherein instability sets in as a direct mode (also called stationary mode). When the Hopf mode (also called oscillatory mode) is the preferred one, like in viscoelastic liquids or as in constrained systems, the GL equation has complex yet constant coefficients. In certain other problems one may also come across a system of GL equations with constant coefficients. There are inhomogeneous GL equations also.

When one considers problems in which gravity experienced in a fluid-based system is perturbed by a time-periodic vibration of the system, then the GL equation turns out to be an equation like the one considered in the paper and the same has been solved here using the HAM, as propounded by Liao[1-5]. One may extend the solution procedure to other types of GL equations mentioned earlier. The great advantage in using the method is that it gives the solution of non-linear equations in the form of a series whose convergence is assured. The method is illustrated here in unabridged form using the example of the GL equation with a time-periodic coeffi-

cient. The readers may refer to Liao [1-5] and others [6-13] for many other versatile applications of the method.

2. The GL Equation with a Time-Periodic Coefficient and its Solution by the HAM

The GL equation with a time-periodic coefficient in the most general form is (see the appendix of the paper for the derivation of the equation in the context of a physical situation):

$$A_\tau = Q_1 [1 + \delta \sin \Omega \tau] A - Q_2 A^3, \qquad (2.1)$$

where A is the amplitude of convection, Ω and δ are the frequency and small amplitude of the gravitational vibration (also called gravity modulation), Q_1 and Q_2 are constant coefficients (real) that are functions of the parameters of a given problem. We have used just δ in place of δ_2 used in the appendix.

The initial condition to solve Equation (2.1) is:

$$A(0) = a_0, \qquad (2.2)$$

where a_0 is a prescribed initial value of the amplitude. The quantities Q_1, Q_2, δ and a_0 in Equations (2.1) and (2.2) have deliberately been left unspecified to keep open the possibility of obtaining a general result from the study that is applicable to different continua. This is one of the salient features of the study.

It must be noted here that in the absence of gravity modulation, *i.e.*, δ =0, Equation (2.1) is the exactly solvable Bernoulli equation, viz., $A_\tau = Q_1 A - Q_2 A^3$ and the unsteady solution of this GL equation, subject to condition (2.2), is given by:

$$A(\tau) = \sqrt{\frac{Q_1}{Q_2}} \left[1 - \left(1 - \frac{Q_1}{Q_2 a_0^2} \right) e^{-2Q_1 \tau} \right]^{-\frac{1}{2}}. \quad (2.3)$$

When the amplitude $A(\tau)$ is small, however, the Bernoulli equation reduces to

$$A_\tau = Q_1 A, \quad (2.4)$$

Whose solution, subject to Equation (2.2), is

$$A(\tau) = a_0 e^{Q_1 \tau}. \quad (2.5)$$

Thus we see that the amplitude grows exponentially to begin with (see solution (2.5)) but the growth is damped by the A^3 term as shown by the solution (2.3).

One can easily see that for the case δ =0, Equation (2.1) has one steady solution $A(\infty) = 0$, for all values of Q_1 and Q_2, and a second steady solution $A(\infty) = \sqrt{\frac{Q_1}{Q_2}}$, for $\frac{Q_1}{Q_2} > 0$. The second steady state solution for δ =0 and the initial condition (2.2) are vital information for our writing down an initial solution $A_0(\tau)$ of the GL equation. The initial solution may be taken in the form:

$$A_0(\tau) = a_1 + (a_0 - a_1) e^{-\gamma \tau}, \quad (2.6)$$

where $a_1 = \sqrt{\frac{Q_1}{Q_2}}$ and γ is as yet unspecified. The determination of γ can and will be dealt with at the time of seeking a series solution of the GL equation with a time-periodic coefficient. Quite obviously $A_0(\tau)$ has been so chosen that it satisfies the conditions $A(\infty) = a_1$ and $A(0) = a_0$. The choice of the form of $A_0(\tau)$ is most important in obtaining a convergent series solution by the HAM and this aspect will be discussed much later in the paper.

Now we discuss the problem in the presence of modulation. One cannot in this case arrive at a useful analytical solution as above, independent of numerical integration, and resorting to numerical methods seems the only option. As a better option, alternately, we propose the HAM to obtain the series solution. To that end we define the following two notations:

$$L[A(\tau)] = A_\tau + \gamma A, \quad (2.7)$$

$$N[A(\tau)] = A_\tau - Q_1(1 + \delta \sin \Omega \tau) A + Q_2 A^3, \quad (2.8)$$

where γ is the same as that used in Equation (2.6) and as said earlier it will be dealt with at the time of obtaining a series solution of the GL equation with a time-periodic coefficient. The particular choice of L in Equation (2.7) is suggested by the fact that the initial solution $A_0(\tau)$ go-es as $e^{-\gamma \tau}$. We throw more light on this in the next section.

The proceeding of the paper thus far does give a feeling that many things are open ended but the reader we reassure that at the end of it all the inevitable postponement of certain discussions to the end of the paper becomes clear. Now we move on to the remaining part of the method involving construction of the solution of the given nonlinear equation by means of concepts borrowed from topology.

In the HAM we obtain the solution of the equation

$$N[A] = 0 \quad (2.9)$$

By constructing a homotopy from $A_0(\tau)$ to the required solution $A_{NL}(\tau)$ of Equation (2.9) using the services of what is called as an *embedding parameter* $p \in [0,1]$. As we see later *p=0* and *p=1* correspond to $A_0(\tau)$ and $A_{NL}(\tau)$ respectively, where the subscript NL indicates that the solution is of the nonlinear problem. Henceforth, we use a suggestive notation $\phi(\tau; p)$ in place of $A(\tau)$ which indicates that the embedding parameter has been brought into picture. Now we use the Maclaurin series for $\phi(\tau; p)$ around *p=0* and this gives us:

$$\phi(\tau; p) = \phi(\tau; 0) + \sum_{m=1}^{\infty} A_m(\tau) p^m, \quad (2.10)$$

where

$$A_m(\tau) = \frac{1}{m!} \left[\frac{\partial^m}{\partial p^m} \{ \phi(\tau; p) \} \right]_{p=0} \quad (m = 1, 2, 3, \cdots). \quad (2.11)$$

Observing Equation (2.10), together with Equation (2.11), we find that $A_{NL}(\tau)$ can be determined provided various *p*-derivatives of $\phi(\tau; p)$ can be found at *p=0*. To obtain these derivatives we need a differential equation for $\phi(\tau; p)$. The required equation for $\phi(\tau; p)$ can be constructed using $L[A(\tau)]$ and $N[A(\tau)]$, and we also remind ourselves at this point that $\phi(\tau; p)$ varies from $A_0(\tau)$ to $A_{NL}(\tau)$ as *p* varies from 0 to 1. The required equation that fits this bill is:

$$(1-p) L[\phi(\tau; p) - A_0(\tau)] = \hbar \, p \, N[\phi(\tau; p)], \quad (2.12)$$

where $\hbar$ is an *auxiliary parameter* whose role in the control of convergence of the HAM-series solution will become apparent further on. In view of the above observation, $\hbar$ may also be called as the *convergence-control parameter*.

We are interested in $\phi(\tau;1) = A_{NL}(\tau)$ which is the solution of the given nonlinear Equation (2.1), now written in the form (2.7). Substituting $p=1$ in Equation (2.10) and noting that $\phi(\tau;0) = A_0(\tau)$, we get

$$A_{NL}(\tau) = A_0(\tau) + \sum_{m=1}^{\infty} A_m(\tau). \qquad (2.13)$$

From Equation (2.13) it is clear that the solution $A_{NL}(\tau)$ of the given nonlinear Equation (2.1), subject to the condition (2.2), will be obtained as a series. Liao [4] proved a convergence theorem that is applicable to the series (2.13). The convergence is certain by the HAM as a consequence of the *convergence theorem* (see p. 18 of Liao [4]) but how to have the best possible convergence, or rather fast convergence, is the question. This is a matter of discussion later on in the paper in the context of discussing the role of a certain to-be-introduced parameter controlling convergence.

The initial condition to solve Equation (2.12) can be obtained from Equation (2.2) as:

$$\phi(0;p) = a_0. \qquad (2.14)$$

Equations (2.12) and (2.14) are called the *zeroth-order deformation equations* as per the notion of deformation, as used in topology. Deformation has been made possible to be used in the method due to the parameter p that allows the varying of $\phi(\tau;p)$ from $A_0(\tau)$ to $A_{NL}(\tau)$ as it varies from 0 to 1.

Now, in order to obtain the p-derivatives of $\phi(\tau;p)$, we differentiate m-times the zeroth-order deformation equations with respect to p. To make use of the notation $A_m(\tau)$ defined in Equation (2.11) we set $p=0$ in the resulting equations and also divide by $m!$. The above procedure results in the following infinite system of linear equations:

$$L\left[A_m(\tau) - \chi_m A_{m-1}(\tau)\right] = \hbar R_m\left(\tilde{A}_{m-1}\right), \quad (m=1,2,3,\cdots) \qquad (2.15)$$

subject to the initial condition

$$A_m(0) = 0, \quad (m=1,2,3,\cdots). \qquad (2.16)$$

In the Equation (2.15) we have used the following notation:

$$\chi_m = \begin{cases} 0 & \text{when } m \leq 1, \\ 1 & \text{when } m > 1, \end{cases} \qquad (2.17)$$

$$\tilde{A}_{m-1}(\tau) \equiv \left\{A_0(\tau), A_1(\tau), A_2(\tau), \cdots, A_{m-1}(\tau)\right\}, m \geq 1 \qquad (2.18)$$

and

$$R_m\left[\tilde{A}_{m-1}(\tau)\right] = \left[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\left\{N\left[\phi(\tau;p)\right]\right\}\right]_{p=0}, m \geq 1. \qquad (2.19)$$

On using the definition of $N\left[\phi(\tau;p)\right]$, from Equation (2.8), we see that Equation (2.19) results in the following equation:

$$R_m\left[\tilde{A}_{m-1}(\tau)\right] = \dot{A}_{m-1}(\tau) + Q_1\left[1+\delta\sin\Omega\tau\right]A_{m-1}(\tau)$$
$$-Q_2\sum_{k=0}^{m-1}A_{m-1-k}(\tau)\sum_{j=0}^{k}A_{k-j}(\tau)A_j(\tau), (m=1,2,3,\cdots). \qquad (2.20)$$

We now make an analysis of both the equation and the possible nature of its solution. Firstly, the to-be-obtained series solution must encompass the solution of Equation (2.1) for the limiting case $\delta=0$, as well as a periodic solution of the Equation (2.1) for small amplitudes, viz., $A_\tau = Q_1\left[1+\delta\sin\Omega\tau\right]A.$. Let the solution of the above limiting cases be denoted respectively by $A_E(\tau)$ and $A_\delta(\tau)$. These have the form:

$$A_E(\tau) = \sum_{m=0}^{\infty} b_m \, e^{-2mQ_1\tau} \qquad (2.21)$$

$$A_\delta(\tau) = \sum_{n=0}^{\infty}\left[\alpha_n \sin n\Omega\tau + \beta_n \cos n\Omega\tau\right], \qquad (2.22)$$

where the coefficients b_m, α_n and β_n are constants that may be functions of the parameters of the problem. The form of the solution (2.21) follows from the fact that $A_E(\tau)$ is essentially the exact solution (2.3) in a binomially expanded form. Secondly, after discussing about the solution of the limiting cases, we note that the solution $A_{NL}(\tau)$ of the full Equation (2.1) must be such that it has an oscillatory and decaying nature as $\tau \to \infty$. The solution $A_{NL}(\tau)$ must, in addition, have the two limiting solutions as part of its series solution. Let us now make a passing remark in regard to the choice of γ. The solution (2.21) does suggest that $\gamma=2Q_1$ but a discussion in favour of this choice by estimating an error is given just before the section on results and discussion.

At this point let us pause a bit and recollect what we have done and what is it that resulted out of it. In seeking the solution of the nonlinear differential equation by HAM we constructed a homotopy using an embedding parameter. The construction of homotopy and the procedure thereafter resulted in an infinite system of inhomogeneous linear equations with the nonlinear term being part of the inhomogeneity.

We now move on to solve the system of linear inhomogeneous equations, as many as are required by a pre-determined convergence criterion. At this juncture while getting ready to solve the system of equations, we notice that $\hbar$ waits to be defined. We will deal with the matter of choice of an appropriate value for $\hbar$ in what follows. As a prologue to what is to follow we may, however, add here that a good choice of $\hbar$ serves the purpose of con-

centrating a major part of the sum of the series (2.12) in its first few terms and thereby rendering the remaining terms vanishingly small. This, in essence, speaks of the convergence of the series (2.12) and its control through $\hbar$. We now move on to solve the system of linear Equation (2.15), with $R_m\left(\tilde{A}_{m-1}\right)$ given by Equation (2.20), and also discuss about the way an appropriate $\hbar$ can be chosen.

On using the definition (2.7) and denoting $A_m(\tau) - \chi_m A_{m-1}(\tau)$ by $B(\tau)$, Equation (2.15) may now be written as:

$$\frac{dB}{d\tau} + \gamma B = \hbar\, R_m\left(\tilde{A}_{m-1}\right). \qquad (2.23)$$

Equation (2.23) is an inhomogeneous linear differential equation that has an integrating factor $e^{\gamma\tau}$. Multiplying Equation (2.23) by $e^{\gamma\tau}$, the equation may be rearranged into the following form:

$$\frac{d}{d\tau}\left\{e^{\gamma\tau}B(\tau)\right\} = \hbar\, e^{\gamma\tau} R_m\left(\tilde{A}_{m-1}\right). \qquad (2.24)$$

Integrating the exact differential equation (2.24), we get

$$B(\tau) - B(0) = \hbar\, e^{-\gamma\tau}\int_0^\tau e^{-\gamma t} R_m\left(\tilde{A}_{m-1}\right)dt, \qquad (2.25)$$

where t is a dummy variable of integration. Reverting back from $B(\tau)$ to the original functions, we get on rearrangement the solution of the Equations (2.15) and (2.16) in the form:

$$A_m(\tau) = \chi_m A_{m-1}(\tau) + \hbar\, e^{-\gamma\tau}\int_0^\tau e^{\gamma t} R_m\left[\tilde{A}_m(t)\right]dt. \qquad (2.26)$$

By means of Mathematica, or such other packages, one may easily complete the integration in Equation (2.26). In fact, the solution (2.26) may be obtained using Mathematica itself and how this is done is demonstrated in the code attached. Thus, we see that it is easy to get the series solution of Equations (2.1) and (2.2) provided we fix the issues relating to $\hbar$ and γ, and also the choice of L.

3. Choice of $\hbar$, γ and L

At various stages in the paper this issue had to be insufficiently addressed and discussions on them had to be postponed to the point at which one is better equipped to handle the matter. The time now is just about right to discuss these matters and we move in that direction. We first take the issue of $\hbar$.

To determine an appropriate $\hbar$, we define a residual error in the form:

$$E_R(\hbar) = \frac{1}{\tau_0}\int_0^{\tau_0}\left[A_\tau - Q_1(1 + \delta\sin\Omega\tau)A + Q_2 A^3\right]^2 d\tau \qquad (3.1)$$

where τ_0 is long enough to capture the error in full. We have to choose an optimum value of $\hbar$, denoted as $\hbar_{\mathrm{opt}}$, that yields the solution $A_{NL}(\tau)$, for a given accuracy, with minimum possible residual error. In other words, $\hbar_{\mathrm{opt}}$ is corresponding to an extrema of the curve $E_R = E_R(\hbar)$. Thus $\hbar_{\mathrm{opt}}$ has to satisfy the following conditions:

$$\frac{dE_R}{d\hbar} = 0 \text{ and } \frac{d^2 E_R}{d\hbar^2} > 0. \qquad (3.2)$$

From the above discussion the rationale behind the introduction of $\hbar$ becomes clear. The most important thing is that $\hbar$ is a helpful parameter that influences convergence of the HAM series solution (2.13) but the convergent solution (the sum of the series 2.13) is independent of the choice of $\hbar$, as proved by Liao [4]. We now move on to the discussion on the choice of γ.

We know from the exact solution (2.3) for the limiting case $\delta = 0$ that the solution decays quicker than $e^{-\gamma\tau}$ as $\tau \to \infty$. By substituting the initial guess (2.6) into Equation (2.1) we can obtain a condition which on being satisfied ensures that the solution decays faster than $e^{-\gamma\tau}$ as $\tau \to \infty$. The initial guess (2.6) cannot obviously satisfy the equation (2.1) exactly and its introduction in the latter equation results in an error Δ that is given by:

$$\Delta = -(a_0 - a_1)(\gamma - 2Q_1)e^{-\gamma\tau} + 3(a_0 - a_1)^2 a_1 Q_2 e^{-2\gamma\tau} + \cdots \qquad (3.3)$$

The choice $\gamma = 2Q_1$ clearly allows Δ to decay faster than $e^{-\gamma\tau}$. In view of this we set $\gamma = 2Q_1$ in the paper.

The choice of an appropriate L involves issues related to the decision of taking the initial solution in a particular fashion. The initial solution $A_0(\tau)$ is chosen in such a way that its nature is akin to the nature of the HAM solution $A_{NL}(\tau)$. Thus the choice of L and the choice of $A_0(\tau)$ are inter-related. Now we dwell on another important related issue involving the method.

The most natural question that comes to the mind of a reader while going through the discussion on the chosen form of the deformation Equation (2.12) is the following:

Why not have the following equation in place of the deformation Equation (2.12)?

$$(1-p)L\left[\phi(\tau;p)\right] = \hbar\, p\, N\left[\phi(\tau;p)\right], \qquad (3.4)$$

where L can be taken to be the linear part of the given nonlinear equation. In the present problem L would in this case be

$$L \equiv \frac{\partial}{\partial\tau} - Q_1. \qquad (3.5)$$

The solution of $L[A(\tau)] = 0$, subject to condition (2.2), would then be given by equation (2.5). Such a choice of

L with the deformation equation given by equation (3.4) would also mean $A_0(\tau) = A_L(\tau)$. Following the procedure of the paper the reader may easily verify that such a choice of $A_0(\tau)$ will lead to a divergent solution! Thus the choice of $A_0(\tau)$, as well as L, is very important in arriving at a series solution with assured convergence by the HAM. This does not, however, mean that this should not work for other problems. This is the rationale behind the particular choice of $A_0(\tau)$ in the present problem and how that helps in homotopically arriving at the solution $A_{NL}(\tau)$. It is quite obvious from the above discussion that Equation (2.12) is, in fact, the right zeroth-order deformation equation for the present problem.

4. Results and Conclusion

Before we venture into the results and discussions, we make a small observation on the given equation. By applying suitable scaling we now show that it is possible to group Q_1, Q_2 and a_0 as a coefficient of the cubic nonlinearity. Dividing Equation (2.1) throughout by Q_1 and denoting $Q_1\tau$ by τ^*, and also $\frac{\Omega}{Q_1}$ by Ω^*, the equation reduces to

$$A_{\tau^*} = A\left[1 + \delta\sin\left(\Omega^*\tau^*\right)\right] - \frac{Q_2}{Q_1}A^3 \qquad (4.1)$$

The condition for solving Equation (4.1) continues to be Equation (2.2). Now replacing A by a_0A in Equations (4.1) and (2.2), we get on simplification the following equation:

$$A_{\tau^*} = A\left[1 + \delta\sin\left(\Omega^*\tau^*\right)\right] - \frac{a_0^2 Q_2}{Q_1}A^3, \qquad (4.2)$$

Subject to

$$A(0) = 1. \qquad (4.3)$$

This illustrates the fact that Equation (2.1) can be scaled to obtain a particular GL-equation. So without loss of generality, we consider the case of $Q_1 = 1, Q_2 = 4$ and $a_0 = 1$ in Equations (2.1) and (2.2) and illustrate the HAM for this case. With the above observation it goes without saying that the method applies to any real GL-equation with a time-periodic coefficient, the latter rendering the GL-equation non-autonomous.

We now introduce the following terminologies before going ahead with the discussion of the results in **Figures 1-4**:

Zeroeth-order HAM solution

$$A_{NL}^{(0)}(\tau) = A_0(\tau)$$

First-order HAM solution
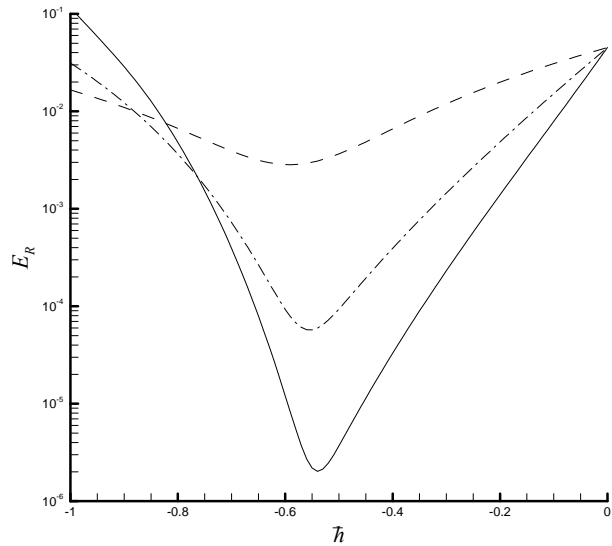
$$A_{NL}^{(1)}(\tau) = A_0(\tau) + A_1(\tau)$$

Second-order HAM solution

$$A_{NL}^{(2)}(\tau) = A_0(\tau) + A_1(\tau) + A_2(\tau)$$

And so on.

The solution of the given equation has been obtained in the following form by the HAM:

$$A_{NL}(\tau) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(c_{m,n}e^{in\Omega\tau} + \overline{c}_{m,n}e^{-in\Omega\tau}\right)e^{-m\gamma\tau}, \quad (4.4)$$

Figure 1. Curve of residual error E_R versus $\hbar$ for $Q_1=1, Q_2=4, a_0=1, \delta=0$. Broken line: $A_{NL}^{(1)}(\tau)$ Dash-dotted line: $A_{NL}^{(3)}(\tau)$ Solid line: $A_{NL}^{(5)}(\tau)$.
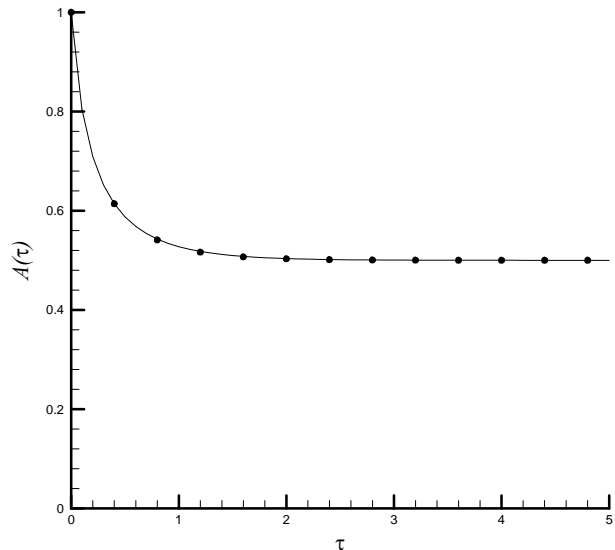
Figure 2. Comparison of the exact solution (2.3) with $A_{NL}^{(3)}(\tau)$ using $\hbar = -1/2$ for $Q_1=1, Q_2=4, a_0=1, \delta=0$. Solid line: exact solution (2.3) Symbols: $A_{NL}^{(3)}(\tau)$.
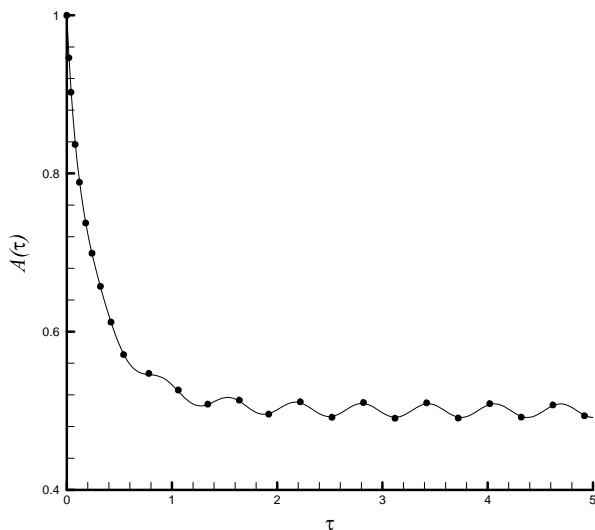
Figure 3. The HAM results using $\hbar = -1/2$ **for** $Q_1=1, Q_2=4, a_0=1, \delta=1/5, \Omega=10$. **Solid line** $A_{NL}^{(3)}(\tau)$; **Symbols** $A_{NL}^{(10)}(\tau)$.

Figure 4. The HAM results $A(\tau)$ **using** $\hbar = -1/2$ **for** $Q_1=1, Q_2=4, a_0=1, \delta=1/5$. **Line in blue:** $\Omega=10$; **Line in red:** $\Omega=50$.

where $i = \sqrt{-1}$. In order that the $A_0(\tau)$ expressions in equations (2.6) and (4.4) match, $c_{0,0}$ and $c_{1,0}$ must take the value a_1 and $(a_0 - a_1)$ respectively. Also, we con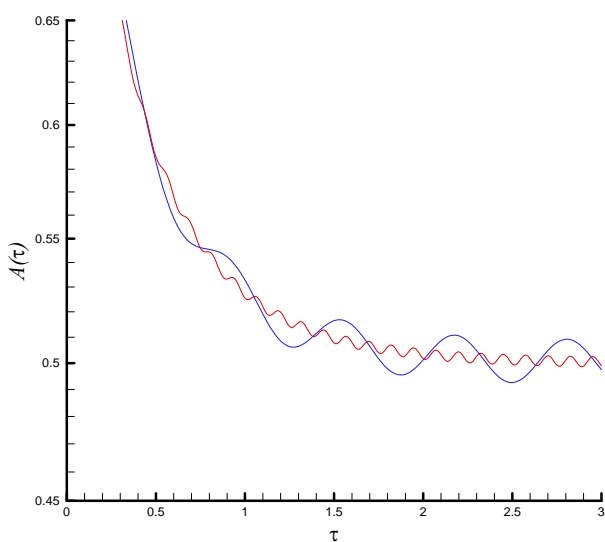firm that the obtained solution has the two solutions $A_E(\tau)$ and $A_\delta(\tau)$ as limiting cases. We now move on to the discussion of the results, firstly we discuss the unmodulated case and secondly the modulated case.

Figure 1 is a plot of the residual error $E_R(\hbar)$ versus

the auxillary parameter $\hbar$ for $\delta = 0$ (unmodulated case). In the discussion that follows the superscript 'n' in the residual error indicates that the nth-order HAM solution has been used in calculating the error. We need to observe the following from the figure and the same is computationally important in the HAM:

• When $\hbar = -1$, increasing the order results in increasing the residual error, thus the series is divergent. Besides, choosing any a value of $\hbar$ in the region $-1 \le \hbar \le -0.8$ results in divergent series. However, choosing any a value of $\hbar$ in the region $-0.6 \le \hbar < 0$ results in convergent series. Obviously, there exists such a region $\hbar_B \le \hbar < 0$, where $\hbar_B$ is a constant, that choosing any a value of $\hbar$ in this region results in convergent HAM series. It is unnecessary to determine the exact value $\hbar_B$ of the boundary. For example, from **Figure 1** it is obvious that the HAM series converge by choosing any a value of $\hbar$ in the region $-0.6 \le \hbar < 0$. Besides, as proved by Liao [4], all of these convergent HAM series converge to the same result for given physical parameters, although the convergence rate is dependent upon the choosing value of $\hbar$.

$\hbar_{opt}$, the value of $\hbar$ that gives the minimum residual error, depends on the order of the solution, the minimum residual error itself being different in each order. The **Table 1** illustrates this point.

We observe from the above table that increasing the order and choosing $\hbar_{opt}$ in each case results in decreasing the minimum error. This means that the series solution by HAM, to a desired accuracy, depends on not only the order but also on $\hbar$. This subtle point in the method is important for the computation. From the above table one might guess $\hbar_{opt} = -1/2$ for high-order approximations.

Computationally speaking, any value of $\hbar$ around its optimum value (such as $\hbar = -1/2$) gives us a convergent series solution with a fast convergence rate but after many more terms than that of the case of $\hbar = \hbar_{opt}$. It is on this reason that $\hbar$ in this paper has also been termed as the convergence-control parameter. The series of functions (4.4) has its terms that are continuous and differentiable and hence converges uniformly to the sum of the series and the control of its rate of convergence (a new feature of the method) is taken care of by $\hbar$. **Figure 2** is also for the unmodulated case using $\hbar = -1/2$. This

Table 1. The value of $\hbar_{opt}$ **for three different values of HAM.**

Order of the HAM (n)	Minimum Error $E_R^{(n)}(\hbar_{opt})$	$\hbar_{opt}$
1	2.84×10^{-3}	-0.59
3	5.72×10^{-5}	-0.55
5	2.02×10^{-6}	-0.54

shows the matching of the third-order HAM solution with the exact solution (2.3).

In the modulated case ($\delta \neq 0$), we use the same procedure for finding $\hbar_{opt}$ as that in the unmodulated case. **Figure 3** shows the matching of the results of the third-order and tenth-order HAM solutions using $\hbar = -1/2$, thereby implying that the third-order solution is good enough for the modulated case also. With an improper choice of $\hbar$ (including the value -1) this would not have been the situation. We would, in that case, have slowing down of convergence or even divergence as the case may be. **Figure 4** shows the convergent solution (using $\hbar = -1/2$) of the tenth-order HAM solution for two values of Ω. Clearly the effect of increasing Ω is to decrease the amplitude.

We observe in **Figures 1-4** that the obtained amplitudes $A_{NL}^{(k)}(\tau)$ are quite small. This can be offered an explanation as given in what follows. For large τ, one can get the asymptotic expression

$$A_0(\tau) \approx a_1 + \left(a_0 - a_1\right)e^{-\gamma\tau} + \alpha_1 \cos\Omega\tau + \beta_1 \sin\Omega\tau$$

From Equation (2.23). Substituting this into Equation (2.1), and neglecting the high-order terms, one gets the error E in the form:

$$E = \left(3Q_1\beta_1 - \Omega\beta_2 - \delta Q_1\right)\sin\Omega\tau + \left(\Omega\beta_1 + 3Q_1\beta_2\right)\cos\Omega\tau.$$

Setting $3Q_1\beta_1 - \Omega\beta_2 - \delta Q_1 = 0$ and $\Omega\beta_1 + 3Q_1\beta_2 = 0$, the amplitude of the harmonic terms turns out to be $\sqrt{\beta_1^2 + \beta_2^2} = \dfrac{\delta}{\sqrt{\Omega^2 / Q_1^2 + 9}}$. Since δ ranges from 0.01 to 0.2 and $\Omega \geq 10$, the amplitude is rather small, as shown in the four figures.

Figure 5 is a plot of the Nusselt number $Nu(\tau)$, defined in Equation (A25) of the appendix, versus slow time τ for two different values of Ω. Clearly the effect of increasing frequency is to decrease the magnitude of the Nusselt number, thereby also meaning that heat transfer in a Rayleigh-Bénard convective system can be regulated using vibration of the system in the vertical direction, viz, gravity modulation. A physical problem involving a nonlinear differential equation with a time-periodic coefficient has been solved in series-form using the HAM and it is possible to do the same with other types of GL-equations as discussed in the introduction to the paper. The Mathematica program that implements the HAM in this paper is recorded in appendix B.

5. Conclusions

The series solution by HAM is a good alternative to the numerical solution of the Ginzburg-Landau equation
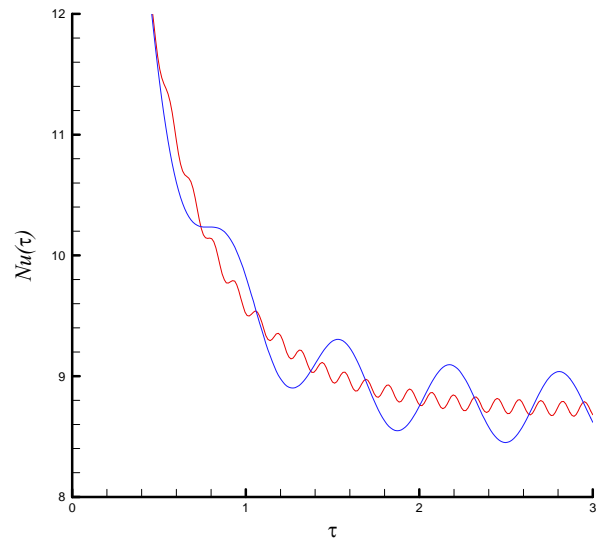
Figure 5. The HAM results $Nu(\tau)$ **using** $\hbar = -1/2$ **for** $Q_1 = 1$, $Q_2 = 4$, $a_0 = 1$, $\delta = 1/5$. **Line in blue:** $\Omega = 10$; **Line in red:** $\Omega = 50$.

with a time-periodic coefficient, as so it is for many other non-linear differential equations—both ordinary and partial (see [2-11]).

6. Acknowledgements

7. References

[1] S. J. Liao, "On the Proposed Homotopy Analysis Techniques for Nonlinear Problems and its Application," Ph. D. Thesis, Shanghai Jiao Tong University, Shanghai, 1992.

[2] S. J. Liao, "Homotopy Analysis Method: A New Analytical Technique for Non-Linear Problems," *Communications in Non-linear Science & Numerical Simulation*, Vol. 2, No. 2, 1997, pp 95-100.

[3] S. J. Liao, "An Explicit, Totally Analytic Approximate Solution for Blasius' Viscous Flow Problems," *International Journal of Non-Linear Mechanics*, Vol. 34, No. 4, 1999, pp. 759-778.

[4] S. J. Liao, "Beyond Perturbation-Introduction to the Homotopy Analysis Method," CRC Press, London, 2003.

[5] S. Liao and Y. Tan, "A General Approach to Obtain Series Solutions of Nonlinear Differential Equations," *Studies in Applied Mathematics*, Vol. 119, No. 4, 2007, pp. 297-354.

[6] S. Abbasbandy, "The Application of the Homotopy Analysis Method to Nonlinear Equations Arising in Heat Transfer," *Physics Letters A*, Vol. 360, No. 1, 2006, pp. 109-113.

[7] S. Abbasbandy, "The Application of Homotopy Analysis Method to Solve a Generalized Hirota-Satsuma Coupled KdV Equation," *Physics Letters A*, Vol. 361, No. 6, 2007 pp. 478-483.

[8] T. Hayat and M. Sajid, "On Analytic Solution for Thin Film Flow of a Fourth Grade Fluid Down a Vertical Cylinder," *Physics Letter A*, Vol. 361, No. 4-5, 2007, pp. 316-322.

[9] M. Sajid, T. Hayat and S. Asghar, "Comparison Between the HAM and HPM Solutions of Thin Film Flows of Non-Newtonian Fluids on a Moving Belt," *Nonlinear Dynamics*, Vol. 50, No. 1-2, 2007, pp. 27-35.

[10] K. Yabushita, M. Yamashita and K. Tsuboi, "An Analytic Solution of Projectile Motion with the Quadratic Resis-tance Law Using the Homotopy Analysis Method," *Journal of Physics A*: *Mathematical and Theoretical*, Vol. 40, 2007, pp. 8403-8416.

[11] F. M. Allan, "Derivation of the Adomian Decomposition Method Using the Homotopy Analysis Method," *Applied Mathematics and Computation*, Vol. 190, No. 1, 2007, pp. 6-14.

[12] A. Alizadeh-Pahlavan, V. Aliakbar, F. Vakili-Farahani, and K. Sadeghy, "MHD Flows of UCM Fluids above Porous Stretching Sheets Using Two-Auxiliary-Parameter Homotopy Analysis Method," *Communication in Nonli-near Science and Numerical Simulation*, Vol. 14, No. 2, 2009, pp. 473-488.

[13] V. Marinca, N. Herisanu, and L. Nemes, "Optimal Ho-motopy Asymptotic Method with Application to Thin Film Flow," *Central European Journal of Physics*, Vol. 6, No. 3, 2008, pp. 648-653.

APPENDIX A

Derivation of the Time-Periodic Ginzburg-Landau Equation

Consider the classical Rayleigh-Bénard problem with gravity modulation (or g-jitter as it is called). The conservation of mass, linear momentum and energy in the problem are respectively governed by:

$$\nabla \cdot \vec{q} = 0 \tag{A1}$$

$$\rho_0 \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = -\nabla p - \rho\left(g + g'\right)\hat{k} + \mu \nabla_2 \vec{q} \tag{A2}$$

$$\frac{\partial T}{\partial t} + \left(\vec{q} \cdot \nabla\right)T = \chi \nabla^2 T \tag{A3}$$

where $\vec{q}$ is the velocity, ρ_0 is the density at the reference temperature T_0 (temperature of the upper plate), p is the pressure, ρ is the density, g is the acceleration due to gravity, g' is the time-dependent gravity modulation due to the vibration of the Rayleigh-Bénard setup, μ is the dynamic coefficient of viscosity, T is the temperature and χ is the thermal diffusivity of the liquid. The equation of state is given by

$$\rho = \rho_0 \left\{1 - \alpha\left(T - T_0\right)\right\} \tag{A4}$$

where α is the coefficient of thermal expansion.

Let us consider the two-dimensional convection in the x-y plane with velocity components in the x and y directions denoted by u and v respectively. Now eliminating the pressure p between the x- and y-components of the linear momentum equation, introducing the stream function ψ and using the equation (A4) in the resulting equation, we get

$$\rho_0 \frac{\partial}{\partial t}\left(\nabla^2 \psi\right) = \alpha g \frac{\partial T}{\partial x} + \left(1 + \frac{g'(t)}{g}\right)\mu g \nabla^4 \psi + \rho_0 \frac{\partial\left(\psi, \nabla^2 \psi\right)}{\partial(x, y)} \tag{A5}$$

Introducing ψ into the equation (A3), we get

$$\left(\frac{\partial}{\partial \tau} - \chi \nabla^2\right)T + \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} = 0, \tag{A6}$$

The equations (A5) and (A6) are rendered dimensionless using the following scaling:

Space coordinates $\qquad\qquad d$

Time $\qquad\qquad\qquad\qquad d^2\big/\chi$

Temperature $\qquad\qquad \Delta T$

where d is the vertical distance between the parallel lower and upper bounding plates of infinite horizontal extent

and ΔT is the temperature difference between the lower hot plate and the upper cold plate. Using the above scaling, equations (A5) and (A6) turn out to be:

$$\frac{1}{Pr}\frac{\partial}{\partial t}\left(\nabla^2 \psi\right) = -\left(1 + gm\right)R\frac{\partial T}{\partial x} + \nabla^4 \psi + \frac{1}{Pr}\frac{\partial\left(\psi, \nabla^2 \psi\right)}{\partial(x, y)} \tag{A7}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T = -\frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x}, \tag{A8}$$

where

$$gm = \frac{g'(t)}{g}. \tag{A9}$$

The boundary condition to solve equations (A7)-(A8) in dimensionless form is:

$$\begin{aligned}\psi = \nabla^2 \psi = 0 \quad &\text{and} \quad T = 1 \quad \text{on} \quad y = 0,\\ \psi = \nabla^2 \psi = 0 \quad &\text{and} \quad T = 0 \quad \text{on} \quad y = 1.\end{aligned} \tag{A10}$$

The conduction profile is

$$\psi = 0, \quad \text{and} \quad T = 1 - y \tag{A11}$$

Now we impose finite-amplitude perturbations on the basic quiescent state (A11) as

$$\psi = \Psi(x, y, t), \ T = 1 - y + \Theta(x, y, t). \tag{A12}$$

$$\begin{aligned}\frac{1}{Pr}&\left[\frac{\partial}{\partial t}\left(\nabla^2 \Psi\right) + \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2 \Psi\right) - \frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2 \Psi\right)\right]\\ &= R\left[1 + gm\right]\frac{\partial \theta}{\partial x} + \nabla^4 \Psi\end{aligned} \tag{A13}$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{\partial \Theta}{\partial y} - \frac{\partial \Psi}{\partial y}\frac{\partial \Theta}{\partial x} - \frac{\partial \Psi}{\partial x} = \nabla^2 \Theta \tag{A14}$$

where Pr and R are respectively the Prandtl and Rayleigh numbers.

The boundary condition (A10) for the perturbations reads as:

$$\begin{aligned}\Psi = 0 = \nabla^2 \Psi \quad &\text{and} \quad \Theta = 1 \quad on \quad y = 0,\\ \Psi = 0 = \nabla^2 \Psi \quad &\text{and} \quad \Theta = 0 \quad on \quad y = 1.,\end{aligned} \tag{A15}$$

We now use the following asymptotic expansion in equations (A13) to (A15):

$$\left.\begin{aligned}R &= R_0 + \varepsilon^2 R_2 + \cdots\\ \Psi &= \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \Psi_3 + \cdots\\ \Theta &= \varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \cdots\end{aligned}\right\} \tag{A16}$$

where R_0 is the critical value of the Rayleigh number at which convection sets in when gravity modulation is absent.

We use the time variations only at the slow time scale $\tau = \varepsilon^2 t$ and $gm(\tau)$ is taken to be:

$$gm(\tau) = \varepsilon^2 \delta_2 \sin(\Omega \tau). \tag{A17}$$

At the lowest order, we have

$$\begin{bmatrix} \nabla^4 & R_0 \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial x} & \nabla^2 \end{bmatrix} \begin{bmatrix} \Psi_1 \\[2mm] \Theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\[2mm] 0 \end{bmatrix} \qquad (A18)$$

The solution of the lowest order system is

$$\left. \begin{aligned} \Psi_1 &= A(\tau) \sin k_c x \, \sin \pi y \\[2mm] \Theta_1 &= \frac{k_c}{\delta^2} A(\tau) \cos k_c x \, \sin \pi y \end{aligned} \right\} \qquad (A19)$$

The system (A18) gives us the critical value of the Rayleigh number and the wave number for stationary onset and their expressions are given below:

$$R_0 = \frac{27\pi^4}{4}, \quad k_c = \frac{\pi}{\sqrt{2}}. \qquad (A20)$$

Amplitude Equation (*Ginzburg-Landau equation*) and Heat Transport for Stationary Instability

At the second order, we have

$$\begin{bmatrix} \nabla^4 & R_0 \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial x} & \nabla^2 \end{bmatrix} \begin{bmatrix} \Psi_2 \\[2mm] \Theta_2 \end{bmatrix} = \begin{bmatrix} \Re_{21} \\[2mm] \Re_{22} \end{bmatrix} \qquad (A21)$$

where

$$\left. \begin{aligned} \Re_{21} &= 0 \\[2mm] \Re_{22} &= \frac{\partial \Psi_1}{\partial x}\frac{\partial \Theta_1}{\partial y} - \frac{\partial \Psi_1}{\partial y}\frac{\partial \Theta_1}{\partial x} \end{aligned} \right\}, \qquad (A22)$$

The second order solution can be obtained as:

$$\left. \begin{aligned} \Psi_2 &= 0, \\[2mm] \Theta_2 &= -\frac{k_c^2 A^2}{8\pi\delta^2} \sin 2\pi y, \end{aligned} \right\}. \qquad (A23)$$

The horizontally-averaged Nusselt number, Nu, for the stationary mode of convection(the preferred mode in this problem) is given by

$$Nu(\tau) = \frac{\left[\dfrac{k_c}{2\pi} \displaystyle\int_{x=0}^{2\pi/k_C} (1 - y + \Theta_2)_y \, dx \right]_{y=0}}{\left[\dfrac{k_c}{2\pi} \displaystyle\int_{\tau=0}^{2\pi/k_C} (1 - y)_y \, dx \right]_{y=0}} \qquad (A24)$$

Substituting equation (A23) in equation (A24) and completing the integration, we get

$$Nu(\tau) = 1 + \frac{k_c^2 \left[A(\tau) \right]^2}{4\delta^2}. \qquad (A25)$$

At the third order, we have

$$\begin{bmatrix} \nabla^4 & R_0 \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial x} & \nabla^2 \end{bmatrix} \begin{bmatrix} \Psi_3 \\[2mm] \Theta_3 \end{bmatrix} = \begin{bmatrix} \Re_{31} \\[2mm] \Re_{32} \end{bmatrix}, \qquad (A26)$$

where

$$\Re_{31} = R_0 \left[\frac{R_2}{R_0} + \delta_2 \sin(\Omega\tau) \right] \frac{\partial \Theta_1}{\partial x} + \frac{1}{Pr}\frac{\partial(\nabla^2 \Psi_1)}{\partial \tau} \qquad (A27)$$

$$\Re_{32} = -\frac{\partial \Theta_1}{\partial \tau} - \frac{\partial \Psi_1}{\partial x}\frac{\partial \Theta_2}{\partial y} \qquad (A28)$$

Substituting Θ_1, Ψ_1 and Θ_2 from equations(A19) and (A23) into equations (A27) and (A28), we get

$$\Re_{31} = -\frac{k_c^2 A}{\delta^2} R_0 \left[\frac{R_2}{R_0} + \delta \sin(\Omega\tau) \right] \sin k_c x \, \sin \pi y \qquad (A29)$$

$$- \frac{\delta^2}{Pr} A_\tau \sin k_c x \sin \pi y$$

$$\Re_{32} =$$
$$-\frac{k_c^3 A^3}{4\delta^2} \cos k_c x \left[\sin 3\pi y - \sin \pi y \right] + \frac{k_c}{\delta^2} A_\tau \cos k_c x \, \sin \pi y \qquad (A30)$$

The *solvability condition* for the existence of the third order solution is:

$$\int_{y=0}^{1} \int_{x=0}^{2\pi/k_c} \left[\hat{\Psi} \, \Re_{31} + R_0 \, \hat{\Theta} \, \Re_{32} \right] dx \, dy = 0, \qquad (A31)$$

where

$$\left. \begin{aligned} \hat{\Psi} &= A(\tau) \sin k_c x \, \sin \pi y, \\[2mm] \hat{\Theta} &= \frac{k_c}{\delta^2} A(\tau) \cos k_c x \, \sin \pi y. \end{aligned} \right\} \qquad (A32)$$

From equation (A31), on substituting equations (A29), (A30) and (A32) into the equation and completing the integration, we get the Ginzburg-Landau equation for stationary instability with a time-periodic coefficient in the form:

$$\frac{dA}{d\tau} = Q_1 \left[\frac{R_2}{R_0} + \delta_2 \sin(\Omega\tau) \right] A - Q_2 A^3, \qquad (A33)$$

where

$$Q_1 = \frac{k_c^2 Pr R_0}{\delta^4 [Pr - 1]} \text{ and } Q_2 = \frac{k_c^2 Pr}{4[Pr - 1]}. \qquad (A34)$$

The main paper deals with the solution of equation (A33) using the HAM, subject to the initial condition $A(0) = a_0$, where a_0 is a chosen initial amplitude of convection. In calculations we may assume $R_2 = R_0$ to keep the parameters to the minimum.

APPENDIX B

Mathematica Program for the Problem

```
(********************************************************************)
(                   *Computer-Assisted HAM Series Solution of                    *)
(*               Ginzburg-Landau equation (2.1) (with g-jitter)                   *)
(*                    subject to initial condition (2.2)                          *)
(********************************************************************)
(*                                                                                *)
(*            A'-Q1*[1+delta*sin(omega*tau)]*A+Q2*A^3 = 0, A(0)=a0                 *)
(*                                                                                *)
(*                          November 16, 2009                                     *)
(********************************************************************)
(********************************************************************)
(*                    Define GetR[k] using equation (2.20)                        *)
(********************************************************************)
GetR[k_]:=Module[{temp,j,n,sint},
sint = (Exp[omega*t*I]-Exp[-omega*t*I])/2/I;
temp[1] = Sum[a[n]*aa[k-1-n],{n,0,k-1}];
temp[2] = at[k-1]-Q1*(1+delta*sint)*a[k-1]+Q2*temp[1];
R[k] = Expand[temp[2]];
];


(********************************************************************)
(*                    Set initial guess A[0] using equation (2.6)                 *)
(********************************************************************)
a1 = Sqrt[Q1/Q2];
a[0] = a1 + (a0-a1)*Exp[-gamma*t];
A[0] = a[0];
At[0] = D[U[0],t];


(********************************************************************)
(*                 Define the function chi_{k} using equation (2.17)              *)
(********************************************************************)
chi[k_]:=If[k<=1,0,1];



(********************************************************************)
(*       Define derivatives of a(t) and an intermediate sum required in equation (2.20)   *)
(********************************************************************)
GetAll[k_]:=Module[{temp},
at[k] = D[a[k],t]//Expand;
aa[k] = Sum[a[n]*a[k-n],{n,0,k}]//Expand;
];

(********************************************************************)
(*                                Define L                                        *)
(********************************************************************)
L[f_]:= D[f,t] + gamma*f;

(********************************************************************)
(*                           MOST IMPORTANT!!                                     *)
(* The code works best if the solution of equation (2.15) is obtained by Mathematica! It saves on time *)
```

```
(* by doing this way and hence this procedure is adopted in the program in the following three *)
(* segments of the code. On a similar reason we have not used the trigonometric sine function directly *)
(*                                  in the code                                                       *)
(****************************************************************************************)
(****************************************************************************************)
(*                            Define the inverse of L                                 *)
(****************************************************************************************)
Linv[f_]:=Block[{temp,g,G,s},
temp = DSolve[L[g[t]] == f,g[t],t]/.C[_]->0;
G = g[t]/.temp;
s = G[[1]]//Expand;
Expand[s]
];


(****************************************************************************************)
(*           Define the linearity of inverse of L, i.e., Linv[f_+g_]  := Linv[f] + Linv[g];    *)
(****************************************************************************************)
Linv[p_Plus] := Map[Linv,p];
Linv[c_*f_] := c*Linv[f] /; FreeQ[c,t];


(****************************************************************************************)
(*                     Define the residual error using equation (3.1)                 *)
(****************************************************************************************)
GetErr[k_,tmax_]:=Module[{temp,sint},
sint = (Exp[omega*t*I]-Exp[-omega*t*I])/2/I;
error[k] = At[k] - Q1*(1+delta*sint)*A[k] + Q2*A[k]^3//Expand;
err[k] = Integrate[error[k]^2,{t,0,tmax}]/tmax;
];


(****************************************************************************************)
(*                                  Main Code                                         *)
(****************************************************************************************)
ham[m0_,m1_]:=Module[{temp,k,n,C1},
For[k=Max[1,m0],k<=m1,k=k+1,
    Print[" k = ",k];
    GetAll[k-1];
GetR[k];
RHS[k] = hbar*R[k]//Expand;
    Special = Linv[RHS[k]];
    temp = Special + chi[k]*a[k-1];
C1 = -temp/.t->0;
a[k] = temp + C1*Exp[-gamma*t]//Expand;
    A[k] = A[k-1] + a[k]//Expand;
At[k] = D[A[k],t];
If[PRN == 1,
GetErr[k,tmax];
Print[" error = ", err[k]//N];
];
];
Print["Successful !"];
];
exact = 1/Sqrt[Q2/Q1-(Q2/Q1-1/a0^2)*Exp[-2*Q1*t]];
(* Input parameters   *)
a0 = 1;
Q1 = 1;
Q2 = 4;
```

```
delta = 1/5;
omega = 10;
hbar = -1/2;
gamma = 2*Q1;
PRN = 0;
tmax = 10;
(* Print out the input parameters *)
Print[" a0 = ",a0];
Print[" Q1 = ",Q1];
Print[" Q2 = ",Q2];
Print[" delta = ",delta];
Print[" omega = ",omega];
Print[" hbar = ",hbar];
(* get fifth-order approximation *)
ham[1,5];
```

AM