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Continuous Maps on Digital Simple Closed Curves

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Abstract

We give digital analogues of classical theorems of topology for continuous functions defined on spheres, for digital simple closed curves. In particular, we show the following: 1) A digital simple closed curve S of more than 4 points is not contractible, *i.e.*, its identity map is not nullhomotopic in S; 2) Let X and Y be digital simple closed curves, each symmetric with respect to the origin, such that $|Y|>5$ (where $|Y|$ is the number of points in Y). Let $f: X \to Y$ be a digitally continuous antipodal map. Then f is not nullhomotopic in Y; 3) Let S be a digital simple closed curve that is symmetric with respect to the origin. Let $f: S \to Z$ be a digitally continuous map. Then there is a pair of antipodes $\{x,-x\} \subset S$ such that $|f(x)-f(-x)| \leq 1$.

Keywords: Digital Image, Digital Topology, Homotopy, Antipodal Point

1. Introduction

A digital image is a set X of lattice points that model a "continuous object" Y, where Y is a subset of a Euclidean space. Digital topology is concerned with developing a mathematical theory of such discrete objects so that, as much as possible, digital images have topological properties that mirror those of the Euclidean objects they model; however, in digital topology we view a digital image as a graph, rather than, e.g., a metric space, as the latter would, for a finite digital image, result in a discrete topological space. Therefore, the reader is reminded that in digital topology, our "nearness" notion is the graphical notion of adjacency, rather than a neighborhood system as in classical topology; usually, we use one of the natural c_l-adjacencies (see Section 2). Early papers in the field, e.g., [1-8], noted that this notion of nearness allows us to express notions borrowed from classical topology, e.g., connectedness, continuous function, homotopy, and fundamental group, such that these often mirror their analogs with respect to Euclidean objects modeled by respective digital images. Applications of digital topology have been found shape description and in image processing operations such as thinning and skeletonization [9].

A. Rosenfeld wrote the following: "The discrete grid (of pixels or voxels) used in digital topology can be regarded as a 'digitization' of (two or three-dimensional) Euclidean space; from this viewpoint, it is of interest to study conditions under which this digitization process preserves topological (or other geometric) properties" [3].

In this spirit, we obtain in this paper several properties of continuous maps on digital simple closed curves, inspired by analogs for Euclidean simple closed curves. In particular, we show that digital simple closed curves of more than 4 points are not contractible, and we obtain several results for continuous maps and antipodal points on digital simple closed curves.

2. Preliminaries

Let Z be the set of integers. Then Z^d is the set of lattice points in d-dimensional Euclidean space. Let $X \subset Z^d$ and let κ be some adjacency relation for the members of X. Then the pair (X, κ) is said to be a (binary) digital image. A variety of adjacency relations are used in the study of digital images. Well known adjacencies include the following.

For a positive integer l with $1 \leq l \leq d$ and two distinct points $p=(p_1, p_2, \ldots, p_d), q=(q_1, q_2, \ldots, q_d) \in Z^d$, p and q are c_l*-adjacent* [10] if

- there are at most l indices i such that $|p_i - q_i| = 1$, and
- for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$. See **Figure 1**.

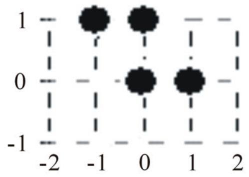

Figure 1. In Z^2, each of the points $p_1 = (1,0)$ and $p_2 = (0,1)$ is both c_1-adjacent and c_2-adjacent to $p_0 = (0,0)$, since both p_1 and p_2 differ from p_0 by 1 in exactly one coordinate and coincide in the other coordinate; $p_3 = (-1,1)$ is c_2-adjacent, but not c_1-adjacent, to p_0, since p_0 and p_3 differ by 1 in both coordinates.

The notation c_l is sometimes also understood as the number of points $q \in Z^d$ that are c_l-adjacent to a given point $p \in Z^d$. Thus, in Z we have $c_1 = 2$; in Z^2 we have $c_1 = 4$ and $c_2 = 8$; in Z^3 we have $c_1 = 6, c_2 = 18,$ and $c_3 = 26$.

More general adjacency relations are studied in [11]. Let κ be an adjacency relation defined on Z^d. A κ-*neighbor* of a lattice point p is κ-adjacent to p. A digital image $X \subset Z^d$ is κ-*connected* [11] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i \in \{0,1,\ldots,r-1\}$.

Let $a, b \in Z$ with $a < b$. A *digital interval* [7] is a set of the form

$$[a,b]_Z = \{z \in Z | a \le z \le b\}.$$

Let $X \subset Z^{d_0}$ and $Y \subset Z^{d_1}$ be digital images with κ_0-adjacency and κ_1-adjacency respectively. A function $f : X \to Y$ is said to be (κ_0, κ_1)-*continuous* [2,8], if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous.

Proposition 2.1 [2,8] *Let $X \subset Z^{d_0}$ and $Y \subset Z^{d_1}$ be digital images with κ_0-adjacency and κ_1-adjacency respectively. Then the function $f : X \to Y$ is (κ_0, κ_1)-continuous if and only if for every κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y.*

This characterization of continuity is what is called an *immersion*, *gradually varied operator*, or *gradually varied mapping* in [12,13].

Given digital images (X_i, κ_i), $i \in \{0,1\}$, suppose there is a (κ_0, κ_1)-continuous bijection $f : X_0 \to X_1$ such that $f^{-1} : X_1 \to X_0$ is (κ_1, κ_0)-continuous. We say X_0 and X_1 are (κ_0, κ_1)-*isomorphic* [14] (this was called (κ_0, κ_1)-*homeomorphic* in [7]) and f is a (κ_0, κ_1)-isomorphism (respectively, a (κ_0, κ_1)-*homeomorphism*).

By a digital κ-path from x to y in a digital image X, we mean a $(2, \kappa)$-continuous function $f : [0,m]_Z \to X$ such that $f(0) = x$ and $f(m) = y$. We say m is the *length* of this path. A simple closed κ-curve of $m \ge 4$ points (for some adjacencies, the minimal value of m may be greater than 4; see below) in a digital image X is a sequence $\{f(0), f(1), \ldots, f(m-1)\}$ of images of the κ-path $f : [0, m-1]_Z \to X$ such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = (i \pm 1) \bmod m$. If $S = \{x_i\}_{i=0}^{m-1}$ where $x_i = f(i)$ for all $i \in [0, m-1]_Z$, we say the points of S are *circularly ordered*.

Digital simple closed curves are often examples of digital images $X \subset Z^d$ for which it is desirable to consider $Z^d \setminus X$ as a digital image with some adjacency (not necessarily the same adjacency as used by X). For example, by analogy with Euclidean topology, it is desirable that a digital simple closed curve $X \subset Z^2$ satisfy the "Jordan curve property" of separating Z^2 into two connected components (one "inside" and the other "outside" X). An example that fails to satisfy this property [10] if we allow $|X| = 4$ is given by (X, c_1), where

$$X = \{(0,0),(1,0),(1,1),(0,1)\}$$

is circularly ordered, and $(Z^2 \setminus X, c_2)$ is c_2-connected. However, this anomaly is essentially due to the "smallness" of X as a simple closed curve; it is known [1,15,16] that for $m_1 = 8$ and $m_2 = 4$, if $X \subset Z^2$ is a digital simple closed c_i-curve such that $|X| \ge m_i$, then $Z^2 \setminus X$ has exactly 2 c_{3-i}-connected components, $i \in \{1,2\}$. Thus, it is customary to require that a digital simple closed curve $X \subset Z^2$ satisfy $|X| \ge 8$ when c_1-adjacency is used; $|X| \ge 4$ when c_2-adjacency is used.

Let $X \subset Z^{d_0}$ and $Y \subset Z^{d_1}$ be digital images with κ_0-adjacency and κ_1-adjacency respectively. Two (κ_0, κ_1)-continuous functions $f, g : X \to Y$ are said to be *digitally* (κ_0, κ_1)-*homotopic* in Y [8] if there is a positive integer m and a function $H : X \times [0,m]_Z \to Y$ such that

- for all $x \in X$, $H(x,0) = f(x)$ and $H(x,m) = g(x)$;
- for all $x \in X$, the induced function $H_x : [0,m]_Z \to Y$ defined by

$H_x(t) = H(x,t)$ *for all* $t \in [0,m]_Z$,

is $(2, \kappa_1)$-continuous; and

- for all $t \in [0,m]_Z$, the induced function $H_t : X \to Y$ defined by

$H_t(x) = H(x,t)$ *for all* $x \in X$,

is (κ_0, κ_1)-continuous.

We say that the function H is a *digital* (κ_0, κ_1)-*homotopy* between f and g.

Additional terminology associated with homotopic maps includes the following.

• If g is a constant map, we say f is (κ_0,κ_1)-*nullhomotopic* [8]; if, further, $(X,\kappa_0)=(Y,\kappa_1)$ and $f=1_X$ (the identity map on X), we say X is κ_0-*contractible* [7,17].

• If $H(x_0,t)=f(x_0)$ for some $x_0\in X$ and all $t\in[0,m]_Z$, we say H is a (κ_0,κ_1)*-pointed homotopy* [18].

• If $H:[0,m_0]_Z\times[0,m_1]_Z\to Y$ is a (c_1,κ)-homotopy between (c_1,κ)-continuous functions $f,g:[0,m_0]_Z\to Y$ such that for all $t\in[0,m_1]_Z$, $H(0,t)=f(0)=g(0)$ and $H(m_0,t)=f(m_0)=g(m_0)$, we say H *holds the endpoints fixed* [18].

Proposition 2.2 [8] *Suppose* $f_0,f_1:X\to Y$ *are* (κ,λ) *-continuous and* (κ,λ) *-homotopic. Suppose* $g_0,g_1:Y\to Z$ *are* (λ,μ)*-continuous and* (λ,μ)*-homotopic. Then* $g_0\circ f_0$ *and* $g_1\circ f_1$ *are* (κ,μ) *-homotopic in* Z.

3. Homotopy Properties of Digital Simple Closed Curves

A classical theorem of Euclidean topology, due to L.E.J. Brouwer, states that a d-dimensional sphere S^d is not contractible [19]. Theorem 3.3, below, is a digital analog, for $d=1$. We also present some related results in this section.

Proposition 3.1 *Let* S_a *be a digital simple closed* κ_a*-curve,* $a\in\{0,1\}$. *Let* $f:S_0\to S_1$ *be a* (κ_0,κ_1)*-continuous function. If* $|S_0|=|S_1|$, *then the following are equivalent.*

a) f is one-to-one.

b) f is onto.

c) f is a (κ_0,κ_1)-isomorphism.

Proof: Since $|S_0|=|S_1|$, the equivalence of a) and b) follows from the fact that S_0 is a finite set. That c) implies both a) and b) follows from the definition of isomorphism. Therefore, we can complete the proof by showing that b) implies c).

Let $S_a=\{x_{a,i}\}_{i=0}^{n-1}$, where the points of S_a are circularly ordered, $a\in\{0,1\}$. Let $x_{1,u}\in S_1$ and let $x_{0,v}=f^{-1}(x_{1,u})$. Then the κ_1-neighbors of $x_{1,u}$ in S_1 are $x_{1,(u-1)\bmod n}$ and $x_{1,(u+1)\bmod n}$, and the κ_0-neighbors of $x_{0,v}$ in S_0 are $x_{0,(v-1)\bmod n}$ and $x_{0,(v+1)\bmod n}$. Since f is a continuous bijection, our choice of $x_{0,v}$ implies

$$f\left(\{x_{0,(v-1)\bmod n},x_{0,(v+1)\bmod n}\}\right)=\{x_{1,(u-1)\bmod n},x_{1,(u+1)\bmod n}\}.$$

Thus,

$$f^{-1}\left(\{x_{1,(u-1)\bmod n},x_{1,(u+1)\bmod n}\}\right)=\{x_{0,(v-1)\bmod n},x_{0,(v+1)\bmod n}\}.$$

Since u was taken as an arbitrary index, f^{-1} is (κ_1,κ_0)-continuous, so f is a (κ_0,κ_1)-isomorphism.

Theorem 3.2 *Let* S *be a simple closed* κ*-curve and let* $H:S\times[0,m]_Z\to S$ *be a* (κ,κ) *-homotopy between an isomorphism* H_0 *and* $H_m=f$, *where* $f(S)\neq S$. *Then* $|S|=4$.

Proof: Let $S=\{x_i\}_{i=0}^{n-1}$, where the points of S are circularly ordered.

There exists $w\in[1,m]_Z$ such that

$$w=\min\{t\in[0,m]_Z|H_t(S)\neq S\}.$$

Without loss of generality, $x_1\notin H_w(S)$. Then the induced function H_{w-1} is a bijection, so there exists $x_u\in S$ such that $H(x_u,w-1)=x_1$. By Proposition 3.1, $H_{w-1}(\{x_{(u-1)\bmod n},x_{(u+1)\bmod n}\})=\{x_0,x_2\}$, and the continuity property of homotopy implies $H(x_u,w)\in\{x_0,x_2\}$. Without loss of generality,

$$H\left(x_{(u-1)\bmod n},w-1\right)=x_0 \tag{1}$$

and

$$H\left(x_u,w\right)=x_2. \tag{2}$$

Suppose $n>4$. Equation (2) implies $H(x_{(u-1)\bmod n},w)\in\{x_1,x_2,x_3\}$, but this is impossible, for the following reasons.

• $H(x_{(u-1)\bmod n},w)\neq x_1$, by choice of x_1.

• $H(x_{(u-1)\bmod n},w)\notin\{x_2,x_3\}$, from Equation (1), because $n>4$ implies neither x_2 nor x_3 is κ-adjacent to x_0.

The contradiction arose from the assumption that $n>4$. Therefore, we must have $n\leq 4$. Since a digital simple closed curve is assumed to have at least 4 points, we must have $n=4$.

In [8], an example is given of a simple closed c_2-curve $S\subset Z^2$ such that $Z^2\setminus S$ has 2 c_1-connected components, with $|S|=4$, such that S is c_2-contractible (see **Figure 2**). By contrast, we have the following.

Theorem 3.3 *Let* (S,κ) *be a simple closed* κ*-curve such that* $|S|>4$. *Then* S *is not* κ*-contractible.*

Proof: It follows from Theorem 3.2 that if $|S|>4$, then there cannot be a (κ,κ)-homotopy between 1_S and a constant map in S.

It is natural to ask whether we can obtain an analog of Theorem 3.3 for higher dimensions. In order to do so, we must decide what is an appropriate digital model for the k-dimensional Euclidean sphere S^k. The literature contains the following.

• Let X_{2k} be the set of all points $p\in Z^k$ such that p is a c_1-neighbor of the origin.

Then ([10], Proposition 4.1) X_{2k} is c_k-contractible. Notice that this example generalizes the contractibility of a 4-point digital simple closed curve [8] (see **Figure 2** for the planar version); the contractibility seems due to the smallness of the image, rather than its form.

• Let $Bd\ I_k$ denote the boundary of a digital k-cube, *i.e.*, for some integer $n>2$,

$$Bd\ I_k=\{(x_1,\ldots,x_k)\in[0,n-1]_Z^k\,|\,\textit{for some } i\in\{1,\ldots,k\},\ x_i\in\{0,n-1\}\}.$$

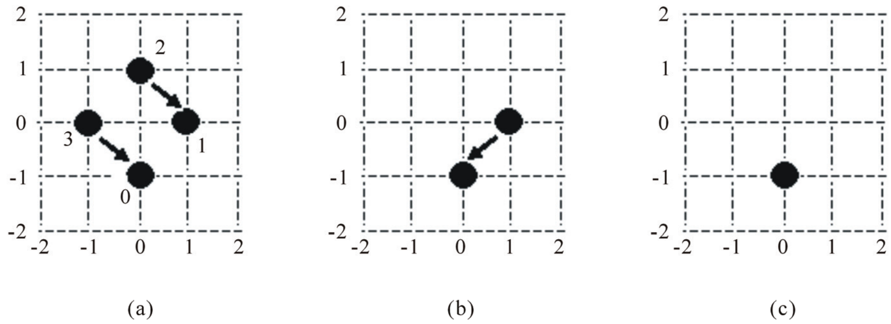

Figure 2. c_2**-contraction of** X_4**, a digital simple closed** c_2**-curve, via** $H: X_4 \times [0,2]_Z \to X_4$**. (a) shows** X_4**. Points are labeled by indices. We have** $H(x,0)=x$ **for all** $x \in X_4$**. Arrows show the "motion" of** x_2, x_3 **at the next step. (b) shows the results of the first step of the contraction:** $H(x_0,1)=H(x_3,1)=x_0$**;** $H(x_1,1)=H(x_2,1)=x_1$**. The arrow shows the motion at the next step of the contraction. (c) shows the results of the final step of the contraction:** $H(x_i,2)=x_0$ **for all** $i \in \{0,1,2,3\}$**.**

See **Figure 3** for the planar version. Then ([7], Corollary 5.9) for $n>2$, $Bd\, I_k$ is not c_1-contractible.

The next result may be interpreted as stating that for $n>4$, a map homotopic to 1_S must be a "rotation" of the points of S.

Theorem 3.4 *Let* S *be a simple closed* κ*-curve such that* $|S|=n>4$. *Let* $f: S \to S$ *be a* (κ,κ)*-continuous function such that* f *is* (κ,κ)*-homotopic to* 1_S. *Then, for some integer* j*, we have*

$f(x_i)=x_{(i+j) \bmod n}$ *for all* $i \in [0,n-1]_Z$.

Proof: Let $H: S \times [0,m]_Z \to S$ be a (κ,κ)-homotopy from 1_S to f. For $t \in [0,m]_Z$, let $H_t: S \to S$ be the induced map. The assertion follows from the following.

Claim 1: For each $t \in [0,m]_Z$, there is an integer j such that $H_t(x_i)=x_{(i+j) \bmod n}$ for all $i \in [0,n-1]_Z$.

To prove Claim 1, we argue by mathematical induction on t. For $t=0$, we can clearly take $j=0$. Now, suppose the claim is valid for $t \in [0,u]_Z$ such that $0 \le u < m$. Then, in particular, there is an integer j such that $H_u(x_i)=x_{(i+j) \bmod n}$ for all $i \in [0,n-1]_Z$. The continuity properties of homotopy imply $H_{u+1}(x_0) \in \{x_{(j-1) \bmod n}, x_j, x_{(j+1) \bmod n}\}$.

Without loss of generality, $H_{u+1}(x_0)=x_j$. This is the initial case of the following:

Claim 2: $H_{u+1}(x_k)=x_{(k+j) \bmod n}$ for all $k \in [0,n-1]_Z$.

Suppose the equation of Claim 2 is true for all $k \in [0,v]_Z$, for some $v \in [0,n-2]_Z$. In particular,

$$H_{u+1}(x_v)=x_{(v+j) \bmod n}. \quad (3)$$

By the continuity properties of homotopy, $H_{u+1}(x_{v+1})$ is adjacent to or coincides with $H_{u+1}(x_v)=x_{(v+j) \bmod n}$ and with $H_u(x_{v+1})=x_{(v+1+j) \bmod n}$. Thus, $H_{u+1}(x_{v+1}) \in \{x_{(v+j) \bmod n}, x_{(v+1+j) \bmod n}\}$. Since H_{u+1} must also be an isomorphism by Theorem 3.2, from Equation (3), $H_{u+1}(x_{v+1}) = x_{(v+1+j) \bmod n}$. This completes the induction proof for the Claim 2, which, in turn, completes the induction proof for the Claim 1. Thus, the assertion is established.

4. Antipodal Maps

A classical theorem of Euclidean topology, due to K. Borsuk, states that a continuous antipodal map $f: S^d \to S^d$ from the d-dimensional unit sphere to itself is not homotopic to a constant map [19]. In this section, we obtain a digital analog, Theorem 4.16, for $d=1$.

We say a set $X \subset Z^d$ is *symmetric with respect to the origin* if X satisfies the property that

$$x \in X \textit{ if and only if } -x \in X.$$

Suppose we have $X \subset Z^{d_0}$, $Y \subset Z^{d_1}$, and X is symmetric with respect to the origin. A function $f: X \to Y$ is called *antipodal-preserving* or an

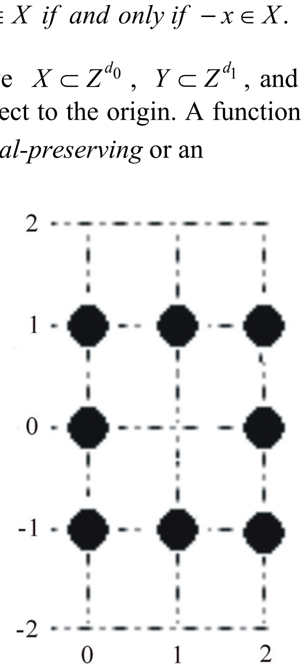

Figure 3. $Bd\, I_2$**, the "boundary" of a digital square.**

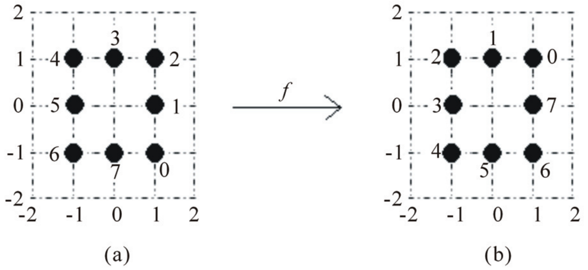

Figure 4. A digital simple closed c_1-curve $S=\{x_i\}_{i=0}^{7}$ and a (c_1,c_1)-continuous map $f:S\to S$ such that f is (c_1,c_1)-homotopic to 1_S. According to Theorem 3.4, such a map f must "rotate" the members of S. In (a), points of S are labeled by indices. In (b), each $y\in S$ is labeled by the index of the point $x_i\in S$ such that $f(x_i)=y$. Here, we have $f(x_i)=x_{(i+2)\bmod 8}$ for all i.

antipodal map if $f(-x)=-f(x)$ for all $x\in X$ [19]. In this section, we study properties of continuous antipodal maps between digital simple closed curves.

Lemma 4.1 *Let p_0,p_1 be c_l-adjacent points in Z^d, $1\le l\le d$. Then p_0 and p_1 are not antipodal.*

Proof: The hypothesis implies that there is an index i such that p_0 and p_1 differ by 1 in the i^{th} coordinate: $|p_{0,i}-p_{1,i}|=1$. If p_0 and p_1 are antipodal, this would imply $\{p_{0,i},p_{1,i}\}=\{-1/2,1/2\}$, which is impossible.

Lemma 4.2 *Let p_0,p_1 be c_l-adjacent points in Z^d, $1\le l\le d$. Then $-p_0$ and $-p_1$ are c_l-adjacent.*

Proof: Elementary, and left to the reader.

Lemma 4.3 *Let $S=\{x_i\}_{i=0}^{n-1}$ be a digital simple closed c_l-curve in Z^d such that the points of S are circularly ordered. If S is symmetric with respect to the origin, then the origin is not a member of S.*

Proof: Suppose the origin is a member of S. Without loss of generality, x_0 is the origin, and therefore is its own antipode.

By Lemma 4.2, x_1 and x_{n-1} are antipodes; by Lemma 4.1, these points are not c_l-adjacent. This establishes the base case of an induction argument: $S_1=\{x_i\}_{i=0}^{1}\cup\{x_{n-j}\}_{j=1}^{1}$ is a connected subset of S, such that x_1 and x_{n-1} are non-adjacent antipodes; hence, S_1 is (c_l,c_1)-isomorphic to a digital interval. Now, suppose for some integer k, $1\le k<\lfloor (n-1)/2\rfloor$, $S_k=\{x_i\}_{i=0}^{k}\cup\{x_{n-j}\}_{j=1}^{k}$ is (c_l,c_1)-isomorphic to a digital interval, with endpoints x_k and x_{n-k}, such that x_m and x_{n-m} are non-adjacent antipodes for all $m\in\{1,\ldots,k\}$. Then, by Lemma 4.2, x_{k+1} and x_{n-k-1} are antipodes, and by Lemma 4.1, these points are not c_l-adjacent. Thus, $S_{k+1}=\{x_i\}_{i=0}^{k+1}\cup\{x_{n-j}\}_{j=1}^{k+1}$ is (c_l,c_1)-isomorphic to a digital interval, with endpoints x_1 and x_{n-1}.

This completes an induction argument from which we conclude that

$$S'=\{x_i\}_{i=0}^{\lfloor (n-1)/2\rfloor}\cup\{x_i\}_{i=n-\lfloor (n-1)/2\rfloor}^{n-1}$$

is (c_l,c_1)-isomorphic to a digital arc, with the endpoints of S' being $x_{\lfloor (n-1)/2\rfloor}$ and $x_{n-\lfloor (n-1)/2\rfloor}$, such that $1\le i\le\lfloor (n-1)/2\rfloor$ implies x_i and x_{n-i} are non-adjacent antipodes.

• If n is odd, then $n-1$ is even, so $S'=S$. This is a contradiction, since S' is not a simple closed c_l-curve.

• If n is even, then $S'=S\setminus\{x_{n/2}\}$. Since S is symmetric with respect to the origin, we must have that $x_{n/2}$ is the origin. But since S is a simple closed c_l-curve in which x_0 is the origin, this is a contradiction.

Whether n is even or odd, the assumption that the origin belongs to S yields a contradiction. Hence, the origin is not a member of S.

Lemma 4.4 *Let $S=\{x_i\}_{i=0}^{n-1}$ be a digital simple closed c_l-curve in Z^d such that the points of S are circularly ordered. If S is symmetric with respect to the origin, then n is even.*

Proof: By Lemma 4.3, the origin is not a member of S, so every member of S is distinct from its antipode. Therefore, n must be even.

Lemma 4.5 *Let $S=\{x_i\}_{i=0}^{n-1}$ be a digital simple closed c_l-curve in Z^d such that the points of S are circularly ordered. If S is symmetric with respect to the origin, then for all i we have $-x_i=x_{(i+n/2)\bmod n}$.*

Proof: Suppose there is a simple closed c_l-curve $S=\{x_i\}_{i=0}^{n-1}$ that is symmetric with respect to the origin such that the points of S are circularly ordered, such that there exist indices u,v such that x_u and x_v are antipodes and $v\ne (u+n/2)\bmod n$. Without loss of generality, we can assume $u=0$, $v\ne n/2$.

Then, from Lemma 4.2, x_1 is antipodal to either x_{v-1} or $x_{(v+1)\bmod n}$. Without loss of generality, x_1 and x_{v-1} are antipodal. If $v-1=1$, this is a contradiction of Lemma 4.3; or, if $v-1=2$, this is a contradiction of Lemma 4.1; otherwise, we inductively repeat the argument above with the antipodal (by Lemma 4.2) pair (x_2,x_{v-2}), etc., until similarly we obtain a contradiction of Lemma 4.3 or of Lemma 4.1. The assertion follows.

We have the following.

Theorem 4.6 *Let $S_i\subset Z^{d_i}$ be simple closed κ_i-curves, $i\in\{0,1\}$, each symmetric with respect to the origin. Let $f:S_0\to S_1$ be a (κ_0,κ_1)-continuous antipodal map. Then f is onto.*

Proof: Let $S_1=\{x_i\}_{i=0}^{n-1}$, where the points of S_1 are circularly ordered. Without loss of generality, there exists $p\in S_0$ such that $f(p)=x_0$. Since f is antipodal, it follows from Lemma 4.5 that $f(-p)=x_{n/2}$. Since f is continuous and S_0 is κ_0-connected, it follows that one of the κ_1-paths in S_1 from x_0 to $x_{n/2}$ is contained in $f(S_0)$. Without loss of generality, $\{x_j\}_{j=0}^{n/2} \subset f(S_0)$. For $n/2<j<n$, there exists $q \in S_0$ such that $f(q)=x_{j-n/2}$, and from Lemma 4.5,

$x_j = -x_{j-n/2} = -f(q)$

$=$ (*since f is antipodal*) $f(-q) \in f(S_0)$.

Thus, $f(S_0)=S_1$.

Corollary 4.7 Let $S_i \subset Z^{d_i}$ be simple closed κ_i-curves, $i \in \{0,1\}$, each symmetric with respect to the origin. Let $f: S_0 \to S_1$ be a (κ_0,κ_1)-continuous antipodal map. If $|S_0|=|S_1|$, then f is a (κ,κ)-isomorphism.

Proof: By Theorem 4.6, f is onto. The assertion follows from Proposition 3.1.

Proposition 4.8 Let $X=\{x_i\}_{i=0}^{n_X-1}$ be a simple closed κ_X-curve with circularly ordered points. Let $Y=\{y_i\}_{i=0}^{n_Y-1}$ be a simple closed κ_Y-curve with circularly ordered points. Suppose $H: X\times[0,m]_Z \to Y$ is a (κ_X,κ_Y)-homotopy between the (κ_X,κ_Y)-continuous maps H_0, H_m such that $H_0(x_0)=H_m(x_0)$. Then there is a (κ_X,κ_Y)-homotopy $G: X\times[0,m]_Z \to Y$ between H_0 and H_m such that $G(x_0,t)= H_0(x_0)$ for all $t\in[0,m]_Z$

Proof: Without loss of generality, $H_0(x_0)=y_0$. Let $a:[0,m]_Z \to [0,m]_Z$ be defined by $a(t)=i$ if $H(x_0,t)=y_i$.

Let $G: X\times[0,m]_Z \to Y$ be defined by

$$G(x_i,t)=y_{[j-a(t)] \bmod n_Y}, \textit{ if } H(x_i,t)=y_j.$$

Roughly, we may think of $G(x_i,t)$ as rotating $H(x_i,t)$ counter to the rotation of $H(x_0,t)$, so that the image under G of x_0 at time t is constant with respect to t. We show below that G is a homotopy. In particular, $a(0)=a(m)=0$, so $G_0=H_0$ and $G_m=H_m$; and, for all $t\in[0,m]_Z$,

$$G(x_0,t)=y_{[a(t)-a(t)] \bmod n_Y}=y_0.$$

For each $x_i \in X$ and $t\in[0,m]_Z$, if $H_t(x_i)=y_j$ then we have the following.

• By the continuity of H_t, we have
$H_t(\{x_{(i-1) \bmod n_X}, x_{(i+1) \bmod n_X}\}) \subset \{y_{(j-1) \bmod n_Y}, y_j, y_{(j+1) \bmod n_Y}\}$.

Therefore, $G_t(x_i)=y_{[j-a(t)] \bmod n_Y}$ and

$$G_t\left(\left\{x_{(i-1) \bmod n_X}, x_{(i+1) \bmod n_X}\right\}\right) \subset \left\{y_{[j-1-a(t)] \bmod n_Y}, y_{[j-a(t)] \bmod n_Y}, y_{[j+1-a(t)] \bmod n_Y}\right\}.$$

Therefore, G_t is (κ_X,κ_Y)-continuous.

• Let $G_{x_i}:[0,m]_Z \to Y$ be the induced function defined by $G_{x_i}(t)=G(x_i,t)$. To simplify the following, let $G_{x_i}(-1)=H_{-1}(x_i)=H_0(x_i)$, $G_{x_i}(m+1)=H_{m+1}(x_i)=H_m(x_i)$, $a(-1)=a(0)=0=a(m)=a(m+1)$. Note that the continuity of H_{x_0} implies that

$$\{a(t-1),a(t+1)\} \subset \{a(t)-1,a(t),a(t)+1\}.$$

Suppose $H_t(x_i)=y_j$, so $G_{x_i}(t)=y_{[j-a(t)] \bmod n_Y}$.

If $a(t-1)=a(t)-1$, then $G_{x_i}(t-1)=y_{[j-a(t-1)] \bmod n_Y}=y_{[j-a(t)+1] \bmod n_Y}$ is κ_Y-adjacent to $G_{x_i}(t)$.

If $a(t-1)=a(t)$, then $G_{x_i}(t-1)=G_{x_i}(t)$.

If $a(t-1)=a(t)+1$, then $G_{x_i}(t-1)=y_{[j-a(t-1)] \bmod n_Y}=y_{[j-a(t)-1] \bmod n_Y}$ is κ_Y-adjacent to $G_{x_i}(t)$.

Similarly, $G_{x_i}(t+1)$ is κ_Y-adjacent to, or equal to, $G_{x_i}(t)$. Therefore, the induced function G_{x_i} is (c_1,κ_Y)-continuous.

Therefore, G is a homotopy between H_0 and H_m such that $G(x_0,t)=H_0(x_0)$ for all $t\in[0,m]_Z$.

Let (X,κ_X) and (Y,κ_Y) be digital images and let $y\in Y$. We denote by $\overline{y}: X\to Y$ (or, for short, $\overline{y}$) the constant map $\overline{y}(x)=y$ for all $x\in X$.

Lemma 4.9 Let (X,κ_X) and (Y,κ_Y) be digital simple closed curves and let $f: X\to Y$ be (κ_X,κ_Y)-homotopic to a constant map. Then for any $x_0\in X$, $f:(X,x_0)\to(Y,f(x_0))$ is (κ_X,κ_Y)-pointed homotopic to the constant map $\overline{f(x_0)}$.

Proof: Let $F: X\times[0,m_0]_Z \to Y$ be a (κ_X,κ_Y)-homotopy between f and the constant map $\overline{y_0}$ for some $y_0\in Y$. Since Y is κ_Y-connected, there is a path $p:[0,m_1]_Z\to Y$ from y_0 to $f(x_0)$. Then the map $H: X\times[0,m_0+m_1]_Z \to Y$ defined by

$$H(x,t)=\begin{cases} F(x,t) & \textit{if } 0\le t\le m_0; \\ p(t-m_0) & \textit{if } m_0\le t\le m_0+m_1, \end{cases}$$

is a homotopy between f and $\overline{f(x_0)}$. The assertion follows from Proposition 4.8.

Given a digital image (X,κ) and a positive integer n, for $x\in X$ we define [20]

$N_\kappa(x,n)=\{x\}\cup$

$\{y\in X \mid \textit{there is a } \kappa-\textit{path in } X$

$\textit{from } y \textit{ to } x \textit{ of length at most } n\}$.

The covering space and the lifting of maps are notions borrowed from algebraic topology that have been important in digital algebraic topology. We have the following.

Definition 4.10 [20] Let (E,κ_E) and (B,κ_B) be digital images. Let $p:E\to B$ be a (κ_E,κ_B)-continuous function. Suppose for each $b\in B$ there exists $\varepsilon\in N$ such that

• for some $\delta \in N$ and some index set M,

$$p^{-1}(N_{\kappa_B}(b,\varepsilon)) = \bigcup_{i \in M} N_{\kappa_E}(e_i,\delta) \text{ with } e_i \in p^{-1}(b);$$

• if $i, j \in M$, $i \neq j$, then $N_{\kappa_E}(e_i,\delta) \cap N_{\kappa_E}(e_j,\delta) = \varnothing$; and

• the restriction map $p|_{N_{\kappa_E}(e_i,\delta)} : N_{\kappa_E}(e_i,\delta) \to N_{\kappa_B}(b,\varepsilon)$ is a (κ_E,κ_B)-isomorphism for all $i \in M$.

Then the map p is a (κ_E,κ_B) - covering map, and the pair (E,p) is a (κ_E,κ_B) - covering (or covering space).

The following is a somewhat simpler characterization of the digital covering than given in Definition 4.10.

Theorem 4.11 [14] *Let (E,κ_E) and (B,κ_B) be digital images. Let $p: E \to B$ be a (κ_E,κ_B)-continuous function. Then the map p is a (κ_E,κ_B)-covering map if and only if for each $b \in B$, there is an index set M such that*

• $p^{-1}(N_{\kappa_B}(b,1)) = \cup_{i \in M} N_{\kappa_E}(e_i,1)$, with $e_i \in p^{-1}(b)$;

• if $i, j \in M$, $i \neq j$, then $N_{\kappa_E}(e_i,1) \cap N_{\kappa_E}(e_j,1) = \varnothing$; and

• the restriction map $p|_{N_{\kappa_E}(e_i,1)} : N_{\kappa_E}(e_i,1) \to N_{\kappa_B}(b,1)$ is a (κ_E,κ_B)-isomorphism for all $i \in M$.

The following is a minor generalization of an example of a digital covering map given in [20].

Example 4.12 *Let $C = \{c_i\}_{i=0}^{m-1} \subset Z^d$ be a circularly ordered simple closed κ-curve. Let $p: Z \to C$ be defined by $p(z) = z \bmod m$ for all $z \in Z$. Then p is a (c_1,κ)-covering map* (see **Figure 5**).

Definition 4.13 [20] *For digital images (E,κ_E), (B,κ_B), and (X,κ_X), let $p: E \to B$ be a (κ_E,κ_B) covering map, and let $f: X \to B$ be (κ_X,κ_B)-continuous. A* lifting of f with respect to p *is a (κ_X,κ_E)-continuous function $F: X \to E$ such that $p \circ F = f$.*

See **Figure 6** for an illustration of Definition 4.13.

Theorem 4.14 [20] *Let (E,κ_E) be a digital image and $e_0 \in E$. Let (B,κ_B) be a digital image and $b_0 \in B$. Let $p: E \to B$ be a (κ_E,κ_B)-covering map such that $p(e_0) = b_0$. Then a κ_B-path $f:[0,m]_Z \to B$ beginning at b_0 has a unique lifting with respect to p to a path $\tilde{f}$ in E starting at e_0.*

For a positive integer n, a (κ_E,κ_B)-covering map $p: E \to B$ is called a *radius n local isomorphism* [21] if for each $b \in B$ and $e \in p^{-1}(b)$, $p|_{N_{\kappa_E}(e,n)} \to N_{\kappa_B}(b,n)$ is an isomorphism. It was observed in [14] that every covering is a radius 1 local isomorphism, but there are coverings that are not radius 2 local isomorphisms.

Theorem 4.15 [21] *Let (E,κ_E) be a digital image and $e_0 \in E$. Let (B,κ_B) be a digital image and $b_0 \in B$. Let $p: E \to B$ be a (κ_E,κ_B)-covering map such that $p(e_0) = b_0$. Suppose p is a radius 2 local isomorphism. For κ_E-paths $g_0, g_1 : [0,m]_Z \to E$ that start at e_0, if there is a κ_B-homotopy in B from $p \circ g_0$ to $p \circ g_1$ that holds the endpoints fixed, then $g_0(m) = g_1(m)$, and there is a κ_E-homotopy in E from g_0 to g_1 that holds the endpoints fixed.*

Theorem 4.16 *Let $X = \{x_i\}_{i=0}^{n_X-1}$ be a simple closed κ_X-curve with points circularly ordered, that is symmetric with respect to the origin. Let $Y = \{y_i\}_{i=0}^{n_Y-1}$ be a simple closed κ_Y-curve with points circularly ordered, that is symmetric with respect to the origin. Let $f: X \to Y$ be a (κ_X,κ_Y)-continuous antipodal map. If $|Y| > 5$, then f is not (κ_0,κ_1)-nullhomotopic.*

Proof: Suppose there is such a function f that is (κ_0,κ_1)-nullhomotopic. From Lemma 4.9, f is point- ed homotopic to $\overline{f(x_0)}$.

Let $b:[0,n_X]_Z \to X$ be defined by $b(t) = x_{t \bmod n_X}$. Let $p: Z \to Y$ be the (c_1,κ_Y)-covering map defined by $p(z) = y_{z \bmod n_Y}$ (see Example 4.12). By Proposition 2.2, $f \circ b$ and $\overline{f(x_0)} \circ b$ are (c_1,κ_Y)-homotopic paths in Y. From Theorem 4.14, these functions have unique

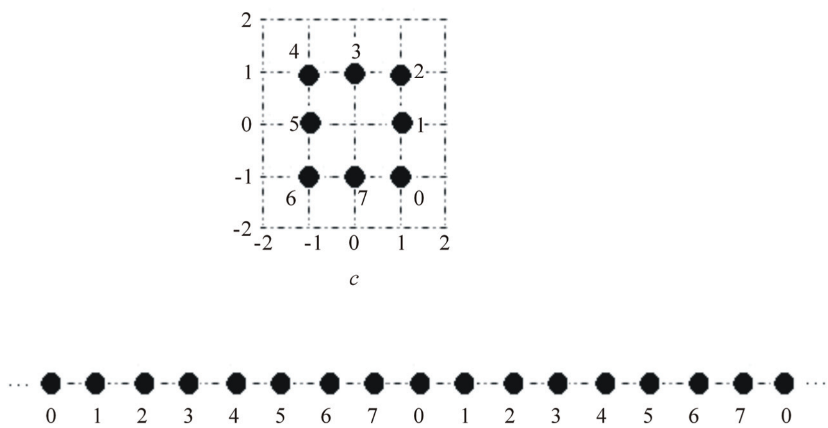

Figure 5. A simple closed c_1-curve C and a covering by the digital line Z. Members of C are labeled by their respective indices. A point $z \in Z$ is labeled by the index of point of C to which the covering map sends z.

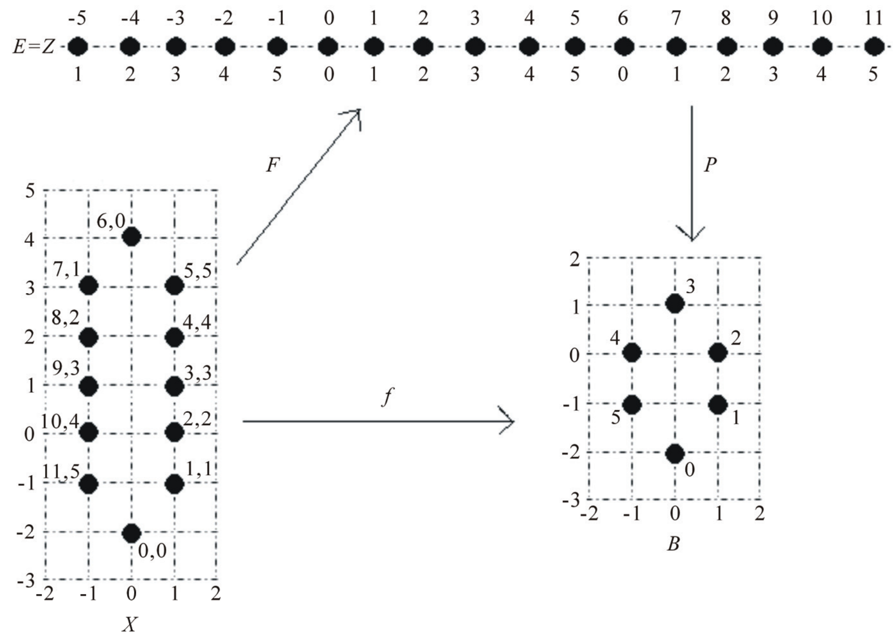

Figure 6. Example of lifting. $B=\{b_i\}_{i=0}^{5}$ is a simple closed c_2-curve whose members are labeled by their indices. $E=Z$ has its points z labeled above by their coordinates and labeled below by the index i such that $p(z)=b_i$ (note p is given by the formula $p(z)=b_{z \bmod 6}$). $X=\{x_m\}_{m=0}^{11}$ is a simple closed c_2-curve that has points labeled by a pair m,n such that m is the index of the point, $f(x_m)=b_n$ (thus, f is defined by $f(x_m)=b_{m \bmod 6}$), and $F(x_m)=m$. Since p is a covering map (by Example 4.12) and $p \circ F = f$, F is a lifting of f with respect to p.

liftings with respect to p to paths F_0 and F_1, respectively, in Z, each starting at $0 \in p^{-1}(f(b(0)))$. Since $|Y|>5$, $N_{\kappa Y}(f(x_0),2) \neq Y$, so p is a radius 2 local isomorphism. From Theorem 4.15, F_0 and F_1 must end at the same point. Indeed, this point must be 0, for the uniqueness of F_1 implies F_1 must be the constant map $\overline{0}$.

Since f is an antipodal map, we must have

$$|F_0(t+n_X/2)-F_0(t)|=n_Y/2 \text{ for all } t\in[0,n_X/2-1]_Z. \quad (4)$$

Since $F_0(0)=0$, $F_0(n_X/2)\in\{-n_Y/2,n_Y/2\}$. Without loss of generality,

$$F_0(n_X/2)=n_Y/2. \quad (5)$$

Since F is continuous and $n_Y>5$, a simple induction argument based on Equations (4) and (5) shows that $F_0(t+n_X/2)>F_0(t)$ for all $t\in[0,n_X/2-1]_Z$. But since $F_0(n_X)=0$, this implies with Equation (4) that $F_0(n_X/2)=-n_Y/2$, which contradicts Equation (5). The assertion follows from the contradiction.

5. Antipodes Mapped Together

A classical result of topology is that if f is a continuous map from the d-dimensional unit sphere S^d to Euclidean d-space R^d, then there is a pair of antipodes $x,-x\in S^d$ such that $f(x)=f(-x)$ [17]. For $d=1$, the following is a digital analog.

Theorem 5.1 *Let S be a digital simple closed c_l-curve with $S=\{x_i\}_{i=0}^{n-1}$, where the points of S are circularly ordered. Suppose S is symmetric with respect to the origin. Suppose $f:S\to Z$ is a (c_l,c_1)-continuous function. Then there is a pair of antipodes $x,-x\in S$ such that $|f(x)-f(-x)|\le 1$.*

Proof: By Lemma 4.4, n is even. Consider the function $g:[0,n-1]_Z\to Z$ defined by $g(i)=f(x_i)-f(x_{(i+n/2)\bmod n})$. By Lemma 4.5, we are done if, for some i, $g(i)=0$. Therefore, we assume for all i that

$$g(i)\neq 0. \quad (6)$$

Clearly, for all i,

$$g[(i+n/2)\bmod n]=-g(i). \quad (7)$$

The continuity of f implies that

$$|g(i)-g(i+1)|\le 2. \quad (8)$$

It follows from Equation (7) that g takes both positive and negative values, so from inequality (6), there is an index j such that $g(j)$ and $g(j+1)$ have opposite sign; without loss of generality, $g(j)>0$ and $g(j+1)<0$. From inequality (8), it follows that

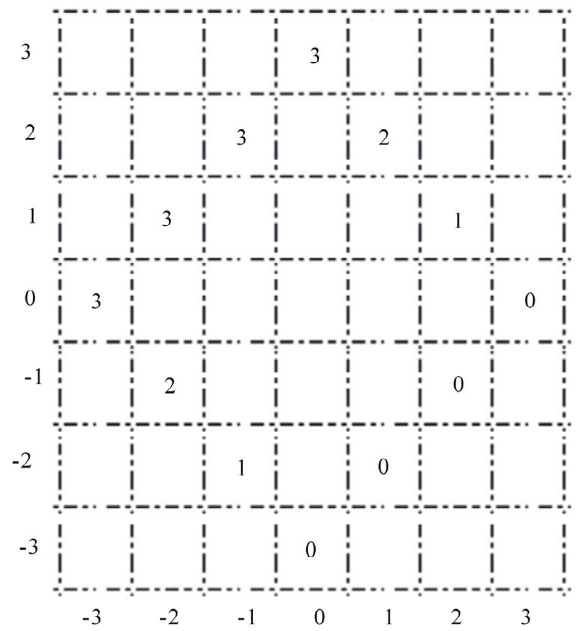

Figure 7. S **and** $f:S\to Z$ **. Each number in the grid labels a point of** S **, showing the image of the grid point under** f **. Note for** $s_0=(1,2)$ **,** $|f(s_0)-f(-s_0)|=1$ **, but there is no** $s\in S$ **for which** $f(s)=f(-s)$ **.**

$g(j)=1$, and the assertion follows.

Theorem 5.1 parallels, in a sense, a result of [2]: In the Euclidean line, a continuous function mapping an interval to itself has a fixed point; in the digital world, a (c_1,c_1)-continuous function $f:[a,b]_Z\to[a,b]_Z$ has a "near-fixed" point, *i.e.*, a point x such that $|x-f(x)|\le 1$.

That we cannot, in general, conclude the existence of antipodes mapped to the same point in Theorem 5.1, is illustrated in the following example (note the simple closed curve has more than 4 points). Let $S=\{(x,y)\mid |x|+|y|=3\}$. Then S is a simple closed c_2-curve in Z^2. The function $f:S\to Z$ given by

$f(-3,0)=f(-2,1)=f(-1,2)=f(0,3)=3$,

$f(-2,-1)=f(1,2)=2$,

$f(-1,-2)=f(2,1)=1$,

$f(0,-3)=f(1,-2)=f(2,-1)=f(3,0)=0$,

is a (c_2,c_1)-continuous function such that $|f(x)-f(-x)|\neq 0$ for each $x\in S$. See **Figure 7**.

6. Further Remarks

In this paper, we have obtained several analogs of classical theorems of Euclidean topology concerning maps on digital simple closed curves. We have shown that digital simple closed curves of more than 4 points are not contractible; that a continuous antipodal map from a digital simple closed curve to itself is not nullhomotopic; and that a continuous map from a digital simple closed curve to the digital line must map a pair of antipodes within 1 of each other. Except as indicated concerning whether or not a digital model of a sphere is contractible, it is not known at the current writing whether these results extend to higher dimensional digital models of Euclidean spheres.

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7. References

[1] A. Rosenfeld, "Digital Topology," *American Mathematical Monthly*, Vol. 86, 1979, pp. 76-87.

[2] A. Rosenfeld, "Continuous Functions on Digital Pictures," *Pattern Recognition Letters*, Vol. 4, No. 3, 1986, pp. 177-184.

[3] A. Rosenfeld, "Directions in Digital Topology," 11*th Summer Conference on General Topology and Applications*, 1995. http://atlas-conferences.com/cgi-bin/abstract/caaf-71

[4] Q. F. Stout, "Topological Matching," *Proceedings* 15*th Annual Symposium on Theory of Computing*, Boston, 1983, pp. 24-31.

[5] T. Y. Kong, "A Digital Fundamental Group," *Computers and Graphics*, Vol. 13, No. 1, 1989, pp. 159-166.

[6] T. Y. Kong, A. W. Roscoe and A. Rosenfeld, "Concepts of Digital Topology," *Topology and Its Applications*, Vol. 46, No. 3, 1992, pp. 219-262.

[7] L. Boxer, "Digitally Continuous Functions," *Pattern Recognition Letters*, Vol. 15, No. 8, 1994, pp. 833-839.

[8] L. Boxer, "A Classical Construction for the Digital Fundamental Group," *Journal of Mathematical Imaging and Vision*, Vol. 10, No. 1, 1999, pp. 51-62.

[9] T. Y. Kong and A. Rosenfeld, "Topological Algorithms for Digital Image Processing," Elsevier, New York, 1996.

[10] L. Boxer, "Homotopy Properties of Sphere-Like Digital Images," *Journal of Mathematical Imaging and Vision*, Vol. 24, No. 2, 2006, pp. 167-175.

[11] G. T. Herman, "Oriented Surfaces in Digital Spaces," *CVGIP*: *Graphical Models and Image Processing*, Vol. 55, No. 1, 1993, pp. 381-396.

[12] L. Chen, "Gradually Varied Surfaces and Its Optimal Uniform Approximation," *SPIE Proceedings*, Bellingham, Vol. 2182 1994, pp. 300-307.

[13] L. Chen, "Discrete Surfaces and Manifolds," Scientific Practical Computing, Rockville, 2004.

[14] L. Boxer, "Digital Products, Wedges, and Covering Spaces," *Journal of Mathematical Imaging and Vision*, Vol. 25, 2006, pp. 159-171.

[15] U. Eckhardt and L. Latecki, "Digital Topology," In:

Current Topics in Pattern Recognition Research, Research Trends, Council of Scientific Information, 1994. http://cosmic.rrz.uni-hamburg.de/webcat/mathematik/eckhardt/eck00001/eck00001.pdf

[16] R. O. Duda, P. E. Hart and J. H. Munson, "Graphical Data Processing Research Study and Experimental Investigation," *Descriptive Note*: *Quarterly Report No*. 7, March 1967, pp. 28-30.

[17] E. Khalimsky, "Motion, Deformation, and Homotopy in Finite Spaces," *Proceedings IEEE International Conference on Systems, Man, and Cybernetics*, Boston, 1987, pp. 227-234.

[18] L. Boxer, "Properties of Digital Homotopy," *Journal of Mathematical Imaging and Vision*, Vol. 22, No. 1, 2005, pp. 19-26.

[19] J. Dugundji, "Topology," Allyn and Bacon, Inc., Boston, 1966.

[20] S. E. Han, "Non-Product Property of the Digital Fundamental Group," *Information Sciences*, Vol. 171, No. 1-3, 2005, pp. 73-91.

[21] S. E. Han, "Digital Coverings and Their Applications," *Journal of Applied Mathematics and Computing*, Vol. 18, No. 1-2, 2005, pp. 487-495.